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A new class of continued fraction expansions

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1. Introduction. Let x be an irrational real number. It is well known that x can be written as an infinite fraction

$$a_0+\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{}{\ldots+\cfrac{1}{a_n+\ldots}}}},\qquad \text{where } a_0\in \mathbf{Z},\ a_n\in \mathbf{N}=\mathbf{Z}_{\geqslant 1},$$

the regular continued fraction expansion of x. Truncation yields the sequence of regular convergents $(P_n/Q_n)_{n\geqslant -1}$, which converges to x. Here $\gcd(P_n, Q_n) = 1$, $Q_n \geqslant 1$ for all $n \geqslant 0$.

Each regular convergent is always a *best approximation* to x, that is, there do not exist better approximations with smaller denominators:

$$\forall r, s\in \mathbf{Z}, 0 < s \leqslant Q_n:\quad \left|x-\frac{r}{s}\right| \leqslant \left|x-\frac{P_n}{Q_n}\right| \Rightarrow \frac{r}{s}=\frac{P_n}{Q_n}.$$

(The converse does not hold, see [Per], §16.)

In this paper we are mainly interested in semi-regular continued fractions which improve the approximation properties of the regular continued fraction. In view of the above-mentioned *best approximation property* we are in particular interested in semi-regular continued fraction expansions of x whose sequences of convergents form subsequences of the sequence of regular convergents of x. A crucial role in obtaining these continued fractions is played by an operation on the numbers a_n, $n \geqslant 1$, the so-called partial quotients of x. This operation, the *singularization of a partial quotient* a_n *equal to* 1, is studied in detail in Section 2.

Using singularizations we then define in Section 4 a new class of semi-regular continued fraction expansions, the so-called S-expansions. This class contains the classical nearest integer continued fraction, Hurwitz' singular continued fraction, Minkowski's diagonal continued fraction and the important α-expansions, introduced by Hitoshi Nakada in [N].

A metrical theory for these S-expansions is developed in Section 5. This general theory contains the metrical results of Rieger [R1] for the nearest integer continued fraction and those of Nakada [N] for his α-expansions as special cases, see also Section 6. Our general theory also yields the analogues of theorems by Legendre, Vahlen, Lévy and Adams, and lead recently to hitherto unknown properties of Minkowski's diagonal expansion [Kr2]. Our proofs are in general considerably shorter than those in the literature for the corresponding case.

We start by giving some definitions and basic results on continued fractions, which we will need for further reference.

(1.1) DEFINITION. In this paper a *continued fraction* is understood to be a pair of two finite or infinite sequences $(\varepsilon_n)_{n\geq 1}$, with $\varepsilon_n \in \{\pm 1\}$, $n \geq 1$, and $(a_n)_{n\geq 0}$, with $a_n \in \mathbf{Z}$, $n \geq 0$, where in the finite case both sequences end with the same index. In the finite case we denote the continued fraction by

$$a_0+\varepsilon_1 \,/\!\!\!\lceil\, a_1+\ldots+\varepsilon_n \,/\!\!\!\lceil\, a_n,$$

in the infinite case we write

$$a_0+\varepsilon_1 \,/\!\!\!\lceil\, a_1+\ldots+\varepsilon_n \,/\!\!\!\lceil\, a_n+\ldots$$

The integers a_n, $n \geq 0$, are called the *partial quotients* of the continued fraction.

(1.2) DEFINITION. A finite or infinite continued fraction is called *semi-regular* when $a_0 \in \mathbf{Z}$; $a_n \in \mathrm{N}$, $n \geq 1$; $\varepsilon_{n+1}+a_n \geq 1$, $n \geq 1$, and, in the infinite case, $\varepsilon_{n+1}+a_n \geq 2$ infinitely often.

(1.3) SOME EXAMPLES.

(i) In case $\varepsilon_n = +1$, $a_n \in N$, $n \geq 1$ we have the so-called *regular* or *simple continued fraction*, see also Section 3. In the sequel we denote the regular continued fraction $a_0+1 \,/\!\!\!\lceil\, a_1+\ldots$ by $[a_0; a_1, \ldots]$.

(ii) In case $a_0 \in \mathbf{Z}$, $a_n \geq 2$, $n \geq 1$ and $\varepsilon_{n+1}+a_n \geq 2$, $n \geq 1$ we have the so-called *nearest integer continued fraction*, introduced by Minnigerode in [Minn] and studied by Hurwitz in [H]. See also [Per], §43, and [R1].

(iii) In case $\varepsilon_1 = +1$, $a_0 \in \mathbf{Z}$, $a_n \in N$, $a_n \equiv 1 \pmod 2$, $\varepsilon_{n+1}+a_n \geq 2$, $n \geq 1$, we have the *continued fraction with odd partial quotients*, introduced by G. J. Rieger in [R2]. For more details, see [Sch] and [Bar].

(iv) Other examples of semi-regular continued fractions are Minkowski's *diagonal continued fraction*, see [Min], [Per], §45, and [Kr2], Wieb Bosma's *optimal continued fraction*, see [Bos], and Hitoshi Nakada's α-*expansions*, see [N]. These α-expansions will be discussed in Section 6 of this paper.

In the sequel we are mainly interested in infinite semi-regular continued fractions. In particular we will introduce and study a class of semi-regular continued fractions obtained from the regular continued fraction via a so-called *singularization process*, see Sections 2 and 4.

(1.4) DEFINITION. Let $a_0+\varepsilon_1 \,/\!\!\!\lceil\, a_1+\ldots$ be a (finite or infinite) continued fraction. The matrices $A_n = A_n(\varepsilon_n, a_n)$, $M_n = M_n(a_0+\varepsilon_1 \,/\!\!\!\lceil\, a_1+\ldots+\varepsilon_n \,/\!\!\!\lceil\, a_n)$, $n \geq 0$, are defined by

$$A_0 := \begin{bmatrix} 1 & a_0 \\ 0 & 1 \end{bmatrix}; \quad A_n := \begin{bmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{bmatrix}, \quad n \geq 1,$$

$$M_n := A_0 A_1 \ldots A_n, \quad n \geq 0,$$

with the obvious restriction on n in the finite case.

Moreover, the numbers $\delta_n \in \{\pm 1\}$, $n \geq 0$, are defined by

$$\delta_n := \det M_n.$$

(1.5) Remarks. Notice that

$$\delta_n = (-1)^n \varepsilon_1 \varepsilon_2 \ldots \varepsilon_n, \quad n \geq 1,$$

since $\det A_n = -\varepsilon_n$, $n \geq 1$. Putting

$$p_{-1} := 1, \quad p_0 := a_0, \quad q_{-1} := 0, \quad q_0 := 1,$$

i.e.

$$M_0 =: \begin{bmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{bmatrix}, \quad \text{and} \quad M_n =: \begin{bmatrix} r_n & p_n \\ s_n & q_n \end{bmatrix}, \quad n \geq 1,$$

we have, due to

(1.6) $$M_0 = A_0, \quad M_n = M_{n-1} A_n, \quad n \geq 1,$$

that

$$M_n = \begin{bmatrix} p_{n-1} & a_n p_{n-1}+\varepsilon_n p_{n-2} \\ q_{n-1} & a_n q_{n-1}+\varepsilon_n q_{n-2} \end{bmatrix}, \quad n \geq 1.$$

Hence we find

(1.7) THEOREM. *Let $a_0+\varepsilon_1 \,/\!\!\!\lceil\, a_1+\ldots$ be a (finite or infinite) continued fraction. Then there exists a (finite or infinite) sequence $(p_n, q_n)_{n\geq -1}$ in $\mathbf{Z}^2$ such that*

$$M_n = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}, \quad n \geq 0.$$

The sequences $(p_n)_{n\geq -1}$ and $(q_n)_{n\geq -1}$ satisfy the recurrence relations:

(1.8) $$\begin{cases} p_{-1} := 1, & p_0 := a_0, \quad p_n = a_n p_{n-1}+\varepsilon_n p_{n-2}, \\ q_{-1} := 0, & q_0 := 1, \quad q_n = a_n q_{n-1}+\varepsilon_n q_{n-2}. \end{cases}$$

One has

$$\gcd(p_n, q_n) = 1, \quad \gcd(q_n, q_{n+1}) = 1, \quad n \geq -1.$$

(1.9) COROLLARY. *Let $a_0+\varepsilon_1\,/\,a_1+\ldots$ be a (finite or infinite) semi-regular continued fraction. Then the sequence $(q_n)_{n\geqslant 1}$ is monotonically increasing if and only if $\varepsilon_n+a_n\geqslant 1$, $n\geqslant 2$.*

(1.10) COROLLARY. *Let $a_0+\varepsilon_1\,/\,a_1+\ldots$ be a (finite or infinite) continued fraction. Then*

$$\det\begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix} = \delta_n, \quad n\geqslant 0,$$

see also Definition (1.4),

$$\det\begin{bmatrix} p_{n-1} & p_{n+1} \\ q_{n-1} & q_{n+1} \end{bmatrix} = a_{n+1}\,\delta_n, \quad n\geqslant 0,$$

$$\det\begin{bmatrix} p_{n-1} & p_{n+2} \\ q_{n-1} & q_{n+2} \end{bmatrix} = (a_{n+1}\,a_{n+2}+\varepsilon_{n+2})\,\delta_n, \quad n\geqslant 0,$$

with the obvious restrictions on n in the finite case.

Hence

$$\left|\det\begin{bmatrix} p_{n-1} & p_{n+1} \\ q_{n-1} & q_{n+1} \end{bmatrix}\right| = 1 \quad \text{if and only if } |a_{n+1}| = 1, \text{ where } n\geqslant 0$$

and, in case of a semi-regular continued fraction, with moreover $\varepsilon_n+a_n\geqslant 1$, $n\geqslant 1$:

$$\left|\det\begin{bmatrix} p_{n-1} & p_{n+2} \\ q_{n-1} & q_{n+2} \end{bmatrix}\right| > 1, \quad n\geqslant 0.$$

(1.11) **Remark.** Let

$$C=\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}, \quad \text{with } \alpha,\beta,\gamma,\delta\in\mathbf{Z},\ \alpha\delta-\beta\gamma=\pm 1.$$

Associated with C we define the Möbius transformation $C\colon \mathbf{R}\cup\{\infty\}\to\mathbf{R}\cup\{\infty\}$ by

$$C(x):=\frac{\alpha x+\beta}{\gamma x+\delta},\quad x\neq\frac{-\delta}{\gamma},\ \infty \quad\text{and}\quad C\left(\frac{-\delta}{\gamma}\right)=\infty;\quad C(\infty)=\frac{\alpha}{\gamma}.$$

Due to Theorem (1.7) we have

$$p_n/q_n = M_n(0), \quad n\geqslant 0.$$

From (1.6) it follows that

$$\begin{aligned} M_n(0) &= M_{n-1}A_n(0) = M_{n-2}A_{n-1}(\varepsilon_n/a_n) = \ldots \\ &= M_0\left(\cfrac{\varepsilon_1}{a_1+\cfrac{\varepsilon_2}{a_2+\ldots+\cfrac{\varepsilon_n}{a_n}}}\right) = a_0+\cfrac{\varepsilon_1}{a_1+\ldots+\cfrac{\varepsilon_n}{a_n}}, \quad n\geqslant 1. \end{aligned}$$

This leads in a natural way to the following definition.

(1.12) DEFINITION. Let $a_0+\varepsilon_1\,/\,a_1+\ldots$ be a (finite or infinite) continued fraction and let p_n, q_n, $n\geqslant 0$, be as in Theorem (1.7). Then we put

$$a_0+\varepsilon_1\,/\,a_1+\ldots+\varepsilon_n\,/\,a_n := p_n/q_n, \quad n\geqslant 0,$$

that is, $a_0+\varepsilon_1\,/\,a_1+\ldots+\varepsilon_n\,/\,a_n$ equals by definition the value of the finite fraction

$$a_0+\cfrac{\varepsilon_1}{a_1+\ldots+\cfrac{\varepsilon_n}{a_n}}, \quad\text{where}\quad \frac{1}{0}:=\infty \text{ and } \frac{1}{\infty}:=0.$$

An infinite continued fraction $a_0+\varepsilon_1\,/\,a_1+\ldots+\varepsilon_n\,/\,a_n+\ldots$ is said to be *convergent* if and only if

(1.13) $$\lim_{n\to\infty} p_n/q_n \text{ exists and is finite.}$$

Let $x\in\mathbf{R}$ be the limit from (1.13). Then x is called the *value* of the continued fraction and we write

$$x = a_0+\varepsilon_1\,/\,a_1+\ldots+\varepsilon_n\,/\,a_n+\ldots$$

The sequence $(p_n/q_n)_{n\geqslant -1}$ is called the *sequence of convergents* of the continued fraction $a_0+\varepsilon_1\,/\,a_1+\ldots$

(1.14) **Remark.** Let $a_0+\varepsilon_1\,/\,a_1+\ldots$ be a (finite or infinite) semi-regular continued fraction and let $(p_n/q_n)_{n\geqslant -1}$ be its sequence of convergents. A simple induction argument shows that

(1.15) $$1\,/\,a_1+\ldots+\varepsilon_n\,/\,a_n\in[1/(a_1+1),\,1], \quad n\geqslant 1,$$

with the obvious restriction on n in the finite case.

Each infinite semi-regular continued fraction converges to an irrational number, see [T], [Per], Ch.V. We will give a proof of this for infinite semi-regular continued fractions $a_0+\varepsilon_1\,/\,a_1+\ldots+\varepsilon_n\,/\,a_n+\ldots$ which satisfy

(1.16) $$\varepsilon_n+a_n\geqslant 1, \quad n\geqslant 1.$$

From Corollary (1.9) and

$$\frac{p_{n-1}}{q_{n-1}}-\frac{p_n}{q_n}=\frac{\delta_n}{q_{n-1}q_n}, \quad n\geqslant 1,$$

it at once follows that $(p_n/q_n)_{n\geqslant -1}$ is a Cauchy sequence, that is, $a_0+\varepsilon_1\,/\,a_1+\ldots+\varepsilon_n\,/\,a_n+\ldots$ is convergent, with limit, say, $x\in\mathbf{R}$. Hence the infinite semi-regular continued fraction

$$\varepsilon_{n+1}\,\underline{/}\,a_{n+1}+\ldots+\varepsilon_{n+m}\,\underline{/}\,a_{n+m}+\ldots$$

is convergent, with limit, say, $t_n \in \mathbf{R}$. In view of (1.15) we have

$$\varepsilon_{n+1}\, t_n \in [1/(a_{n+1}+1),\, 1], \qquad n \geqslant 0.$$

It follows from Definition (1.12) that

(1.17) $$t_n = \frac{\varepsilon_{n+1}}{a_{n+1}+t_{n+1}}, \qquad n \geqslant 0,$$

and from Definition (1.2) and (1.15):

(1.18) $$\varepsilon_{n+1}\, t_n \in \left[1/(a_{n+1}+1),\, 1\right), \qquad n \geqslant 0.$$

Moreover, we have

$$x = M_{n-1}\begin{bmatrix} 0 & \varepsilon_n \\ 1 & a_n+t_n \end{bmatrix}(0), \qquad n \geqslant 1,$$

hence

(1.19) $$x = \frac{(a_n+t_n)\,p_{n-1}+\varepsilon_n\, p_{n-2}}{(a_n+t_n)\,q_{n-1}+\varepsilon_n\, q_{n-2}} = \frac{p_n+t_n\, p_{n-1}}{q_n+t_n\, q_{n-1}}, \qquad n \geqslant 0,$$

(1.20) $$x-\frac{p_n}{q_n} = \frac{\delta_n\, t_n}{q_n\,(q_n+t_n\, q_{n-1})}, \qquad n \geqslant 0.$$

From (1.18) and (1.19) it follows that

$$|q_n x - p_n| < |q_{n-1} x - p_{n-1}|, \qquad n \geqslant 1.$$

From this one can see that x is irrational.

(1.21) Remark. Since every infinite semi-regular continued fraction converges to an irrational number one finds that (1.17), (1.18), (1.19) and (1.20) hold for every infinite semi-regular continued fraction.

(1.22) LEMMA. *Let $a_0+\varepsilon_1\,\underline{/}\,a_1+\ldots$ be a (finite or infinite) continued fraction. Then*

$$q_{n-1}/q_n = 0+1\,\underline{/}\,a_n+\varepsilon_n\,\underline{/}\,a_{n-1}+\ldots+\varepsilon_2\,\underline{/}\,a_1, \qquad n \geqslant 1,$$

with the obvious restriction on n in the finite case.

Proof. This lemma follows directly from the second recurrence relation in (1.8) or from the observation that

$$M_n^T = \begin{bmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{bmatrix}^T = \begin{bmatrix} 0 & 1 \\ \varepsilon_n & a_n \end{bmatrix}\begin{bmatrix} 0 & 1 \\ \varepsilon_{n-1} & a_{n-1} \end{bmatrix}\cdots\begin{bmatrix} 0 & 1 \\ \varepsilon_1 & a_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a_0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}\begin{bmatrix} \varepsilon_n & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & a_{n-1} \end{bmatrix}\cdots\begin{bmatrix} 0 & 1 \\ 1 & a_1 \end{bmatrix}\begin{bmatrix} \varepsilon_1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ a_0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & a_n \end{bmatrix}\begin{bmatrix} 0 & \varepsilon_n \\ 1 & a_{n-1} \end{bmatrix}\cdots\begin{bmatrix} 0 & \varepsilon_2 \\ 1 & a_1 \end{bmatrix}\begin{bmatrix} \varepsilon_1 & 0 \\ a_0 & 1 \end{bmatrix}.$$

Hence

$$M_n^T(0) = q_{n-1}/q_n = 0+1\,\underline{/}\,a_n+\varepsilon_n\,\underline{/}\,a_{n-1}+\ldots+\varepsilon_2\,\underline{/}\,a_1, \qquad n \geqslant 1$$

since

$$\begin{bmatrix} \varepsilon_1 & 0 \\ a_0 & 1 \end{bmatrix}(0) = 0. \ \blacksquare$$

(1.23) DEFINITION. Let $x \in \mathbf{R}$, $p/q \in \mathbf{Q}$ where we assume that $q > 0$, $\gcd(p, q) = 1$. Then we define

$$\theta(x, p/q) := q\,|qx-p|.$$

A direct consequence of this definition, (1.8) and the formulas (1.17), (1.18) and (1.20) is:

(1.24) LEMMA. *Let $a_0+\varepsilon_1\,\underline{/}\,a_1+\ldots$ be an infinite semi-regular continued fraction, with value x. Then*

$$\theta(x, p_n/q_n) = \varepsilon_{n+1}\, t_n/(1+t_n v_n), \qquad n \geqslant 0,$$

$$\theta(x, p_{n-1}/q_{n-1}) = v_n/(1+t_n v_n), \qquad n \geqslant 1,$$

where

$$v_n := q_{n-1}/q_n, \qquad n \geqslant 0.$$

(1.25) Remarks.

(i) Instead of $\theta(x, p_n/q_n)$ we often write $\theta_n(x)$, or shortly θ_n. For the regular continued fraction we moreover write Θ_n instead of θ_n. Notice that we always have: $0 < \Theta_n < 1$, $n \geqslant 0$.

(ii) The numbers θ_n, $n \geqslant 0$, connected with the (finite or infinite) semi-regular continued fraction $a_0+\varepsilon_1\,\underline{/}\,a_1+\ldots$ are called the *approximation coefficients* of $a_0+\varepsilon_1\,\underline{/}\,a_1+\ldots$ For more results on the regular approximation coefficients, see Section 3.

2. The singularization process. The theory which we develop in this paper is based upon a process by which a continued fraction can sometimes be transformed into another one, with better approximation properties. Fundamental are the following matrix identities, which are easily checked:

(2.1) $$\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & \varepsilon \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} 1 & a+\varepsilon \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & -\varepsilon \\ 1 & b+1 \end{bmatrix},$$

(2.2) $$\begin{bmatrix} 0 & \varepsilon \\ 1 & a \end{bmatrix}\begin{bmatrix} 0 & \mu \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & b \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon \\ 1 & a+\mu \end{bmatrix}\begin{bmatrix} 0 & -\mu \\ 1 & b+1 \end{bmatrix},$$

with arbitrary a, b, ε and μ.

(2.3) DEFINITION. Let

$$a_0+\varepsilon_1 \lrcorner\!\ /\ a_1+\dots \tag{2.4}$$

be a (finite or infinite) continued fraction with for some $n \geq 0$:

$$a_{n+1} = 1, \quad \varepsilon_{n+2} = 1, \quad \text{i.e.}$$

$$a_0+\varepsilon_1 / a_1+\dots = a_0+\varepsilon_1 / a_1+\dots+\varepsilon_n / a_n+\varepsilon_{n+1} / 1+1 / a_{n+2}+\dots$$

The transformation σ_n which changes this continued fraction into the continued fraction

$$a_0+\varepsilon_1 / a_1+\dots+\varepsilon_n / (a_n+\varepsilon_{n+1})+(-\varepsilon_{n+1}) / (1+a_{n+2})+\dots \tag{2.5}$$

is called a *singularization*. We say that we have singularized the pair $a_{n+1} = 1$, $\varepsilon_{n+2} = 1$.

We will now study the effect of this singularization σ_n upon the sequence of vectors

$$\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k \geq -1}$$

connected with (2.4). Let $(M_k)_{k\geq 0}$ be the sequence of matrices as defined in (1.4), connected with (2.4) and let $(M_k^*)_{k\geq 0}$ be the sequence connected with (2.5). We will write

$$\begin{pmatrix} p_k^* \\ q_k^* \end{pmatrix} = M_k^* \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad k \geq 0.$$

Clearly

$$M_k^* = M_k, \quad k = 0, 1, \dots, n-1,$$

and, in view of the identities (2.1) and (2.2)

$$M_k^* = M_{k+1}, \quad k = n+1, n+2, \dots$$

The two matrices M_n and M_{n+1} are replaced by M_n^*, where

$$M_n^* = M_{n-1}\begin{bmatrix} 0 & \varepsilon_n \\ 1 & a_n+\varepsilon_{n+1} \end{bmatrix} = M_{n+1}\begin{bmatrix} 0 & \varepsilon_{n+1} \\ 1 & 1 \end{bmatrix}^{-1}\begin{bmatrix} 0 & \varepsilon_n \\ 1 & a_n \end{bmatrix}^{-1}\begin{bmatrix} 0 & \varepsilon_n \\ 1 & a_n+\varepsilon_{n+1} \end{bmatrix}.$$

From this we see that

$$M_n^* = M_{n+1}\begin{bmatrix} -\varepsilon_{n+1} & 0 \\ \varepsilon_{n+1} & 1 \end{bmatrix}$$

and hence that

$$\begin{pmatrix} p_n^* \\ q_n^* \end{pmatrix} = M_{n+1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} p_{n+1} \\ q_{n+1} \end{pmatrix}$$

since

$$\begin{bmatrix} -\varepsilon_{n+1} & 0 \\ \varepsilon_{n+1} & 1 \end{bmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

(2.6) Thus the sequence $\begin{pmatrix} p_k^* \\ q_k^* \end{pmatrix}_{k\geq -1}$ is obtained from $\begin{pmatrix} p_k \\ q_k \end{pmatrix}_{k\geq -1}$ by removing the term $\begin{pmatrix} p_n \\ q_n \end{pmatrix}$ from the latter.

(2.7) Remark. Suppose that $a_{n+1} = 1$, $\varepsilon_{n+2} = 1$ and $a_{m+1} = 1$, $\varepsilon_{m+2} = 1$ for two integers n and m which are not consecutive. Then clearly one may singularize both pairs independently of the order, or what comes down to the same, at the same time. Similarly for more than two, or even infinitely many non-consecutive blocks $a_{i+1} = 1$, $\varepsilon_{i+2} = 1$.

On the other hand, one can singularize a block of consecutive pairs $a_{n+1} = 1$, $\varepsilon_{n+2} = 1; \dots; a_{n+l} = 1$, $\varepsilon_{n+l+1} = 1$, $l > 0$ step by step, but then the order in which we singularize subsequent pairs definitely determines the outcome, as is shown by the following example.

(2.8) EXAMPLE. Consider the continued fraction $1 / 2+(-1) / 1+1 / 1 +1 / 1+1 / 0+\dots$ Singularizing the pair $a_3 = 1$, $\varepsilon_4 = 1$ we get $1 / 2+(-1) / 2+(-1) / 2+1 / 0+\dots$, while successively singularizing $a_2 = 1$, $\varepsilon_3 = 1$ and $a_4 = 1$, $\varepsilon_5 = 1$ yields $1 / 1+1 / 2+1 / 1+1 / 0+\dots$ and $1 / 1+1 / 3 +(-1) / 1+\dots$, or in reversed order $1 / 2+(-1) / 1+1 / 2+(-1) / 1+\dots$ and $1 / 1+1 / 3+(-1) / 1+\dots$

(2.9) DEFINITION. A *singularization process* consists of a *set* of continued fractions and a *law* which determines in an unambiguous way the pairs $a_{n+1} = 1$, $\varepsilon_{n+2} = 1$ from each continued fraction of the given set that will be singularized.

(2.10) Remark. The sequence of convergents of the »final« continued fraction is a subsequence of the sequence of convergents of the original one. Hence, if the original continued fraction converges to x, so does the new one. The new continued fraction converges faster than the original one, see also Section 4.

In Section 4 we give more precise definitions, but to illustrate here already the idea we give some examples.

(2.11) EXAMPLES OF SINGULARIZATION PROCESSES. (i) The class of continued fractions to be singularized is that of the regular continued fractions. The law is:

From every block of m consecutive pairs $a_{n+1} = 1$, $\varepsilon_{n+2} = 1; \dots; a_{n+m} = 1$, $\varepsilon_{n+m+1} = 1$, *where* $m \in \mathbb{N} \cup \{\infty\}$, $a_{n+m+1} \neq 1$ *and* $a_n \neq 1$ *in case* $n > 0$, *we singularize the first, third, fifth etc. pair.*

Applying this law yields a semi-regular continued fraction $a_0+\varepsilon_1 \lrcorner\!\!\!\int a_1 +\ldots+\varepsilon_n \lrcorner\!\!\!\int a_n+\ldots$ which satisfies:

$$\varepsilon_n\in\{\pm 1\},\ n\geqslant 1;\quad a_0\in\mathbf{Z};\quad a_n\geqslant 2,\ n\geqslant 1,\quad \text{and}\quad \varepsilon_{n+1}+a_n\geqslant 2,\ n\geqslant 1.$$

Thus we see that this singularization process yields the nearest integer continued fraction, see (1.3) (ii) and [Per], Satz 5.17, p. 160.

(ii) Again the class of continued fractions to be singularized is the class of regular continued fractions. Now the law is:

From every block of m consecutive pairs $a_{n+1}=1,\ \varepsilon_{n+2}=1;\ \ldots;\ a_{n+m}=1,\ \varepsilon_{n+m+1}=1$, *where* $m\in\mathbf{N}\cup\{\infty\}$, $a_{n+m+1}\neq 1$ *and* $a_n\neq 1$ *in case* $n>0$, *we singularize the first, third, etc. pair in case m is odd and we singularize the second, fourth etc. in case m is even.*

Applying this law yields a semi-regular continued fraction $a_0+\varepsilon_1 \lrcorner\!\!\!\int a_1 +\ldots+\varepsilon_n \lrcorner\!\!\!\int a_n+\ldots$ which satisfies:

$$\varepsilon_n\in\{\pm 1\},\ n\geqslant 1;\quad a_0\in\mathbf{Z};\ a_n\geqslant 2,\ n\geqslant 1,\quad \text{and}\quad \varepsilon_n+a_n\geqslant 2,\ n\geqslant 1.$$

Hence this law gives *Hurwitz' singular continued fraction*, see [Per], §44.

(iii) The class of continued fractions to be singularized consists of the continued fractions with odd partial quotients. The law is now:

Singularize each pair $a_{n+1}=1,\ \varepsilon_{n+2}=1,\ n\geqslant 0$ *for which* $\varepsilon_{n+1}=-1$.

Each irrational number x has a unique continued fraction expansion with odd partial quotients, see e.g. [R2] and [Sch]. In [Bar] it is shown that applying this law one obtains a semi-regular continued fraction whose sequence of convergents is contained in the sequence of regular convergents of x and which contains the sequence of nearest integer continued fraction convergents of x.

(2.12) Remark. Let $a_0+\varepsilon_1 \lrcorner\!\!\!\int a_1+\ldots$ be a (finite or infinite) semi-regular continued fraction with value $x\in\mathbf{R}$, such that the sequence of its convergents forms a subsequence of the sequence of regular convergents of x. As in [Bos] we denote this by

$$\mathrm{SRCF}(x)\subseteq\mathrm{RCF}(x).$$

From the fact that we always can invert a singularization, cf. [Per], §40, it follows that we have:

(2.13) THEOREM. *Let* $a_0+\varepsilon_1 \lrcorner\!\!\!\int a_1+\ldots+\varepsilon_n \lrcorner\!\!\!\int a_n+\ldots$ *be an infinite semi-regular continued fraction, with value* x, *which satisfies* (1.16). *Then*

$$\mathrm{SRCF}(x)\subseteq\mathrm{RCF}(x)\quad \textit{if and only if}\quad \varepsilon_n+\varepsilon_{n+1}>2-2a_n,\quad n\geqslant 1.$$

3. The regular continued fraction. In this section we will describe briefly some properties of the regular continued fraction which we shall need in subsequent sections. Recall that in case of the regular continued fraction we have by definition: $\varepsilon_n=+1,\ a_n\in\mathbf{N},\ n\geqslant 1$, see (1.3) (i).

(3.1) DEFINITION. The *regular* or *simple continued fraction operator* T: $[0,1)\to[0,1)$ is defined by

$$Tx:=1/x-[1/x],\quad x\neq 0;\qquad T0:=0,$$

where $[\cdot]$ denotes the entier (or floor) function. The function B: $[0,1)\to\mathbf{N}\cup\{\infty\}$ is defined by

$$B(x):=[1/x],\quad x\neq 0;\qquad B(0):=\infty.$$

Let $\omega\in\mathbf{R}$, $B_0:=[\omega]$ and $x:=\omega-B_0$. Put $T_0:=x$, $T_n:=T^n x$, $n\geqslant 1$, and $B_n:=B(T_{n-1})$, $n\geqslant 1$. Notice that when $T_i\neq 0$, $0\leqslant i\leqslant n-1$, we have

$$\omega=B_0+\cfrac{1}{B_1+\cfrac{1}{B_2+\ldots+\cfrac{1}{B_n+T_n}}}. \tag{3.2}$$

In case $\omega\in\mathbf{Q}$ it follows from the Euclidean algorithm that there exists a non-negative integer n such that $T_n=0$. Then ω equals the finite regular fraction

$$B_0+\cfrac{1}{B_1+\ldots+\cfrac{1}{B_n}},$$

with $B_n\geqslant 2$, i.e.

$$\omega=[B_0;B_1,\ldots,B_n],\quad \text{see also Definition (1.12).}$$

In the sequel we assume that $\omega\in\mathbf{R}\backslash\mathbf{Q}$. Then $T_n\in[0,1)\backslash\mathbf{Q}$ for all $n\in\mathbf{N}$ and therefore $B_n\in\mathbf{N}$ for all $n\in\mathbf{N}$.

From Remark (1.14) it follows that $[B_0;\ B_1,\ldots,B_n,\ldots]$, the regular continued fraction obtained in this way from $\omega\in\mathbf{R}\backslash\mathbf{Q}$, is convergent. Due to (1.20) and (3.2) we moreover have

$$\omega-\frac{P_n}{Q_n}=\frac{(-1)^n T_n}{Q_n(Q_n+T_nQ_{n-1})},\quad n\geqslant 0, \tag{3.3}$$

where $(P_n/Q_n)_{n\geqslant -1}$ denotes the sequence of (regular) convergents of $[B_0;\ B_1,\ldots,B_n,\ldots]$. Since $(Q_n)_{n\geqslant -1}$ is a monotonically increasing sequence in $\mathbf{N}$ and $T_n\in[0,1)\backslash\mathbf{Q}$, for all $n\in\mathbf{N}$, we at once have

$$\omega=[B_0;\ B_1,\ldots,B_n,\ldots].$$

Moreover

$$T_n=[0;\ B_{n+1},\ldots,B_{n+m},\ldots],\quad n\geqslant 0,$$

$$V_n := Q_{n-1}/Q_n = [0; B_n, \ldots, B_1], \quad n \geq 1,$$

see also Lemma (1.22). A simple but useful consequence of Definition (3.1) is

(3.4) LEMMA. *Let* $\xi = [B_0; B_1, \ldots, B_n, \ldots]$ *and* $\eta = [B_0; B_1, \ldots, B_n^*, \ldots]$, *where* $B_n \neq B_n^*$, $n \geq 0$. *If* n *is even, then*

$$B_n < B_n^* \Leftrightarrow \xi < \eta,$$

if n *is odd, then*

$$B_n < B_n^* \Leftrightarrow \xi > \eta.$$

Three classical results in the theory of the regular continued fraction are the theorems of Legendre, Vahlen and Borel which we will state now for further reference.

(3.5) THEOREM ([Leg]). *Let* $x \in \boldsymbol{R}\backslash\boldsymbol{Q}$, $p/q \in \boldsymbol{Q}$ *with* $\gcd(p,q) = 1$, $q > 0$, *such that*

$$\theta(x, p/q) < 1/2.$$

Then there exists a non-negative integer n *such that*

$$\binom{p}{q} = \binom{P_n}{Q_n},$$

or in words: p/q *is a regular convergent of* x. *The constant* $1/2$ *is best possible.*

(3.6) Remark. By "the constant 1/2 is best possible" we mean that for every $\varepsilon > 0$ there exist an irrational number x and a rational number p/q, $\gcd(p, q) = 1$, $q > 0$, such that p/q is not a regular convergent of x and $\theta(x, p/q) < 1/2 + \varepsilon$.

(3.7) THEOREM ([V]). *Let* $x \in \boldsymbol{R}\backslash\boldsymbol{Q}$ *and let* Θ_n, $n \geq 0$, *be defined as in* (1.25) (i). *Then for every* $n \geq 0$ *one has*

$$\min(\Theta_n, \Theta_{n+1}) < 1/2,$$

and the constant $1/2$ *is best possible.*

(3.8) THEOREM ([Bor]). *For every* $n \geq 1$ *one has*

$$\min(\Theta_{n-1}, \Theta_n, \Theta_{n+1}) < 1/\sqrt{5},$$

and the constant $1/\sqrt{5}$ *is best possible.*

A basic result in the *metrical* theory of continued fractions is:

(3.9) THEOREM. *Let* $\Omega := [0, 1)$ *and let* $\mathscr{B}$ *be the collection of Borel subsets of* Ω. *Define the Gauss-measure* μ *on* $(\Omega, \mathscr{B})$ *by*

$$\mu(E) := \frac{1}{\log 2} \int_E \frac{dx}{1+x}, \quad E \in \mathscr{B}.$$

Then $(\Omega, \mathscr{B}, \mu, T)$ *forms an ergodic system, with* T *the operator from Definition* (3.1).

For a proof of this, see e.g. [Bil].

(3.10) Remark. It was discovered by Doeblin and Ryll-Nardzewski that many classical probabilistic theorems from the theory of the regular continued fraction can be proved by using Theorem (3.9). See [D], [R-N] and also [Bil]. To mention some of them:

For almost all x we have:

$$\lim_{n\to\infty} \frac{1}{n} \log q_n = \frac{\pi^2}{12 \log 2} \quad \text{(Lévy, 1929)},$$

$$\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{P_n}{Q_n} \right| = \frac{-\pi^2}{6 \log 2} \quad \text{(Lévy, 1929)},$$

$$\lim_{n\to\infty} \sqrt[n]{B_1 \ldots B_n} = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n(n+1)}\right)^{\log n/\log 2} = 2.685452\ldots \quad \text{(Khintchine, 1934)}.$$

In 1983 the application of ergodic theory in this part of number theory got a new impetus by a paper of W. Bosma, H. Jager and F. Wiedijk, see [BJW]. Essential in this paper is the use of the *natural extension* of the ergodic system $(\Omega, \mathscr{B}, \mu, T)$ from Theorem (3.9), see also [NIT] and [N].

(3.11) THEOREM. *Let* $\underline{\Omega} := \Omega \times [0, 1]$, $\mathscr{B}$ *the collection of Borel subsets of* $\underline{\Omega}$. *Define the two-dimensional Gauss-measure* μ *on* $(\underline{\Omega}, \mathscr{B})$ *by*

$$\mu(E) := \frac{1}{\log 2} \iint_E \frac{dx\, dy}{(1+xy)^2}, \quad E \in \mathscr{B}.$$

Define the operator $\bar{T}$: $\underline{\Omega} \to \underline{\Omega}$ *by*

$$\bar{T}(x, y) := \left(Tx, \frac{1}{B(x)+y}\right), \quad (x, y) \in \underline{\Omega},$$

where B *is the function defined in* (3.1). *Then* $(\underline{\Omega}, \mathscr{B}, \mu, \bar{T})$ *forms an ergodic system.*

(3.12) Remark. Theorem (3.11) constitutes a fundamental ingredient of the theory to be developed hereafter. A proof of Theorem (3.11) can be found in [NIT] and in [N], but can also be obtained by applying [CFS], Th. 1, p. 241 to Theorem (3.9).

In the sequel we will use several times the following important consequence of Theorem (3.11):

(3.13) THEOREM. *For almost all irrational numbers* x *the two-dimensional sequence*

$$\left(T_n, \frac{Q_{n-1}}{Q_n}\right)_{n\geq 0}$$

is distributed over $\underline{\Omega}$ according to the density function $d(x, y) := (\log 2)^{-1} \times (1+xy)^{-2}$.

For a proof, see [J2]. A proof of its analogue for the nearest integer continued fraction can be found in [Kr1], p. 182.

4. S-expansions. In this section, which forms the main part of this paper, we introduce a certain class $\mathscr{C}$ of singularization processes (see Definition (2.9)). All these singularization processes work on the set of the regular continued fractions. The semi-regular continued fractions obtained by a singularization process from the class $\mathscr{C}$ will be called *singularization expansions*, or simply *S-expansions.*

Without giving here already the exact definition of the class $\mathscr{C}$ we mention that the two singularization processes described in (2.11) (i) and (ii) both belong to $\mathscr{C}$, hence the nearest integer continued fraction and Hurwitz' singular continued fraction are two examples of these S-expansions. Other examples of S-expansions are Hitoshi Nakada's α-expansions, see Section 6, Minkowski's diagonal expansion, [Kr2], and Wieb Bosma's optimal continued fraction, [Bos], [BK1]. A continued fraction which is not an S-expansion is the continued fraction with odd partial quotients, [R2], [Sch], [Bar].

(4.1) Remark. From now on we will apply singularizations to the regular continued fraction, where all the ε_n are equal to $+1$. Therefore we will speak of "singularizing $B_{n+1} = 1$" instead of "singularizing the pair $B_{n+1} = 1$, $\varepsilon_{n+2} = 1$".

In case of the nearest integer continued fraction it follows from the definition of the operator $\bar{T}$ in Theorem (3.11) that the law in (2.11) (i):

»singularize in each block of m consecutive partial quotients equal to 1, where $m \in N \cup \{\infty\}$, the first, third,... etc. partial quotient«,

is equivalent to

»singularize B_{n+1} if and only if $(T_n, V_n) \in S_{\mathrm{I}}$, $n \geqslant 0$«

where T_n, V_n are as in Section 3, i.e. $T_n = [0; B_{n+1}, \ldots]$, $V_n = [0; B_n, \ldots, B_1]$ and where S_{I} is the following subset of $\underline{\Omega}$:

$$S_{\mathrm{I}} := [1/2, 1) \times [0, g]. \tag{4.2}$$

Here and in the sequel the numbers g and G are defined by:

$$g := \tfrac{1}{2}(\sqrt{5}-1), \qquad G := g+1.$$

In a similar way one can verify that the law from (2.11) (ii), leading to Hurwitz' singular continued fraction, is equivalent to

»singularize B_{n+1} if and only if $(T_n, V_n) \in S_{\mathrm{II}}$, $n \geqslant 0$«

where

$$S_{\mathrm{II}} := [g, 1) \times [0, 1]. \tag{4.3}$$

These two examples lead to the idea to prescribe by a subset S of $\underline{\Omega}$ which partial quotients are to be singularized in the regular continued fraction, in the form of the condition $(T_n, V_n) \in S$. Such a set S cannot be arbitrary but must satisfy the condition

$$S \subseteq [1/2, 1) \times [0, 1]$$

since otherwise B_{n+1} would not be equal to 1, and must also satisfy the condition

$$S \cap \bar{T} S = \emptyset,$$

otherwise one would prescribe to singularize two consecutive partial quotients equal to 1, which is impossible.

We are thus led in a natural way to the two central definitions of this paper, which describe exactly the above mentioned class $\mathscr{C}$:

(4.4) Definition. Let $(\underline{\Omega}, \mathscr{B}, \mu, \bar{T})$ be the ergodic system from Theorem (3.11). A subset S of $\underline{\Omega}$ is called a *singularization area* if it satisfies the following conditions:

(I) $S \in \mathscr{B}$ and S is a μ-continuity set;
(II) $S \subseteq [1/2, 1) \times [0, 1]$;
(III) $S \cap \bar{T} S = \emptyset$.

(4.5) Definition. Let S be a singularization area and x an irrational number. The *S-expansion of* x is that semi-regular continued fraction expansion converging to x which is obtained from the regular expansion of x by singularizing the partial quotients B_{n+1} if and only if $(T_n, V_n) \in S$.

(4.6) Remarks. (i) As is usual, by a μ-continuity set we mean a set S with the property that $\mu(\partial S) = 0$. We need this condition on S to be able to draw the following conclusion:

Let x be an irrational number with regular continued fraction expansion

$$[B_0; B_1, \ldots, B_n, \ldots]$$

and let $A(S, N)$ be defined by

$$A(S, N) := \# \{j \leqslant N; (T_j, V_j) \in S\}.$$

Then for almost all x we have

$$\lim_{N \to \infty} \frac{A(S, N)}{N} = \mu(S),$$

and as is well known, see [KN], p. 174, 175 and also [BP], the most general condition in this respect is μ-continuity. Here we need of course also Theorem (3.13).

(ii) The sets S_{I} and S_{II} from (4.2) resp. (4.3) do not in fact satisfy condition (III). Indeed, $S \cap \bar{T} S$ consists in both cases of a line segment.

Taking

$$S_{\mathrm{I}}^* := [1/2, g] \times [0, g] \cup (g, 1) \times [0, g), \quad S_{\mathrm{II}}^* := [g, 1) \times [0, g] \cup (g, 1) \times (g, 1]$$

instead of S_{I} resp. S_{II} we have

$$S_i^* \cap \bar{T} S_i^* = \{(g, g)\}, \quad i = \mathrm{I}, \mathrm{II}.$$

Notice that (g, g) is a fixed point of $\bar{T}$, a very fundamental property of $\bar{T}$, see [JK]. Since for all $x \in \boldsymbol{R}$ and $n \geqslant 0$,

$$(T_n, V_n) \neq (g, g),$$

because $V_n \in \boldsymbol{Q}$, $g \in \boldsymbol{R}/\boldsymbol{Q}$, we will admit that instead of (III) we have

(III)′ $S \cap \bar{T} S \subseteq \{(g, g)\}$

and still call S a singularization area.

Since

$$\mu([\tfrac{1}{2}, 1] \times [0, 1]) = (\log 2)^{-1} \log \tfrac{4}{3} = 0.41503\ldots,$$

a singularization area can, in view of condition (II) in its definition, never have a μ-measure greater than 0.41503... But condition (III) in the definition of singularization area causes the maximal possible μ-measure of a singularization area to be essentially smaller, as is shown in the next theorem.

(4.7) THEOREM. *Let S be a singularization area. Then*

$$\mu(S) \leqslant 1 - \frac{\log G}{\log 2} = 0.30575\ldots$$

This constant is best possible.

Proof. Define $M_1 := S_{\mathrm{I}}^*$ with S_{I}^* as before and $M_2 := [0, g) \times (g, 1] \cup [g, 1) \times [g, 1]$. One easily verifies that $\bar{T}(M_1) = M_2$ and that

$$\mu(M_1) = \mu(M_2) = 1 - \frac{\log G}{\log 2}.$$

Next we put $S_1 := S \cap M_1$, $S_2 := S \cap M_2$. Clearly

$$\bar{T}(S_1) \cup S_2 \subseteq M_2$$

and, in view of condition (4.4) (III),

$$\bar{T}(S_1) \cap S_2 = \varnothing,$$

see also Figure 1.

Now one sees that

$$\mu(S) = \mu(S_1) + \mu(S_2) = \mu(\bar{T} S_1) + \mu(S_2) = \mu(\bar{T} S_1 \cup S_2) \leqslant \mu(M_2) = 1 - \frac{\log G}{\log 2}.$$

That a singularization area actually can have this maximal measure $1 - (\log 2)^{-1} (\log G)$ is shown by the two examples S_{I}^* and S_{II}^*. ■

(4.8) DEFINITION. A singularization area S is called *maximal* if

$$\mu(S) = 1 - \frac{\log G}{\log 2}.$$

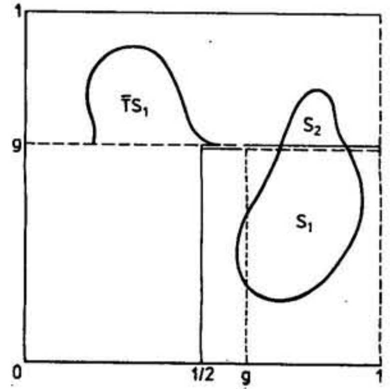


Fig. 1

Let S be a singularization area and let $x = [0; B_1, \ldots, B_n, \ldots]$. Suppose that B_S is a subset of $\underline{\Omega}$ such that for all x we have:

The partial quotient B_{n+1} equals 1 and is unchanged by the S-singularization

$$\Leftrightarrow (T_n, V_n) \in B_S.$$

Clearly such a subset always exists and has the properties

(1) $B_S \subseteq [1/2, 1) \times [0, 1]$ since $B_{n+1} = 1$;

(2) $B_S \cap S = \varnothing$ since B_{n+1} is not singularized;

(3) $\bar{T}^{-1} B_S \cap S = \varnothing$ since B_n is not singularized;

(4) $\bar{T} B_S \cap S = \varnothing$ since B_{n+2} is not singularized.

This subset B_S determines the occurrence of partial quotients equal to 1 in the S-expansion of x. Its μ-measure indicates e.g. the probability that a partial quotient be equal to 1.

(4.9) DEFINITION. Let S be a singularization area. Then the subset B_S of $\underline{\Omega}$, defined by

$$B_S := ([1/2, 1) \times [0, 1]) \setminus (S \cup \bar{T}^{-1} S \cup \bar{T} S)$$

is called the *area of the preservation of* 1's.

(4.10) Remark. In [Kr2] the area of the preservation of ones for Minkowski's diagonal continued fraction is explicitly given. Its μ-measure equals

$$\frac{1}{\log 2}(\log(\sqrt{2}-1)+\sqrt{2}-\tfrac{1}{2}) = 0.0473\ldots$$

In (2.11) (i) & (ii) we saw that the partial quotients in the nearest integer continued fraction and Hurwitz' singular continued fraction are never equal to 1. Further we saw that the singularization areas which yield these particular expansions are both maximal. Indeed, in both cases $B_S = \varnothing$. That for S-expansions with a maximal S the partial quotients are with probability 1 always greater than 1 is shown by the next theorem.

(4.11) THEOREM. *Let S be a singularization area and let B_S be as defined in* (4.9). *Then*

$$S \text{ is maximal} \Rightarrow \mu(B_S) = 0.$$

Proof. Let M_1, M_2, S_1 and S_2 be as in the proof of Theorem (4.7). Put moreover

$$B_1 := B_S \cap M_1; \quad B_2 := B_S \cap M_2.$$

It is now easy to see that

$$\bar{T} B_1 \cap (\bar{T} S_1 \cup S_2) = \varnothing, \quad \bar{T} B_1 \cup \bar{T} S_1 \cup S_2 \subseteq M_2,$$
$$B_2 \cap (\bar{T} S_1 \cup S_2) = \varnothing, \quad B_2 \cup \bar{T} S_1 \cup S_2 \subseteq M_2.$$

Hence we at once have, since S is maximal,

$$\mu(B_2) = 0, \quad \mu(B_1) = \mu(\bar{T} B_1) = 0,$$

which proves the theorem. ∎

(4.12) Remark. It is not difficult to show that the converse of this theorem does not hold. We hope to return to this in the future.

Let S be a singularization area and x an irrational number. Then $(r_k/s_k)_{k\geq -1}$, the sequence of S-convergents of x forms a subsequence of the sequence of regular convergents $(P_n/Q_n)_{n\geq -1}$ of x, see also (2.10). Thus there exists a monotonic function n_S: $N \to N$ such that

$$\begin{pmatrix} r_k \\ s_k \end{pmatrix} = \begin{pmatrix} P_{n_S(k)} \\ Q_{n_S(k)} \end{pmatrix}, \quad k \geq 1.$$

(4.13) THEOREM. *Let S be a singularization area. Then for almost all x we have*

$$\lim_{k\to\infty} \frac{n_S(k)}{k} = \frac{1}{1-\mu(S)}.$$

Proof. From the definition of n_S it follows that

$$n_S(k) = k + \sum_{j=1}^{n_S(k)} \chi_S(T_j, V_j),$$

where χ_S denotes the indicator function of the set S. Hence from Theorem (3.11) and the μ-continuity of S we see that for almost all x

$$1 = \lim_{k\to\infty} \frac{k}{n_S(k)} + \lim_{k\to\infty} \frac{1}{n_S(k)} \sum_{j=1}^{n_S(k)} \chi_S(T_j, V_j) = \lim_{k\to\infty} \frac{k}{n_S(k)} + \mu(S). \; ∎$$

(4.14) Remark. Notice that Theorem (4.13) implies that

$$\lim_{k\to\infty} \frac{n_S(k)}{k} \leq \frac{\log 2}{\log G} = 1.4404\ldots,$$

the maximum being attained if and only if S is maximal. In words: the sparsest subsequences are given by the maximal singularization areas. Since the S for the nearest integer continued fraction is maximal, we have proved a theorem by William W. Adams, see [Ad] and also [J1], [N].

The following corollary gives the analogues for S-expansions of the two classical theorems of Lévy, quoted in (3.10).

(4.15) COROLLARY. *Let S be a singularization area and let $(r_k/s_k)_{k\geq -1}$ be the sequence of S-convergents of the irrational number x. Then for almost all x we have*

$$\lim_{k\to\infty} \frac{1}{k}\log s_k = \frac{1}{1-\mu(S)} \frac{\pi^2}{12\log 2},$$
$$\lim_{k\to\infty} \frac{1}{k}\log\left|x - \frac{r_k}{s_k}\right| = \frac{1}{1-\mu(S)} \frac{-\pi^2}{6\log 2}.$$

Proof. This is an immediate consequence of (3.10) and Theorem (4.13); we have for almost all x

$$\lim_{k\to\infty} \frac{1}{k}\log s_k = \lim_{k\to\infty} \frac{n_S(k)}{k} \frac{1}{n_S(k)} \log Q_{n_S(k)} = \frac{1}{1-\mu(S)} \frac{\pi^2}{12\log 2}$$

and similarly for the second equality. ∎

The function n_S introduced above is a measure for the number of regular convergents that are missing in the S-expansion. On the other hand, by the mechanism of singularization itself, this number equals the number of ε's in the S-expansion that are equal to -1. It is easily seen that the precise relationship is expressed by

$$n_S(k) - k = \tfrac{1}{2}\Big(k - \sum_{x=1}^{k} \varepsilon_x\Big).$$

Hence we have the following

(4.16) COROLLARY. *Let $a_0 + \varepsilon_1 \,\lrcorner\!\!/\, a_1 + \ldots + \varepsilon_n \,\lrcorner\!\!/\, a_n + \ldots$ be the S-expansion of the irrational number x. Then for almost all x one has*

$$\lim_{k\to\infty} \frac{1}{k} \sum_{x=1}^{k} \varepsilon_x = \frac{1-3\mu(S)}{1-\mu(S)}.$$

(4.17) Remarks. (i) Using a formula of Spence and Abel for the dilogarithm, see [Lew], p.8, G. J. Rieger gave a proof of Corollary (4.15) for the special case of the nearest integer continued fraction, see [R1], Satz 5, 6. But we see here that these transcendent techniques can be avoided. This was also observed in [J1].

(ii) The minimum of the expression in Corollary (4.16) is attained when S is maximal, the minimum being

$$\frac{1}{\log G} \log \frac{G^3}{4} = 0.11915\ldots$$

This value was already found by Rieger in [R1] for the nearest integer continued fraction, see also [Ad], [J1].

We conclude this section by giving the analogue of Legendre's Theorem (3.5) for S-expansions.

(4.18) Theorem. *Define for a real number z with $0 < z \leqslant 1$ the set $A(z)$ by*

$$A(z) := \{(T, V) \in \Omega;\ T/(1+TV) < z,\ V \in Q\}$$

and the real number c_S by

$$c_S := \sup\{z \in (0, 1];\ A(z) \cap S = \emptyset\}.$$

Put

$$L_S := \min(c_S, 1/2).$$

Then for every rational number P/Q where $\gcd(P, Q) = 1$, $Q > 0$, and every irrational number x we have

$$\theta(x, P/Q) < L_S \Rightarrow P/Q \text{ is an } S\text{-convergent of } x.$$

The constant L_S is best possible.

Proof. Suppose that $\theta(x, P/Q) < L_S$ and that P/Q is not an S-convergent of x. Since $L_S \leqslant 1/2$ we have, due to Legendre's Theorem (3.5), that P/Q is a regular convergent of x, i.e. there exists an integer n such that $P/Q = P_n/Q_n$. By definition of S-expansions we now have, since P_n/Q_n is not an S-convergent, that $(T_n, V_n) \in S$. By definition of L_S we have

$$\frac{T_n}{1+T_n V_n} \geqslant c_S \geqslant L_S.$$

Due to (1.24) and (1.25) we then have

$$\theta(x, P/Q) = \Theta_n \geqslant L_S,$$

contrary to the hypothesis.

From the definition of L_S and Theorem (3.13) it follows at once that L_S is best possible. ∎

(4.19) Example. Of course, the expression for L_S in Theorem (4.18) is rather theoretical. But in many cases it yields without any effort the numerical value of the Legendre constant. Let us consider for example the case of the nearest integer continued fraction. Here the S is the S_1^* from (4.6) (ii), and one finds at once that c_S is determined by the point $(1/2, g)$ and that in fact

$$c_S = (1/2)(1+(1/2)g)^{-1} = g^2 = 0.38196\ldots$$

Recently, Shunji Ito [I] developed a theory to determine the Legendre constants for a whole class of continued fractions. Ito's method is more general and it contains our result on S-expansions. However, his method is much more complicated.

The value g^2 for the Legendre constant in case of the nearest integer continued fraction was also found recently by still another method, see [JK]. For further examples see Section 6 and [Bos], Theorem (4.31).

5. The two-dimensional ergodic system connected with the shift operator for S-expansions. In this section we will show that for each S-expansion there exists an »underlying« two-dimensional ergodic system. These ergodic systems will be obtained via an induced transformation from $(\Omega, \mathscr{B}, \mu, \bar{T})$, the two-dimensional ergodic system underlying the regular continued fraction. From the thus obtained ergodic systems we will then deduce further metrical theorems for the corresponding S-expansions.

Let S be a singularization area and let $x = [B_0; B_1, \ldots, B_n, \ldots]$ be an irrational number. Let $a_0 + \varepsilon_1 \,\underline{/}\, a_1 + \ldots + \varepsilon_n \,\underline{/}\, a_n + \ldots$ be the S-expansion of x. Recall that this is a semi-regular continued fraction expansion satisfying $\varepsilon_n + a_n \geqslant 1$, $n \geqslant 1$.

Let T_n and t_k be as in Definition (3.1) resp. Remark (1.14), i.e.

$$T_n = [0; B_{n+1}, \ldots, B_{n+m}, \ldots], \quad n \geqslant 0,$$

$$t_k = \varepsilon_{k+1} \,\underline{/}\, a_{k+1} + \ldots + \varepsilon_{k+m} \,\underline{/}\, a_{k+m} + \ldots, \quad k \geqslant 0.$$

We have, see Lemma (1.22):

$$(5.1) \quad \begin{aligned} V_n &:= Q_{n-1}/Q_n = [0; B_n, \ldots, B_1], & n \geqslant 1, \\ v_k &:= s_{k-1}/s_k = 1 \,\underline{/}\, a_k + \varepsilon_k \,\underline{/}\, a_{k-1} + \ldots + \varepsilon_2 \,\underline{/}\, a_1, & k \geqslant 1, \end{aligned}$$

where $(P_n/Q_n)_{n \geqslant -1}$ is the sequence of regular convergents of x and $(r_k/s_k)_{k \geqslant -1}$ is the sequence of S-convergents of x.

In (1.19) it was observed that

$$(5.2) \quad x = \begin{cases} \dfrac{P_n + T_n P_{n-1}}{Q_n + T_n Q_{n-1}}, & n \geqslant 1, \\[2ex] \dfrac{r_k + t_k r_{k-1}}{s_k + t_k s_{k-1}}, & k \geqslant 1. \end{cases}$$

Finally we put

$$\Delta := \underline{\Omega}\backslash S, \quad \Delta^- := \bar{T}\,S \quad \text{and} \quad \Delta^+ := \Delta\backslash\Delta^-.$$

(5.3) THEOREM. *With the above notations we have:*

(i) $(T_n, V_n) \in S \Leftrightarrow P_n/Q_n$ *is not an S-convergent;*

(ii) P_n/Q_n *is not an S-convergent* $\Rightarrow$ *both* P_{n-1}/Q_{n-1} *and* P_{n+1}/Q_{n+1} *are S-convergents;*

(iii) $(T_n, V_n) \in \Delta^+ \Leftrightarrow \exists k: \begin{cases} r_{k-1} = P_{n-1},\ r_k = P_n \\ s_{k-1} = Q_{n-1},\ s_k = Q_n \end{cases}$ *and* $\begin{cases} t_k = T_n \ (\textit{hence } \varepsilon_{k+1} := \operatorname{sgn}(t_k) = +1), \\ v_k = V_n; \end{cases}$

(iv) $(T_n, V_n) \in \Delta^- \Leftrightarrow \exists k: \begin{cases} r_{k-1} = P_{n-2},\ r_k = P_n \\ s_{k-1} = Q_{n-2},\ s_k = Q_n \end{cases}$ *and* $\begin{cases} t_k = -T_n/(1+T_n) \ (\textit{hence } \varepsilon_{k+1} = -1), \\ v_k = 1-V_n. \end{cases}$

Proof. (i) This follows directly from Definition (4.5) of S-expansions and from (2.6).

(ii) This follows from the first and third equality of Corollary (1.10); in the sequence of regular continued fraction convergents we cannot remove two or more consecutive convergents and still have a sequence of convergents of a semi-regular continued fraction.

(iii) In case $(T_n, V_n) \in \Delta^+$ we have by definition of Δ^+:

$$(T_{n-1}, V_{n-1}) \notin S, \quad (T_n, V_n) \notin S.$$

Hence neither B_n nor B_{n+1} are singularized and therefore both P_{n-1}/Q_{n-1} and P_n/Q_n are S-convergents. But then there exists a non-negative integer k such that

$$\frac{r_{k-1}}{s_{k-1}} = \frac{P_{n-1}}{Q_{n-1}}, \quad \frac{r_k}{s_k} = \frac{P_n}{Q_n}.$$

Since all these fractions are in their lowest terms and their denominators are positive we even have

$$r_{k-1} = P_{n-1}, \quad r_k = P_n,$$

$$s_{k-1} = Q_{n-1}, \quad s_k = Q_n.$$

From this and (5.2) it follows that

$$\frac{P_n + T_n P_{n-1}}{Q_n + T_n Q_{n-1}} = \frac{P_n + t_k P_{n-1}}{Q_n + t_k Q_{n-1}},$$

hence $t_k = T_n$. Finally,

$$v_k = \frac{s_{k-1}}{s_k} = \frac{Q_{n-1}}{Q_n} = V_n.$$

(iv) In case $(T_n, V_n) \in \Delta^-$ we have, by definition of Δ^-,

$$(T_{n-1}, V_{n-1}) \in S, \quad (T_n, V_n) \notin S.$$

Then $B_n = 1$ and must be singularized by Definition (4.5). Due to (ii) we then have that P_{n-2}/Q_{n-2} and P_n/Q_n are consecutive S-convergents. Again there exists a non-negative integer k such that

$$r_{k-1} = P_{n-2}, \quad r_k = P_n,$$

$$s_{k-1} = Q_{n-2}, \quad s_k = Q_n.$$

Since

$$Q_n = B_n Q_{n-1} + Q_{n-2} = Q_{n-1} + Q_{n-2}$$

we have

$$v_k = \frac{Q_{n-2}}{Q_n} = \frac{Q_n - Q_{n-1}}{Q_n} = 1 - V_n.$$

From (5.2) it follows that

$$\frac{P_n + T_n P_{n-1}}{Q_n + T_n Q_{n-1}} = \frac{P_n + t_k Q_{n-2}}{Q_n + t_k Q_{n-2}}$$

and from this, using the first and second equality of Corollary (1.10) and the fact that $B_n = 1$ we arrive at

$$t_k + t_k T_n + T_n = 0$$

or, equivalently,

$$t_k = \frac{-T_n}{1+T_n}, \quad T_n = \frac{-t_k}{1+t_k}. \ \blacksquare$$

(5.4) DEFINITION. The transformation $\mathscr{S}: \Delta \to \Delta$ is defined by:

$$\mathscr{S}(x,y) := \begin{cases} \bar{T}(x, y), & \bar{T}(x, y) \notin S, \\ \bar{T}^2(x, y), & \bar{T}(x, y) \in S. \end{cases}$$

Since $\mathscr{S}$ is an induced transformation, we have at once, see [Pet], Sections 2.3 and 2.4:

(5.5) THEOREM. *With the above definition and notations we have:*

$$(\Delta, \mathscr{B}, \mu_\Delta, \mathscr{S}) \text{ forms an ergodic system.}$$

Here $\mathscr{B}$ is the collection of Borel subsets of Δ and μ_Δ is the probability measure on $(\Delta, \mathscr{B})$ with density

$$\frac{1}{\mu(\Delta)\log 2}\,\frac{1}{(1+TV)^2}.$$

(5.6) **Remark.** Since the entropy $h(T)$ of the regular continued fraction operator T equals

$$\frac{\pi^2}{6\log 2},$$

see e.g. [Bil], [N], we easily obtain the entropy $h(\mathscr{S})$ for each S-expansion by applying the *formula of Abramov*, see [Ab], [Pet], p. 257.

We have

$$h(\mathscr{S}) = \frac{h(T)}{\mu(\Delta)} = \frac{1}{1-\mu(S)}\,\frac{\pi^2}{6\log 2}.$$

Note that the entropy is maximal for maximal singularization areas.

In view of Theorem (5.3) it is natural to consider the following definition.

(5.7) Definition. The map $M\colon \Delta \to \mathbf{R}^2$ is defined by

$$M(T, V) := \begin{cases} (T, V), & (T, V) \in \Delta^+, \\ (-T/(1+T), 1-V), & (T, V) \in \Delta^-. \end{cases}$$

We define the space Ω_S by $\Omega_S := M(\Delta)$. Hence Ω_S consists of $\Delta^+ = \Omega \backslash (S \cup \bar{T} S)$ and of the image of $\Delta^- = \bar{T} S$ under the above defined map M. Writing $(t, v) := M(T, V)$, the image of Δ^- lies in the second quadrant of the (t, v)-plane, see Figure 2. Notice that M is an injection.

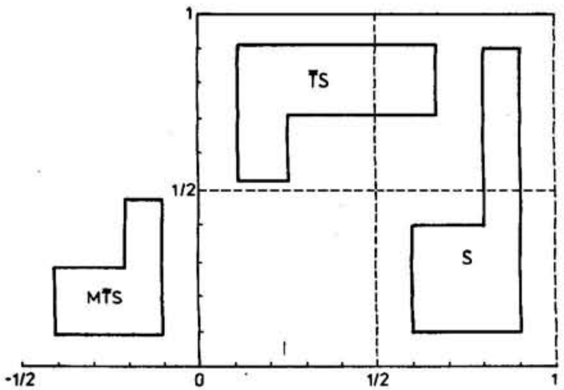


Fig. 2

(5.8) Definition. The operator $\tau\colon \Omega_S \to \Omega_S$ is defined by

$$\tau(t, v) := M\,\mathscr{S}\,M^{-1}(t, v), \qquad (t, v) \in \Omega_S.$$

Let $a_0 + \varepsilon_1 / a_1 + \ldots + \varepsilon_n / a_n + \ldots$ be the S-expansion of the irrational number x and let t_k, v_k be as in Remark (1.14), resp. (5.1). Then it follows from Theorem (5.3) that we have

$$\tau(t_k, v_k) = (t_{k+1}, v_{k+1}), \qquad k \geq 0.$$

A simple calculation shows that the determinant of the Jacobian J of $M|_{\Delta^-}$ equals

$$J = 1/(1+T)^2 > 0.$$

Putting

$$c := \frac{1}{1-\mu(S)}\,\frac{1}{\log 2}$$

we have for $(T, V) \in \Delta^-$

$$|J|^{-1}\frac{c}{(1+TV)^2} = c\,\frac{(1+T)^2}{(1+TV)^2} = c\left(\frac{1+T}{T}\right)^2\left(\frac{1}{1/T+V}\right)^2$$

$$= \frac{c}{t^2}\,\frac{1}{((1+T)/T-1+V)^2} = \frac{c}{t^2}\,\frac{1}{(1/t+v)^2} = \frac{c}{(1+tv)^2}.$$

From this, Theorem (5.5) and the definition of M and τ it is now obvious that we have:

(5.9) Theorem. *Let Ω_S be the set defined as above, $\mathscr{B}$ the collection of its Borel subsets. Denote by ϱ the probability measure on $(\Omega_S, \mathscr{B})$ with density function*

$$\frac{1}{1-\mu(S)}\,\frac{1}{\log 2}\,\frac{1}{(1+tv)^2}.$$

Then $(\Omega_S, \mathscr{B}, \varrho, \tau)$ forms an ergodic system.

(5.10) **Remark.** Due to the way in which it is constructed it follows that $(\Omega_S, \mathscr{B}, \varrho, \tau)$ is the two-dimensional ergodic system underlying the corresponding S-expansion. We moreover have

$$h(\tau) = \frac{1}{1-\mu(S)}\,\frac{\pi^2}{6\log 2}.$$

(5.11) Theorem. *Let the map $f\colon \Omega_S \to \mathbf{R} \cup \{\infty\}$ be defined by*

$$f(t, v) := |t^{-1}| - \tau_1(t, v), \qquad (t, v) \in \Omega_S,$$

where τ_1 is the first coordinate function of τ. Let $B\colon [0, 1) \to \mathbf{N} \cup \{\infty\}$ be the function defined in (3.1). *We then have:*

$$\text{(i)} \quad f(t, v) = \begin{cases} B(t), & \textit{when } \operatorname{sgn}(t) = 1, \quad \bar{T}(t, v) \notin S, \\ B(t)+1, & \textit{when } \operatorname{sgn}(t) = 1, \quad \bar{T}(t, v) \in S, \\ B(-t/(1+t))+1, & \textit{when } \operatorname{sgn}(t) = -1, \quad \bar{T}(M^{-1}(t, v)) \notin S, \\ B(-t/(1+t))+2, & \textit{when } \operatorname{sgn}(t) = -1, \quad \bar{T}(M^{-1}(t, v)) \in S. \end{cases}$$

(ii) $\tau(t, v) = \left(|t^{-1}| - f(t, v), \left(\operatorname{sgn}(t)\, v + f(t, v)\right)^{-1}\right), \quad (t, v) \in \Omega_S.$

A consequence of this is:

(5.12) COROLLARY. (i) $f(t, v) \in N$, $(t, v) \in \Omega_S$, $t \neq 0$.
(ii) $a_{k+1} = f(t_k, v_k)$, $k \geqslant 0$, *where* $(t_0, v_0) = (x - a_0, 0)$.

(5.13) Proof of (5.11). We distinguish four cases, of which two will be proved here. The other two are proved in an analogous way.

(I) Let $(t, v) \in \Delta^+$ and $\bar{T}(t, v) \in S$. Then

$$\begin{aligned}\mathscr{S}(M^{-1}(t, v)) &= \bar{T}^2(t, v)\\ &= \bar{T}\left(\frac{1}{t} - B(t), \frac{1}{B(t)+v}\right) = \left(\frac{1}{t^{-1} - B(t)} - 1, \frac{1}{1 + 1/(B(t)+v)}\right)\\ &= \left(\frac{t - 1 + tB(t)}{1 - tB(t)}, \frac{B(t)+v}{B(t)+1+v}\right) \in \Delta^-.\end{aligned}$$

Therefore

$$\tau(t, v) = M\left(\mathscr{S}(M^{-1}(t, v))\right) = \left(\frac{-\dfrac{t-1+tB(t)}{1-tB(t)}}{1 + \dfrac{t-1+tB(t)}{1-tB(t)}}, 1 - \frac{B(t)+v}{B(t)+1+v}\right)$$

$$= \left(\frac{1 - t(B(t)+1)}{t}, \frac{1}{(B(t)+1)+v}\right) = \left(\frac{1}{t} - (B(t)+1), \frac{1}{v + (B(t)+1)}\right).$$

Thus we see that

$$\tau(t, v) = \left(\left|\frac{1}{t}\right| - f(t, v), \frac{1}{\operatorname{sgn}(t) \cdot v + f(t, v)}\right)$$

where we have in this case

$$f(t, v) = B(t) + 1 \in N_{\geqslant 2} \cup \{\infty\}.$$

(II) Let $(t, v) \in M(\Delta^-)$ and $\bar{T}(M^{-1}(t, v)) \notin S$. Then $\varepsilon(t) = -1$ and we have:

$$\begin{aligned}\tau(t, v) &= M\,\bar{T}\,M^{-1}(t, v) = \bar{T}\,M^{-1}(t, v) = \bar{T}\left(\frac{-t}{1+t}, 1 - v\right)\\ &= \left(\frac{1}{-t/(1+t)} - B\left(\frac{-t}{1+t}\right), \frac{1}{1 - v + B(-t/(1+t))}\right)\\ &= \left(\frac{-1}{t} - \left(B\left(\frac{-t}{1+t}\right) + 1\right), \frac{1}{v \operatorname{sgn}(t) + \left(B(-t/(1+t)) + 1\right)}\right)\\ &= \left(\left|\frac{1}{t}\right| - f(t, v), \frac{1}{v \operatorname{sgn}(t) + f(t, v)}\right),\end{aligned}$$

where by definition we have in this case

$$f(t, v) = B\left(\frac{-t}{1+t}\right) + 1 \in N_{\geqslant 2} \cup \{\infty\}. \quad \blacksquare$$

6. Nakada's α-expansions viewed as S-expansions. In this section we will see that the so-called α-expansions, introduced and studied by Hitoshi Nakada in 1981, are examples of S-expansions. Once we have established this we can apply the general results of the previous sections on S-expansions. In this way we will obtain Nakada's results in a very simple way.

For sake of convenience we put here and in the rest of the paper: $\Omega := [0, 1]$, $\underline{\Omega} := \Omega^2$.

Let α be a real number satisfying $g < \alpha \leqslant 1$. (We remind the reader that g is the smaller of the two golden numbers, $g := (1/2)(\sqrt{5} - 1)$.) Put

(6.1) $$S_\alpha := [\alpha, 1] \times [0, 1].$$

It is easily checked that S_α is a singularization area: Obviously it is μ-continuous, it is contained in $[1/2, 1] \times [0, 1]$ and $S_\alpha \cap \bar{T} S_\alpha = \varnothing$ since, as is easily verified,

$$\bar{T} S_\alpha = [0, (1-\alpha)/\alpha] \times [1/2, 1],$$

and

(6.2) $$(1-\alpha)/\alpha < \alpha.$$

A simple calculation shows that

$$\mu(S_\alpha) = 1 - \frac{\log(1+\alpha)}{\log 2},$$

thus for the values of α we consider here, the singularization area S_α is never maximal. Instead of S_α-expansions we will simply speak of α-expansions.

Next we observe that

$$M\bar{T}(S_\alpha) = [\alpha - 1, 0] \times [0, 1/2].$$

Finally, if we denote

(6.3) $$\Omega_\alpha := \left(\underline{\Omega} \setminus (S_\alpha \cup \bar{T} S_\alpha)\right) \cup \left(M\bar{T} S_\alpha \setminus (\{0\} \times [0, 1/2])\right)$$

and if we denote by f_α the f from (5.11) in this special case, we see that $f_\alpha \colon \Omega_\alpha \to N$ is such that

$$|t^{-1}| - f_\alpha(t, v) \in [\alpha - 1, \alpha), \quad t \in [\alpha - 1, \alpha) \setminus \{0\}.$$

Since there exists only one positive integer n such that

$$|t^{-1}|-n \in [\alpha-1, \alpha)$$

we find that $f_\alpha(t, v)$ is independent of v, in fact $f_\alpha(t, v) = [|t^{-1}|+1-\alpha]$. Hence, $t \to |t^{-1}|-f_\alpha(t, v)$ is the operator from [N]. We denote it here by T_α. Thus for the above values of α we now have found Nakada's main result:

(6.4) THEOREM (Nakada, 1981). *Let $g < \alpha \leqslant 1$. Put*

$$\underline{\Omega}_\alpha := [\alpha-1, 0) \times [0, 1/2] \cup [0, (1-\alpha)/\alpha] \times [0, 1/2) \cup ((1-\alpha)/\alpha, \alpha) \times [0, 1].$$

Denote by $\mathscr{B}$ the collection of Borel subsets of $\underline{\Omega}_\alpha$ and define $\underline{\mu}_\alpha$ as the probability measure on $(\underline{\Omega}_\alpha, \mathscr{B})$ with density

$$\frac{1}{\log(1+\alpha)} \frac{1}{(1+tv)^2}.$$

Finally, define the map $\bar{T}_\alpha$: $\underline{\Omega}_\alpha \to \underline{\Omega}_\alpha$ by

$$\bar{T}_\alpha(t, v) := \left(|t^{-1}|-[|t^{-1}|+1-\alpha], \frac{1}{\operatorname{sgn}(t)\cdot v+[|t^{-1}|+1-\alpha]}\right), \qquad (t, v) \in \underline{\Omega}_\alpha.$$

Then $(\underline{\Omega}_\alpha, \mathscr{B}, \underline{\mu}_\alpha, \bar{T}_\alpha)$ forms an ergodic system.

For a picture of S_α and $\underline{\Omega}_\alpha$, see Figure 3 ($\alpha = 3/4$)

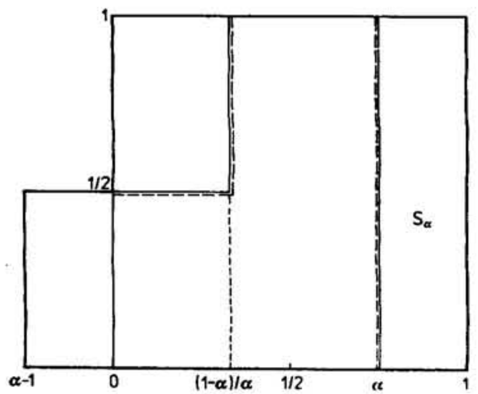


Fig. 3

Due to (5.6) we moreover have

(6.5) COROLLARY (Nakada, 1981). *Let $g < \alpha \leqslant 1$ and let $(\underline{\Omega}_\alpha, \mathscr{B}, \underline{\mu}_\alpha, \bar{T}_\alpha)$ be as in (6.4). Then $h(\bar{T}_\alpha)$, the entropy of $\bar{T}_\alpha$, equals $\pi^2/(6\log(1+\alpha))$* ([N], p. 417, Theorem 3).

By projection on the first axis, we at once have from Theorem (6.4):

(6.6) COROLLARY (Nakada, 1981). *Let $g < \alpha \leqslant 1$ and put $\Omega_\alpha := [\alpha-1, \alpha)$. Define μ_α as the probability measure on $(\Omega_\alpha, \mathscr{B})$ with density $(\log(1+\alpha))^{-1} h_\alpha(t)$, where*

$$h_\alpha(t) = \begin{cases} 1/(t+2), & t \in [\alpha-1, (1-\alpha)/\alpha], \\ 1/(t+1), & t \in ((1-\alpha)/\alpha, \alpha). \end{cases}$$

Then $(\Omega_\alpha, \mathscr{B}, \mu_\alpha, T_\alpha)$ forms an ergodic system.

See also [N], Corollary 1, p. 412.

We conclude this part of Section 6 with some results on these α-expansions which are not to be found in Nakada's paper.

Clearly the vertex $(\alpha, 1)$ of S_α determines the value of the Legendre constant L_α. As $\min(\alpha/(1+\alpha), 1/2) = \alpha/(1+\alpha)$ we have the next theorem.

(6.7) THEOREM. *Let $g < \alpha \leqslant 1$ and let x be an irrational number and $P/Q \in \mathbf{Q}$,* $\gcd(P, Q) = 1$, *$Q > 0$, such that*

$$\theta(x, P/Q) < \alpha/(\alpha+1).$$

Then P/Q is a convergent of the α-expansion of x. The constant $\alpha/(\alpha+1)$ is best possible.

For a picture of L_α, see Figure 5.

It is again easily checked that $\bar{T}^{-1} S_\alpha \cap ([1/2, 1] \times [0, 1]) = [1/2, 1/(1+\alpha)] \times [0, 1]$. Since for our values of α we have $(1-\alpha)/\alpha < 1/(1+\alpha)$ we find that the set B_{S_α}, or shortly B_α, from Definition (4.9) equals $(1/(1+\alpha), \alpha) \times [0, 1]$.

Now

$$\mu_\alpha(B_\alpha) = 2-\frac{\log(2+\alpha)}{\log(1+\alpha)},$$

thus we find

(6.8) THEOREM. *Let $g < \alpha \leqslant 1$ and let x be an irrational number, with α-expansion $a_0+\varepsilon_1 \lrcorner a_1+\ldots+\varepsilon_n \lrcorner a_n+\ldots$ Then for almost all x we have*

$$\lim_{n\to\infty} \frac{1}{n} \# \{j;\, j \leqslant n,\, a_j = 1\} = 2-\frac{\log(2+\alpha)}{\log(1+\alpha)}.$$

The case $\alpha = 1$ gives the classical result.

(6.9) Remark. Note that on the interval $[g, 1]$ the function $\alpha \to 2-\dfrac{\log(2+\alpha)}{\log(1+\alpha)}$ increases monotonically from 0 to $2-\log 3/\log 2 = 0.4150\ldots$, the last number being the classical result for the number of partial quotients equal to 1 in the regular continued fraction. For $\alpha = 0.76292\ldots$ we have just lost half of the original 1's.

From Corollary (4.16) it follows that for each α-expansion with $\alpha \in (g, 1]$ we have

$$\lim_{k\to\infty} \frac{1}{k} \sum_{\varkappa=1}^{k} \varepsilon_\varkappa = 3 - \frac{\log 4}{\log(1+\alpha)}, \quad \text{for almost all } x.$$

If we take in (6.1) a parameter α with $\alpha \leqslant g$, then (6.2) is no longer true. For $1/2 \leqslant \alpha \leqslant g$, we define the set S_α by

(6.10) $S_\alpha := [\alpha, g) \times [0, g) \cup [g, (1-\alpha)/\alpha] \times [0, g] \cup ((1-\alpha)/\alpha, 1] \times [0, 1]$.

Then

(6.11) $$\bar{T} S_\alpha = [0, (2\alpha-1)/(1-\alpha)) \times [1/2, 1] \cup [(2\alpha-1)/(1-\alpha), g] \times [g, 1] \cup (g, (1-\alpha)/\alpha] \times (g, 1].$$

Hence $S_\alpha \cap \bar{T} S_\alpha = \{(g, g)\}$, so we may consider S_α as a singularization area, see also Remark (4.6) (ii).

Now we find

$$\mu(S_\alpha) = 1 - \frac{\log G}{\log 2}$$

hence the S_α from (6.10), where $1/2 \leqslant \alpha \leqslant g$, is always maximal.

Notice that

$$M\bar{T} S_\alpha = [\alpha-1, g-1) \times [0, 1-g) \cup [g-1, (1-2\alpha)/\alpha] \times [0, 1-g] \cup ((1-2\alpha)/\alpha, 0] \times [0, 1/2].$$

Define Ω_α as in (6.3) and denote by f_α the f from (5.11). Analogously to the case $g < \alpha \leqslant 1$ we find that $f_\alpha(t, v)$ is independent of v and that in fact we have

$$f_\alpha(t, v) = [|t^{-1}| + 1 - \alpha], \quad (t, v) \in M_\alpha,\ t \neq 0.$$

Thus we have found

(6.12) THEOREM (Nakada, 1981). *Let $1/2 \leqslant \alpha \leqslant g$. Put*

$$\underline{\Omega}_\alpha := [\alpha-1, g-1) \times [0, 1-g) \cup [g-1, (1-2\alpha)/\alpha] \times [0, 1-g] \cup ((1-2\alpha)/\alpha, 0) \times [0, 1/2] \cup [0, (2\alpha-1)/(1-\alpha)] \times [0, 1/2) \cup ((2\alpha-1)/(1-\alpha), \alpha) \times [0, g).$$

Define $\underline{\mu}_\alpha$ as the probability measure on $(\underline{\Omega}_\alpha, \mathscr{B})$ with density

$$\frac{1}{\log G} \frac{1}{(1+tv)^2}.$$

Furthermore, define the map $\bar{T}_\alpha$: $\underline{\Omega}_\alpha \to \underline{\Omega}_\alpha$ as in (6.4). Then $(\underline{\Omega}_\alpha, \mathscr{B}, \underline{\mu}_\alpha, \bar{T}_\alpha)$ forms an ergodic system.

For a picture of S_α and $\underline{\Omega}_\alpha$, see Figure 4 ($\alpha = 0.55$).

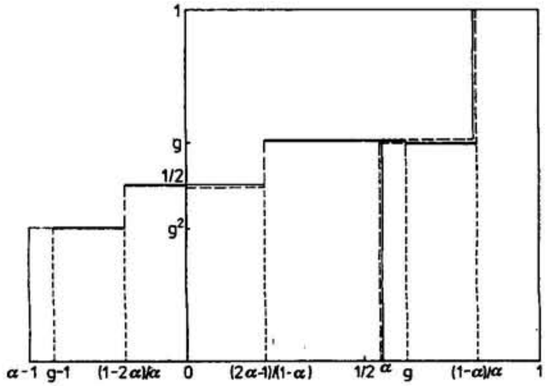


Fig. 4

We have, due to (5.6):

(6.13) COROLLARY (Nakada, 1981). *Let $1/2 \leqslant \alpha \leqslant g$ and let $(\underline{\Omega}_\alpha, \mathscr{B}, \underline{\mu}_\alpha, \bar{T}_\alpha)$ be as in (6.12). Then $h(\bar{T}_\alpha)$, the entropy of $\bar{T}_\alpha$, equals $\pi^2/(6 \log G)$.*

Again by projection on the first axis, we have from Theorem (6.12):

(6.14) COROLLARY (Nakada, 1981). *Let $1/2 \leqslant \alpha \leqslant g$ and put $\Omega_\alpha := [\alpha-1, \alpha)$. Define μ_α as the probability measure on $(\Omega_\alpha, \mathscr{B})$ with density $(\log G)^{-1} h_\alpha(t)$, where*

$$h_\alpha(t) = \begin{cases} \dfrac{1}{t+G+1}, & t \in \left[\alpha-1, \dfrac{1-2\alpha}{\alpha}\right], \\[2ex] \dfrac{1}{t+2}, & t \in \left(\dfrac{1-2\alpha}{\alpha}, \dfrac{2\alpha-1}{1-\alpha}\right), \\[2ex] \dfrac{1}{t+G}, & t \in \left[\dfrac{2\alpha-1}{1-\alpha}, \alpha\right). \end{cases}$$

Then $(\Omega_\alpha, \mathscr{B}, \mu_\alpha, T_\alpha)$ forms an ergodic system.

See also [N], Corollary 1, p. 412.

(6.15) Remark. If we take $\alpha = 1/2$ we obtain the one-dimensional ergodic system of the nearest integer continued fraction, a result obtained independently by G. J. Rieger and A. M. Rockett, see [R1] and [Roc]. Taking $\alpha = g$ we obtain the one-dimensional ergodic system of Hurwitz' singular continued fraction.

From Figure 4 it is obvious that the vertices (α, g) and $((1-\alpha)/\alpha, 1)$ determine the value of the Legendre constant L_α. A short calculation yields

(6.16) THEOREM. *Let $1/2 \leqslant \alpha \leqslant g$ and let x be an irrational number and $P/Q \in \mathbb{Q}$, $\gcd(P, Q) = 1$, $Q > 0$ such that*

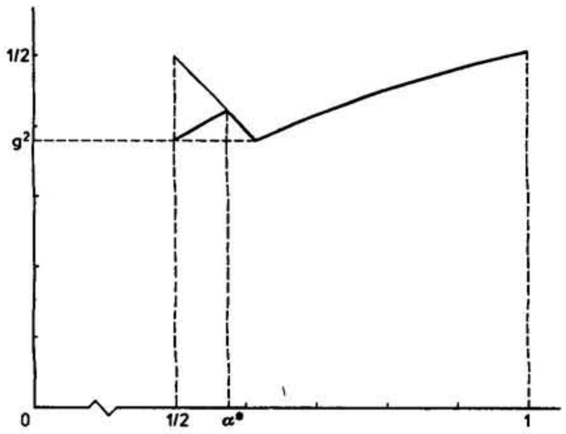
$$\theta(x, P/Q) < L_\alpha$$

where

$$L_\alpha = \min\{\alpha/(1+\alpha g),\ 1-\alpha\}.$$

Then P/Q is a convergent of the α-expansion of x. The constant L_α is best possible.

For a picture of L_α, see Figure 5.



$$\alpha^* = \frac{-2+g+\sqrt{5-g}}{2g}$$

Fig. 5

Notice that for $1/2 \leqslant \alpha \leqslant g$ we have

$$\mathcal{T}([1/2, \alpha) \times [0, g)) \subseteq S_\alpha.$$

From this and (6.10) it at once follows that $B_\alpha = \emptyset$, which confirms Theorem (4.11).

We conclude this section by giving the analogue of Vahlen's Theorem (3.7) for α-expansions. For the nearest integer continued fraction and Hurwitz' singular continued fraction the analogue of Vahlen's Theorem (3.7) was independently given by Kurosu and Sendov, see [Kur], [Sen]. For these continued fraction expansions we have, for all irrational numbers x and all positive integers n,

$$\min(\theta_n, \theta_{n+1}) < 2g^3 = 0.4721\ldots,$$

where the constant $2g^3$ is best possible.

One might ask whether there are α's for which still smaller values can be obtained. Note that one can never find a value smaller than $1/\sqrt{5} = 0.447\ldots$, due to a theorem of Hurwitz. In [J2] it was shown that the point (θ_n, θ_{n+1}) always lies in the interior of the triangle with vertices (0, 0), (1, 0) and (0, 1). In [Kr1] the Kurosu–Sendov result was proved by giving the area in which the point (θ_n, θ_{n+1}) always lies. The method described in [Kr1] can easily be adapted for S-expansions. As an example we will sketch here the case of the α-expansions.

(6.17) Some notations. Let $x \in [\alpha-1, \alpha)\backslash \mathbf{Q}$, where $1/2 \leqslant \alpha \leqslant 1$ and put

$$T_{\alpha,n} := T_\alpha^n(x), \quad \varepsilon_{\alpha,n+1} := \operatorname{sgn}(T_{\alpha,n}), \quad V_{\alpha,n} := Q_{\alpha,n-1}/Q_{\alpha,n}, \quad n \geqslant 0,$$

where $(P_{\alpha,n}/Q_{\alpha,n})_{n\geqslant -1}$ is the sequence of α-convergents of x.

Writing $\theta_{\alpha,n}$ instead of $\theta(x, P_{\alpha,n}/Q_{\alpha,n})$ we have, due to Lemma (1.24) and Remark (1.25),

$$\theta_{\alpha,n} = \frac{\varepsilon_{\alpha,n+1} T_{\alpha,n}}{1+T_{\alpha,n} V_{\alpha,n}}, \quad n \geqslant 0,$$

$$\theta_{\alpha,n-1} = \frac{V_{\alpha,n}}{1+T_{\alpha,n} V_{\alpha,n}}, \quad n \geqslant 1.$$

This leads in a natural way to the introduction of the function $F\colon \Omega_\alpha \to \mathbf{R}^2$, see also [J2] and [Kr1], p. 183:

$$F(t, v) := \left(\frac{v}{1+tv}, \frac{|t|}{1+tv}\right) =: (\xi, \eta), \quad tv \neq -1.$$

Put

$$\Gamma_\alpha := F(\Omega_\alpha),$$

where Ω_α is defined as in (6.3), $1/2 \leqslant \alpha \leqslant 1$. Then for all $x \in \mathbf{R}\backslash\mathbf{Q}$ and $n \geqslant 0$ we have

(6.18) $$F(T_{\alpha,n}, V_{\alpha,n}) = (\theta_{\alpha,n-1}, \theta_{\alpha,n}) \in \Gamma_\alpha.$$

Let

$$\Gamma_\alpha^+ := F(M_\alpha^+); \quad \Gamma_\alpha^- := F(M_\alpha^-)$$

where

$$M_\alpha^+ := \{(t, v) \in M_\alpha;\ t \geqslant 0\}; \quad M_\alpha^- := \{(t, v) \in M_\alpha;\ t < 0\}.$$

For a picture of Γ_α^+ see Figure 6 ($\alpha = 0.55$) and Figure 8 ($\alpha = 0.7$), and for a picture of Γ_α^- see Figure 7 ($\alpha = 0.55$) and Figure 9 ($\alpha = 0.7$).

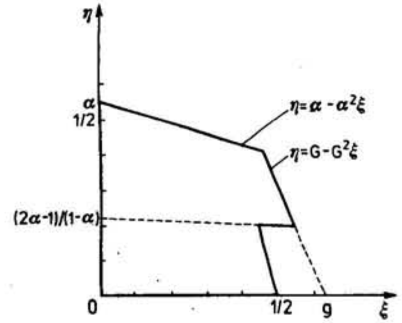


Fig. 6. Γ_α^+ $(1/2 \leqslant \alpha \leqslant g)$

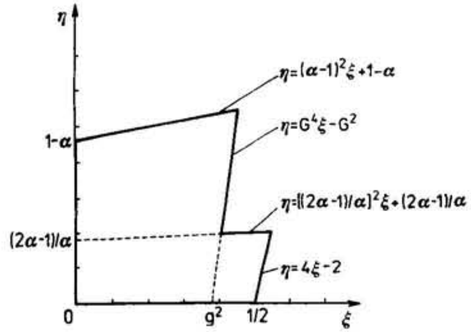


Fig. 7. Γ_α^- $(1/2 \leqslant \alpha \leqslant g)$

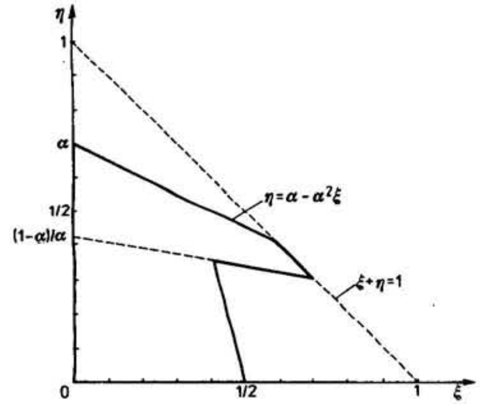


Fig. 8. Γ_α^+ $(g < \alpha \leqslant 1)$

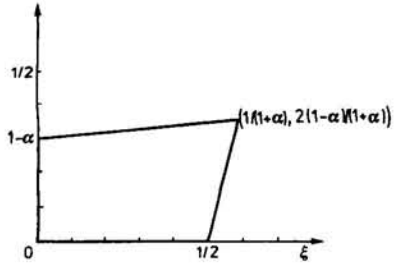


Fig. 9. Γ_α^- $(g < \alpha \leqslant 1)$

From Figures 6–9 it at once follows that

(6.19) THEOREM. *For all irrational numbers x and all positive integers n we have*

$$\theta_{\alpha,n} < c(\alpha) \quad \textit{and} \quad \min(\theta_{\alpha,n-1}, \theta_{\alpha,n}) < V(\alpha)$$

where the functions c, V: $[1/2, 1] \to \mathbf{R}^2$ *are defined by:*

$$c(\alpha) := \max\left(G\frac{1-\alpha}{1+g\alpha}, \alpha\right), \quad 1/2 \leqslant \alpha \leqslant 1,$$

and

$$V(\alpha) := \begin{cases} \max\left(\dfrac{g}{1+\alpha g}, 4\alpha-2\right), & 1/2 \leqslant \alpha \leqslant g, \\ \max\left(2\dfrac{1-\alpha}{1+\alpha}, \dfrac{\alpha}{1+\alpha^2}\right), & g < \alpha \leqslant 1. \end{cases}$$

The constants $c(\alpha)$ *and* $V(\alpha)$ *are best possible.*

For a picture of $c(\alpha)$ and $V(\alpha)$, see Figure 10 resp. Figure 11.

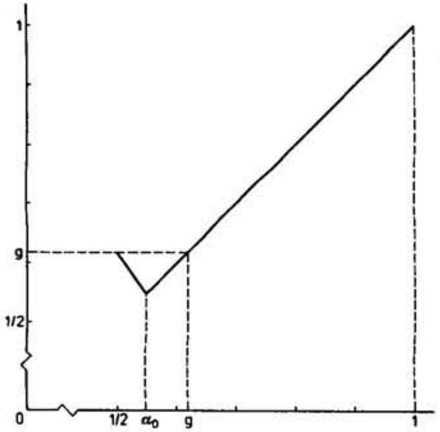


Fig. 10

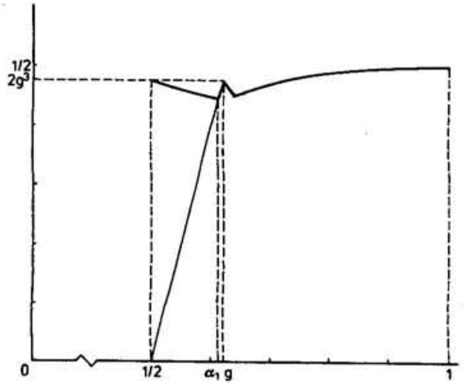


Fig. 11

(6.20) Remarks. (i) Taking $\alpha = 1$ in Theorem (6.19) we again have Vahlen's Theorem (3.7), taking $\alpha = 1/2$ or $\alpha = g$ we again have the Kurosu–Sendov analogue of Vahlen's theorem for the nearest integer continued fraction resp. Hurwitz' singular continued fraction.

(ii) A simple calculation yields, see also [BJW], p. 292:

$$\min_\alpha c(\alpha) = c(\alpha_0) = \alpha_0$$

with

$$\alpha_0 = \tfrac{1}{2}\left(-2-\sqrt{5}+\sqrt{6\sqrt{5}+15}\right) = 0.5473\ldots$$

Moreover, we have

$$\min_\alpha V(\alpha) = V\left(\frac{1-3g+\sqrt{10-11g}}{4g^2}\right) = 0.4484\ldots,$$

a constant slightly larger than $1/\sqrt{5}$, with

$$\alpha_1 := \frac{1-3g+\sqrt{10-11g}}{4g^2} = 0.6121\ldots < g.$$

7. Two other examples. In this final section we will briefly describe two other examples of S-expansions: the classical diagonal continued fraction of Minkowski and Wieb Bosma's optimal continued fraction. The singularization areas of both expansions will be given here, and some conclusions will be drawn from this.

(7.1) Minkowski's diagonal expansion. Let x be a real number such that $x \notin \mathbf{Z}$ and $2x \notin \mathbf{Z}$. Consider the sequence σ of all irreducible fractions P/Q, with $Q > 0$, satisfying

$$\left| x - \frac{P}{Q} \right| < \frac{1}{2}\frac{1}{Q^2},$$

ordered in such a way that the denominators form an increasing sequence.

It can be shown, see [Per], §45, [Min], that there exists a unique semi-regular continued fraction expansion of x such that σ is the sequence of convergents of this expansion. This unique expansion is Minkowski's diagonal expansion.

From Legendre's Theorem (3.5) we see that we take precisely those regular convergents for which $\theta_n < 1/2$. Using the first formula from (1.24) it now at once follows that Minkowski's diagonal expansion is an S-expansion, with singularization area

$$S_{\mathrm{DCF}} = \{(T, V) \in \underline{\Omega};\ T/(1+TV) > 1/2\},$$

see Figure 12.

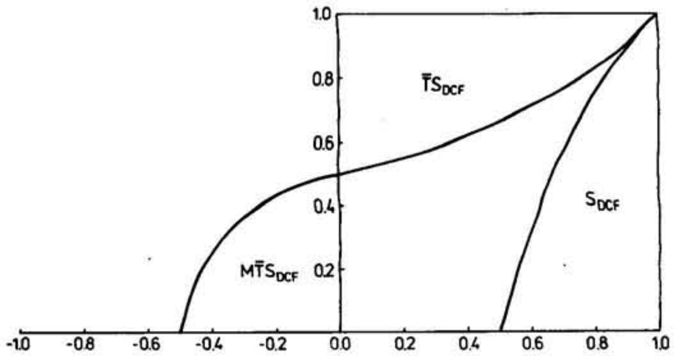


Fig. 12

Due to Vahlen's Theorem (3.7) the diagonal expansion picks at least one out of any two consecutive regular convergents of x. Since

$$\mu(S_{\mathrm{DCF}}) = 1 - \frac{1}{2\log 2},$$

see [BJW], p. 286, we have

$$\lim_{N\to\infty} \frac{k(N)}{N} = 2\log 2 = 1.3862\ldots, \quad \text{for almost all } x,$$

with $k\colon \mathbf{N} \to \mathbf{N}$ defined as in Section 4.

For more details and results, see [Kr2].

(7.2) Bosma's optimal continued fraction expansion. In 1895, F. Klein [K] gave a geometrical interpretation of the regular continued fraction algorithm. In [Bos], §4, Wieb Bosma shows how a similar interpretation can be given for Nakada's α-expansions. Inspired by this Bosma developed a new continued fraction algorithm which yields a semi-regular continued fraction expansion, previously found in a different way by C. O. Selenius [Sel].

Since this continued fraction has »various« optimal properties, the name optimal continued fraction was chosen for it. We mention two of these properties:

(i) the optimal continued fraction is fastest; the growth rate of the denominators of the optimal convergents is maximal.

(ii) $\min(\theta_n, \theta_{n+1}) < 1/\sqrt{5}$.

In [BK1] it is shown that the optimal continued fraction is in fact an S-expansion, its singularization area being

$$S_{\mathrm{OCF}} = \{(T, V) \in \underline{\Omega};\ V < T \text{ and } V < (2T-1)/(1-T)\},$$

see Figure 13.

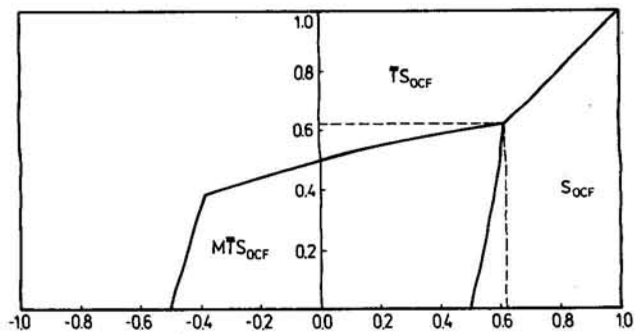


Fig. 13

Notice that S_{DCF} is contained in S_{OCF}, hence the sequence of optimal convergents of an irrational number x forms a subsequence of the sequence of diagonal convergents σ of x. Simple calculations yield furthermore that the analogue of the Legendre value $L_{S_{\mathrm{OCF}}}$ equals $1/\sqrt{5}$, and that S_{OCF} is maximal, which is a probabilistic interpretation of (i). Notice that $B_{S_{\mathrm{OCF}}}$, the set defined in Section 4, is empty. Hence each optimal partial quotient is greater than 1.

For more details, results and proofs, see [Bos], [BK1], [BK2].

(7.3) Remark. Both in case of the diagonal continued fraction and the optimal continued fraction we have, see Figures 12 and 13, that the underlying two-dimensional ergodic system has curved boundaries. Related with this we have that in both cases the function f from Theorem (5.11) depends on t and v, and not only on t, as in the case of Nakada's α-expansions. Due to this a one-dimensional ergodic system cannot be given.

References

[Ab] L. M. Abramov, *Entropy of induced automorphisms*, Dokl. Akad. Nauk SSSR 128 (1959), 647–650.

[Ad] William W. Adams, *On a relationship between the convergents of the nearest integer and regular continued fractions*, Math. Comp. 33 (1979), 1321–1331.

[Bar] Dominique Barbolosi, *Fractions Continues à Quotients Partiels Impairs*, Thèse, Université de Provence, 1988.

[Bil] P. Billingsley, *Ergodic Theory and Information*, John Wiley and Sons, 1965.

[Bor] É. Borel, *Contribution à l'analyse arithmétique du continu*, J. Math. Pures Appl. (5) 9 (1903), 329–375.

[Bos] Wieb Bosma, *Optimal continued fractions*, Indag. Math. 50 (1988), 353–379.

[BJW] W. Bosma, H. Jager and F. Wiedijk, *Some metrical observations on the approximation by continued fractions*, ibid. 45 (1983), 281–299.

[BK1] Wieb Bosma and Cor Kraaikamp, *Metrical theory for optimal continued fractions*, J. Number Theory 34 (1990), 251–270.

[BK2] –, – *Optimal approximation by continued fractions*, J. Austral. Math. Soc., to appear.

[BP] N. G. de Bruijn and K. A. Post, *A remark on uniformly distributed sequences and Riemann integrability*, Indag. Math. 30 (1968), 149–150.

[CFS] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinaĭ, *Ergodic Theory*, Springer-Verlag, 1981.

[D] W. Doeblin, *Remarques sur la théorie métrique des fractions continues*, Comp. Math. 7 (1940), 353–371.

[H] A. Hurwitz, *Über eine besondere Art der Kettenbruchentwicklung reeller Grössen*, Acta Math. 12 (1889), 367–405.

[I] Sh. Ito, Preprint, Tsuda College.

[J1] H. Jager, *On the speed of convergence of the nearest integer continued fractions*, Math. Comp. 39 (1982), 555–558.

[J2] – *The distribution of certain sequences connected with the continued fraction*, Indag. Math. 48 (1986), 61–69;
– *Continued fractions and ergodic theory*, in *Transcendental Numbers and Related Topics*, RIMS Kokyuroku 599, Kyoto University, Kyoto, Japan, 1986, 55–59.

[JK] H. Jager and C. Kraaikamp, *On the approximation by continued fraction*, Indag. Math. 51 (1989), 289–307.

[K] F. Klein, *Über eine geometrische Auffassung der gewöhnlichen Kettenbruchentwicklung*, Nachr. Göttingen 1895, 357–359.

[Kr1] Cor Kraaikamp, *The distribution of some sequences connected with the nearest integer continued fraction*, Indag. Math. 49 (1987), 177–191.

[Kr2] – *Statistic and ergodic properties of Minkowski's diagonal continued fraction*, Theoret. Comput. Sci. 65 (1989), 197–212.

[KN] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley and Sons, 1974.

[Kur] K. Kurosu, *Notes on some points in the theory of continued fractions*, Japan J. Math. 1 (1924), 17–21.

[Leg] A. M. Legendre, *Essai sur la théorie des nombres*, Paris 1798.

[Lew] L. Lewin, *Polylogarithms and Associated Functions*, North-Holland, 1981.

[Min] H. Minkowski, *Über die Annäherung an eine reelle Grösse durch rationale Zahlen*, Math. Ann. 54 (1901), 91–124.

[Minn] B. Minnigerode, *Über eine neue Methode, die Pellsche Gleichung aufzulözen*, Nachr. Göttingen 1873.

[N] Hitoshi Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. 4 (1981), 399–426.

[NIT] Hitoshi Nakada, Shunji Ito and Shigeru Tanaka, *On the invariant measure for the transformations associated with some real continued fractions*, Keio Engrg. Rep. 30 (1977), 159–175.

[Per] O. Perren, *Die Lehre von den Kettenbrüchen*, Band I, 3. verb. u. etw. Aufl., B. G. Teubner, Stuttgart.

[Pet] Karl Peterson, *Ergodic Theory*, Cambridge University Press, 1983.

[R1] G. J. Rieger, *Mischung und Ergodizität bei Kettenbrüchen nach nächsten Ganzen*, J. Reine Angew. Math. 310 (1979), 171–181.

[R2] – *Über die Länge von Kettenbrüchen mit ungeraden Teilnennern*, Abh. Braunschweig Wiss. Ges. 32 (1981), 61–69;
– *On the metrical theory of the continued fractions with odd partial quotients*, in *Topics in Classical Number Theory*, vol. II, Budapest 1981, 1371–1481. (Colloq. Math. Soc. János Bolyai, 34, North-Holland 1984.)

[Roc] A. M. Rockett, *The metrical theory of continued fractions to the nearer integer*, Acta Arith. 38 (1980), 97–103.

[R-N] C. Ryll-Nardzewski, *On the ergodic theorems (II) (Ergodic theory of continued fractions)*, Studia Math. 12 (1951), 74–79.

[Sch] F. Schweiger, *Continued fractions with odd and even partial quotients*, Arbeitsbericht Math. Instit. Univ. Salzburg 4 (1982), 59–70;
– *On the approximation by continued fractions with odd and even partial quotients*, Arbeitsbericht Math. Institut Univ. Salzburg 1–2 (1984), 105–114.

[Sel] C. O. Selenius, *Konstruktion und Theorie halbregelmässiger Kettenbrüche mit idealer relativer Approximation*, Acta Acad. Abo., ser. B, XXII. 2 (1960), 1–75.

[Sen] B. Sendov, *Der Vahlensche Satz über die singulären Kettenbrüche und die Kettenbrüche nach nächsten Ganzen*, Ann. Univ. Sofia, Fac. Sci. Math. Livre 1 Math. 54 (1959/1960), 251–258.

[T] H. Tietze, *Über die raschesten Kettenbruchentwicklungen reeller Zahlen*, Monatsh. Math. Phys. 24 (1913), 209–241.

[V] K. Th. Vahlen, *Über Näherungswerte und Kettenbrüche*, J. Reine Angew. Math. 115 (1895), 221–233.

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