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An infinite product with bounded partial quotients

by

J.-P. ALLOUCHE, M. MENDÈS FRANCE (Talence) and
A. J. VAN DER POORTEN* (Macquarie)

A Roger Apéry en toute amitié

1. Introduction. Continued fractions of formal power series are not well understood and there is a dearth of instructive examples. We illustrate a technique for finding the continued fraction expansion of certain products satisfying a suitable functional equation and show for the principal example that all its partial quotients are linear.

The infinite products

$$\prod_{h=0}^{\infty}(1+X^{-k^h}), \qquad k=4,6,8,\ldots$$

viewed as Laurent series in X^{-1} over a ground field $\boldsymbol{K}$ of arbitrary characteristic have partial quotients which can readily be listed explicitly, and which are of rapidly increasing degree [4]. But the techniques of [4] are less informative for $k=3,5,\ldots$ odd and report no more than that the truncations of the product yield convergents, allowing the computation of the degree of just a subsequence of partial quotients. For $k=3$ the partial quotients of that subsequence all have degree 1. Explicit computation in characteristic zero yields

$$\begin{aligned}F=\prod_{h=0}^{\infty}(1+X^{-3^h})=[1, X, -X+1, -\tfrac{1}{2}X-\tfrac{1}{4}, 8X+4, \tfrac{1}{16}X-\tfrac{1}{16}, -16X+16,\\ -\tfrac{1}{32}X-\tfrac{1}{16}, 32X-32, \tfrac{1}{64}X+\tfrac{5}{256}, \tfrac{1024}{5}X-\tfrac{256}{5},\\ -\tfrac{25}{2048}X+\tfrac{25}{2048}, -\tfrac{2048}{35}X-\tfrac{4096}{245}, \tfrac{343}{4096}X+\tfrac{245}{4096}, \ldots].\end{aligned}$$

The partial quotients all appear to be linear, but their coefficients grow in complexity at a furious rate — the 30th partial quotient is

$$-\tfrac{13743895347200}{15737111}X-\tfrac{137438953472000}{456376219}$$

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— and seem quite intractable. Nevertheless, we prove that these partial quotients are indeed all linear and, implicitly, we give a relatively easy technique for the recursive computation of the coefficients.

The product F arises naturally from a geometrical construction which is the motivating example of [3].

It appears that in positive characteristic, other than characteristic 3 when F is a quadratic irrational, not all the partial quotients are linear. For all primes $p \neq 3$ small enough to fall within the range of our computations the sequence of partial quotients has bad reduction at p, and this entails nonlinearity. It is also almost explicit in our argument that for arbitrary odd $k \geqslant 5$ two of every three partial quotients of

$$\prod_{h=0}^{\infty} (1+X^{-k^h})$$

are linear — this holds in characteristic zero and characteristic 3 — and the degrees of the intervening partial quotients are readily computable. By the way, it is rather easy to see that, in the 'generic' case, an infinite series has all its partial quotients linear — though it is debatable whether a series with coefficients 0 or 1 is 'generic'. Indeed, we have remarked that, of all the nontrivial products

$$\prod_{h=0}^{\infty} (1+X^{-k^h}),$$

only that with $k = 3$ has all its partial quotients linear. In any case, the 'surprise' in our principal example is not so much the result, as the fact that we can prove it.

We are indebted to the program PARI — and to Henri Cohen and Michel Olivier for implementing it on our behalf — for providing experimental data. However, the computations and manipulations explicitly used in the body of this paper were done by hand to show that not all the old skills need atrophy.

There are open related questions. It might be interesting to understand the continued fraction expansion of F in positive characteristic other than 3. Computation suggests that the partial quotients of

$$G = \prod_{h=0}^{\infty} (1-X^{-2^h})$$

in characteristic zero all are of degree at most 2. This fact may be vulnerable to our techniques.

2. Terminology. Given a field $\boldsymbol{K}$ let $\boldsymbol{L} = \boldsymbol{K}((X^{-1}))$ denote the field of formal Laurent series in X^{-1} over $\boldsymbol{K}$. Then each $F \in \boldsymbol{L}$ is of shape

$$\sum_{h=-d}^{\infty} a_h X^{-h}, \quad a_{-d} \neq 0, \quad d \in \boldsymbol{Z}$$

and we say that the *degree* of F is $\deg F = d$. The *integral part* of F is the polynomial $\sum_{h=-d}^{0} a_h X^{-h}$ in $\boldsymbol{K}[X]$.

It is straightforward to see that each $F \in \boldsymbol{L}$ has a unique continued fraction expansion

$$c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \ddots}}} = [c_0, c_1, c_2, c_3, \ldots],$$

where the *partial quotients* c_h are polynomials in X of degree at least 1 once $h \geqslant 1$. From the general formal theory of continued fractions one has the *fundamental correspondence* whereby for $h = 0, 1, 2, \ldots$

$$\frac{p_h}{q_h} = [c_0, c_1, \ldots, c_h]$$

if and only if

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_h & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_h & p_{h-1} \\ q_h & q_{h-1} \end{pmatrix},$$

defining the *convergents* p_h/q_h by matrix products. For our present purpose it will be convenient to view these products a little differently. Accordingly, set

$$J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

noticing that $JR = LJ$, $JL = RJ$ and $J^2 = I$. Then, on observing that for $d \in \boldsymbol{Z}$,

$$R^d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \quad L^d = \begin{pmatrix} 1 & 0 \\ d & 1 \end{pmatrix},$$

we may write, once again with a formal interpretation intended,

$$\begin{pmatrix} c_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots = R^{c_0} L^{c_1} R^{c_2} \cdots,$$

that is, with $\leftrightarrow$ denoting the fundamental correspondence between matrix products and continued fractions,

$$[c_0, c_1, c_2, \ldots] \leftrightarrow R^{c_0} L^{c_1} R^{c_2} \ldots$$

The reader can find a discussion of approximation properties, criteria for recognising convergents, and the like appropriate to the present context in [4]; or from a rather different viewpoint, in [1].

The correspondence of continued fractions and so-called *R–L sequences* is particularly convenient in multiplying, or dividing, a continued fraction by a rational quantity. We will not need the general principles, for which see Raney [6], but it will be helpful if we introduce our terminology by a simple numerical example:

Accordingly, take $D=61$ and set $\beta=\sqrt{D}$, $\alpha=\frac{1}{2}(1+\sqrt{D})$. The regular continued fraction expansion of β is obtained by noting $7<\beta<8$ and computing (setting $\beta'=-\sqrt{D}$)

$$\begin{aligned}\beta &= 7-(\beta'+7),\\ (\beta+7)/12 &= 1-(\beta'+5)/12,\\ (\beta+5)/3 &= 4-(\beta'+7)/3,\\ (\beta+7)/4 &= \dots\end{aligned}$$

where

$$\begin{pmatrix}7&1\\1&0\end{pmatrix}\begin{pmatrix}1&1\\1&0\end{pmatrix}\begin{pmatrix}4&1\\1&0\end{pmatrix}=\begin{pmatrix}39&8\\5&1\end{pmatrix}.$$

But, noting that $\operatorname{Norm}\alpha=-15$ and $\operatorname{Trace}\alpha=1$ we see that

$$39^2-61\cdot 5^2=-4$$

is just

$$\left(\tfrac{39+5}{2}\right)^2-1\cdot\left(\tfrac{39+5}{2}\right)5+\left(\tfrac{1-61}{4}\right)5^2=\operatorname{Norm}(22-5\alpha)=-1,$$

showing by

$$\begin{pmatrix}15\cdot 5&22\\22-5\cdot 1&5\end{pmatrix}=\begin{pmatrix}4&1\\1&0\end{pmatrix}\begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix}2&1\\1&0\end{pmatrix}\begin{pmatrix}3&1\\1&0\end{pmatrix}$$

that

$$\alpha=[4,\overline{2,2,3+4}]\leftrightarrow R^4L^2R^2L^7R^2L^2R^7\dots$$

For background to and elucidation of these cryptic remarks see [7].

We obtain the continued fraction of β from that of α by multiplying by 2 (and subtracting 1). Accordingly, set

$$A=\begin{pmatrix}2&0\\0&1\end{pmatrix},\qquad A'=\begin{pmatrix}1&0\\0&2\end{pmatrix}.$$

Then

$$\beta\leftrightarrow R^{-1}AR^4L^2R^2L^7R^2L^2R^7\dots$$

Of course the matrix product on the right does not correspond to a continued fraction, but using the *transition formulae*

$$\begin{aligned}AR&=R^2A, & A'L&=L^2A',\\ AL^2&=LA, & A'R^2&=RA',\\ ALR&=RLA', & A'RL&=LRA\end{aligned}$$

we can transit the multiplier A through the R–L sequence until it disappears into the ... on the right. Our way of describing this process will be to say, for example, that the operation 'multiplication by 2' acting on LR *enters* in *state* A, *transduces* LR to RL, and *exits* in state A'.

Multiplying α by 2 yields, conveniently tabulated,

$$\begin{array}{lcccccccccccc} \alpha & R^4 & L^2 & R^2 & L^6 & LR & RL & LR & R^6 & L^2 & R^2 & L^7 & R^2\\ & A & A & A & A & A & A' & A & A' & A' & A' & A' & A'\dots\\ 1+\beta & R^8 & L & R^4 & L^3 & RL & LR & RL & R^3 & L^4 & R & L^{14} & R\end{array}$$

Thus

$$\beta=[7,\overline{1,4,3,1,2,2,1,3,4,1,14}].$$

Conversely, we divide $1+\beta$ by 2 by entering in state A'

$$\begin{array}{lcccccccccccc} 1+\beta & R^8 & L & R^4 & L^3 & RL & LR & RL & R^3 & L^4 & R & L^{14} & \\ & A' & A' & A' & A' & A' & A & A' & A & A & A & A & \dots\\ \alpha & R^4 & L^2 & R^2 & L^6 & LR & RL & LR & R^6 & L^2 & R^2 & L^7 & \end{array}$$

In each transduction we have economised by stopping halfway in the periodic process, since the period of β, respectively α, is already evident.

In this example there are just finitely many states, in fact two, of the operation; we have a so-called *finite-state transducer*. In our application below the operations induce infinitely many states, but these fall into finitely many classes – which we refer to as *shapes* – and that permits us to give a finite list of the relevant transition formulae. Moreover, our transductions have good reduction modulo 3 and that reduction yields a finite state transducer.

3. Main result.

THEOREM. *The infinite product*

$$F(X)=\prod_{h=0}^{\infty}(1+X^{-3^h})$$

considered as an element of $\mathbf{K}((X^{-1}))$, *with* $\mathbf{K}$ *a field of characteristic zero (or of characteristic 3) has a continued fraction expansion*

$$[1,c_1,c_2,\dots]$$

with each partial quotient c_h *a polynomial in* X *of degree* 1.

Proof. Obviously the zeroth partial quotient is 1 as alleged. We note the functional equation

$$F(X)=(1+X^{-1})F(X^3)$$

and set $f_h(X)=c_{2h-1}(X^3)$, $g_h(X)=c_{2h}(X^3)$. Recalling the matrix correspondence to continued fractions, we obtain the matrix identity

$$\begin{pmatrix} X+1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} RL^{f_1}R^{g_1}L^{f_2}\ldots = RL^{c_1}R^{c_2}L^{c_3}\ldots$$

But

$$\begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix} R = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & X \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -X & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} X+1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} = \begin{pmatrix} X+1 & 0 \\ X & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ X & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & X+1 \end{pmatrix} = RL^X\begin{pmatrix} 1 & -1 \\ 0 & X+1 \end{pmatrix}.$$

Thus, apparently, $c_1 = X$ so $f_1 = X^3$.

We interrupt the proof of the theorem to detail auxiliary results.

LEMMA 1 (Transition formulae). *Let a, b, c, d denote constants and f, g polynomials in $\mathbf{K}[X]$. We set*

$$\bar f = (f - f(0))/X, \quad \tilde f = (\bar f - \bar f(-1))/(X+1)$$

and similarly for g. Quantities that are inverted in the course of the claims below are assumed nonzero:

$$\begin{pmatrix} b & a \\ -a^{-1}X & 0 \end{pmatrix} L^f = \begin{pmatrix} b & a \\ -a^{-1}X & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} = \begin{pmatrix} af+b & a \\ -a^{-1}X & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -a^2\bar f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} af(0)+b & a \\ -a^{-1}X & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -a^2\bar f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -(af(0)+b)^{-1}a^{-1}X & 1 \end{pmatrix}\begin{pmatrix} af(0)+b & a \\ 0 & (af(0)+b)^{-1}X \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ 0 & a^{-1}(X+1) \end{pmatrix}\begin{pmatrix} 1 & -c\bar f \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} a & -ac\bar f+b \\ 0 & a^{-1}(X+1) \end{pmatrix} = \begin{pmatrix} 1 & -a^2c\tilde f \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & -ac\bar f(-1)+b \\ 0 & a^{-1}(X+1) \end{pmatrix},$$

$$\begin{pmatrix} a & b \\ 0 & a^{-1}(X+1) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ dX & 1 \end{pmatrix} = \begin{pmatrix} bdX+a & b \\ a^{-1}dX(X+1) & a^{-1}(X+1) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ a^{-1}b^{-1}(X+1)-b^{-2}d^{-1} & 1 \end{pmatrix}\begin{pmatrix} bdX+a & b \\ b^{-2}d^{-1}(a-bd) & b^{-1}d^{-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ a^{-1}b^{-1}(X+1)-b^{-2}d^{-1} & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & b^2d(a-bd)^{-1}(bdX+a) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -b^2d(a-bd)^{-1}(X+1) \\ b^{-2}d^{-1}(a-bd) & b^{-1}d^{-1} \end{pmatrix}.$$

Similarly

$$\begin{pmatrix} a & b \\ 0 & -a^{-1}X \end{pmatrix} R^g = \begin{pmatrix} a & b \\ 0 & -a^{-1}X \end{pmatrix}\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & ag+b \\ 0 & a^{-1}X \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a^2\bar g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & ag(0)+b \\ 0 & a^{-1}X \end{pmatrix}$$

$$= \begin{pmatrix} 1 & a^2\bar g \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ (ag(0)+b)^{-1}a^{-1}X & 1 \end{pmatrix}\begin{pmatrix} a & ag(0)+b \\ -(ag(0)+b)^{-1}X & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -a^{-1}(X+1) \\ a & b \end{pmatrix}\begin{pmatrix} 1 & c\bar g \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -a^{-1}(X+1) \\ a & -ac\bar g+b \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -a^2c\tilde g & 1 \end{pmatrix}\begin{pmatrix} 0 & -a^{-1}(X+1) \\ a & ac\bar g(-1)+b \end{pmatrix},$$

$$\begin{pmatrix} 0 & -a^{-1}(X+1) \\ a & b \end{pmatrix}\begin{pmatrix} 1 & 0 \\ dX & 1 \end{pmatrix} = \begin{pmatrix} a^{-1}dX(X+1) & -a^{-1}(X+1) \\ bdX+a & b \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -a^{-1}b^{-1}(X+1)+b^{-2}d^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} b^{-2}d^{-1}(bd-a) & -b^{-1}d^{-1} \\ bdX+a & b \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -a^{-1}b^{-1}(X+1)+b^{-2}d^{-1} \\ 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & 0 \\ b^2d(bd-a)^{-1}(bdX+a) & 1 \end{pmatrix}\begin{pmatrix} b^{-2}d^{-1}(bd-a) & -b^{-1}d^{-1} \\ 0 & b^2d(bd-a)^{-1}(X+1) \end{pmatrix}.$$

Proof. This is just brute computation whilst carefully preserving pattern. In particular, the second triad of formulae is readily obtainable from the first by strategic multiplications by J and the like. ■

We proceed with the proof of the theorem by noting that if f is of degree 3 then $\tilde f$ is linear; we recall that $f_1 = X^3$.

The operation 'division by X' acts on L^{f_1} in state

$$\begin{pmatrix} 1 & 1 \\ -X & 0 \end{pmatrix}$$

and exits the first triad of transitions in a state of the shape

$$\begin{pmatrix} A & B \\ 0 & A^{-1}X \end{pmatrix}$$

with $B \neq 0$, since it is a constant inverted in the course of the transition. Similarly, 'multiplication by $(X+1)$' enters in state

$$\begin{pmatrix} 1 & -1 \\ 0 & X+1 \end{pmatrix}$$

and exits in a state of the shape

$$\begin{pmatrix} 0 & -A^{-1}(X+1) \\ A & B \end{pmatrix},$$

again with $B \neq 0$. The matrix L^{f_1} is transduced to a product

$$R^{c_2} L^{c_3} R^{c_4}$$

with, in the notation of the respective transition,

$$\begin{aligned} c_2 &= -a^2 c\tilde{f}_1 = -X+1, && a=1, \quad c=1, \\ c_3 &= a^{-1}b^{-1}(X+1) - b^{-2}d^{-1}\tilde{f}_1 = -\tfrac{1}{2}X - \tfrac{1}{4}, && a=1, \quad b=-2, \quad d=-1, \\ c_4 &= b^2 d(a-bd)^{-1}(bdX+a) = 8X+4. \end{aligned}$$

We complete this part of the argument by induction on n, supposing that the partial quotients c_h have already been shown to be of degree 1 for $h \leqslant 6n-2$ and that the operations exit their action on L^{f_n} in states of the shape detailed at entry to the second triad of transitions in Lemma 1. As remarked immediately above, these assumptions hold for $n=1$.

Inspection of the second triad of transitions shows that the operations exit their action on R^{g_n} in states of the shape detailed at entry to the first triad. Since c_{2n} is linear, so is $\tilde{g}_n$. The matrix R^{g_n} is transduced to a product

$$L^{c_{6n-1}} R^{c_{6n}} L^{c_{6n+1}},$$

with each c_h linear.

We have already described the consequences of the operations acting on $L^{f_{n+1}}$ since the detail peculiar to the case $n=0$ does not interfere with the nature of the result in the general case. In particular, the exit states are of the shape detailed at entry to the second triad of transitions, and the partial quotients c_{6n+2}, c_{6n+3}, and c_{6n+4} are linear.

Thus the process marches ([1]) and the theorem is proved subject only to our establishing the nonvanishing of the constants inverted en route.

We do that by reducing to a finite problem:

([1]) This word is used in deference to the edict that French civil servants use *the* language whenever practicable.

LEMMA 2 (Reduction modulo 3). *If the ground field* **K** *has characteristic* 3 *then*

$$F = \prod_{h=0}^{\infty} (1+X^{-3^h}) = (1+X^{-1})^{-1/2}$$

and

$$F = [1, X, \overline{-X+1, X-1}].$$

Proof. We have

$$F = \prod_{h=0}^{\infty} (1+X^{-1})^{3^h} = (1+X^{-1})^{\sum_{h=0}^{\infty} 3^h}.$$

But, because of the formal context, we have

$$\lim_{h\to\infty} (1+X^{-1})^{3^h} = \lim_{h\to\infty} 1+X^{-3^h} = 1$$

and it follows that

$$F = \lim_{h\to\infty} (1+X^{-1})^{(1-3^h)/(1-3)} = (1+X^{-1})^{-1/2}$$

as asserted. (The sceptical reader may verify the allegation synthetically.) Accordingly, $(F-1)^{-1}$ is a quadratic irrational. It satisfies 'Pell's equation'

$$\mathrm{Norm}(-X^2+X+1-(F-1)^{-1}(-X+1)) = 1$$

and has norm $N = X+1$ and trace $T = X+1$. But

$$\begin{pmatrix} -N(-X+1) & -X^2+X+1 \\ -X^2+X+1-T(-X+1) & -X+1 \end{pmatrix} = \begin{pmatrix} X & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -X+1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix},$$

proving that

$$(F-1)^{-1} = [X, \overline{-X+1, -1+X}],$$

which readily yields the second claim. ∎

Remark. The hocus pocus of the first argument is elucidated by Mendès France and van der Poorten in [2]. The proof of periodicity is an instance of a general proof of Lagrange's theorem; see [7].

With the extra information provided by the lemma we can analyse the transitions and verify that all inverted constants are nonzero modulo 3 and *a fortiori* do not vanish in characteristic zero.

It turns out that in characteristic 3 the transduction is periodic with a preperiod of three transitions (the first two coincide with those detailed at the beginning of the proof) and a period of 12 transitions. Though tedious, detail seems more convenient than analysis. We start with the third transition: recall that $c_1 = X$.

$$\begin{pmatrix}1 & 1\\ -X & \end{pmatrix}\begin{pmatrix}1 & \\ X^3 & 1\end{pmatrix}=\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ & X\end{pmatrix},$$

$$\begin{pmatrix}1 & -1\\ & X+1\end{pmatrix}\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}=\begin{pmatrix}1 & -X+1\\ & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ & X+1\end{pmatrix},$$

$$\begin{pmatrix}1 & 1\\ & X+1\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}=\begin{pmatrix}1 & \\ X-1 & 1\end{pmatrix}\begin{pmatrix}1 & -X+1\\ & 1\end{pmatrix}\begin{pmatrix} & -(X+1)\\ 1 & -1\end{pmatrix},$$

$$\begin{pmatrix}1 & 1\\ & X\end{pmatrix}\begin{pmatrix}1 & -X^3+1\\ & 1\end{pmatrix}=\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ X & \end{pmatrix},$$

$$\begin{pmatrix} & -(X+1)\\ 1 & -1\end{pmatrix}\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}=\begin{pmatrix}1 & \\ X-1 & 1\end{pmatrix}\begin{pmatrix} & -(X+1)\\ 1 & 1\end{pmatrix},$$

$$\begin{pmatrix} & -(X+1)\\ 1 & 1\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}=\begin{pmatrix}1 & -X+1\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ X-1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ & -(X+1)\end{pmatrix}.$$

This last exit state is the additive inverse of the entry state at the second transition above, so the second half of the transition period is little different from the first:

$$\begin{pmatrix}1 & -1\\ & X\end{pmatrix}\begin{pmatrix}1 & \\ X^3-1 & 1\end{pmatrix}=\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ & X\end{pmatrix},$$

$$\begin{pmatrix}-1 & 1\\ & -(X+1)\end{pmatrix}\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}=\begin{pmatrix}1 & -X+1\\ & 1\end{pmatrix}\begin{pmatrix}-1 & -1\\ & -(X+1)\end{pmatrix},$$

$$\begin{pmatrix}-1 & -1\\ & -(X+1)\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}=\begin{pmatrix}1 & \\ X-1 & 1\end{pmatrix}\begin{pmatrix}1 & -X+1\\ & 1\end{pmatrix}\begin{pmatrix} & X+1\\ -1 & 1\end{pmatrix},$$

$$\begin{pmatrix}1 & 1\\ X & \end{pmatrix}\begin{pmatrix}1 & -X^3+1\\ & 1\end{pmatrix}=\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ X & \end{pmatrix},$$

$$\begin{pmatrix} & X+1\\ -1 & 1\end{pmatrix}\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}=\begin{pmatrix}1 & \\ X-1 & 1\end{pmatrix}\begin{pmatrix} & X+1\\ -1 & -1\end{pmatrix},$$

$$\begin{pmatrix} & X+1\\ -1 & -1\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}=\begin{pmatrix}1 & -X+1\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ X-1 & 1\end{pmatrix}\begin{pmatrix}1 & -1\\ & X+1\end{pmatrix},$$

$$\begin{pmatrix}1 & -1\\ X & \end{pmatrix}\begin{pmatrix}1 & \\ X^3-1 & 1\end{pmatrix}=\begin{pmatrix}1 & -X^2\\ & 1\end{pmatrix}\begin{pmatrix}1 & \\ -X & 1\end{pmatrix}\begin{pmatrix}1 & 1\\ & X\end{pmatrix},$$

completing our listing of the transition period given the periodicity of the partial quotients. This verification demonstrates the validity of the inversions in the course of the transition process and completes the proof of the theorem. ∎

COROLLARY. *The infinite products*

$$\prod_{h=0}^{\infty}(1+sX^{-3^h})$$

considered over a ground field $\boldsymbol{K}$ *of characteristic zero (or of characteristic* 3) *have all their partial quotients linear in* X *with coefficients rational functions in* s. *If* $s \equiv \pm 1 \pmod 3$ *the coefficients specialise to nonzero elements of* $\boldsymbol{K}$.

Proof.

$$1+sX^{-1}=\frac{X+s}{X}=\frac{s^{-1}X+1}{s^{-1}X}. \ ∎$$

Remark. In other words, work with $Y=s^{-1}X$ in place of X and remember, since the c_h will turn out to be linear, that now

$$f_h(Y)=c_{2h-1}(s^2Y^3), \quad g_h(X)=c_{2h}(s^2Y^3).$$

Under specialisation nothing changes in the argument reduced modulo 3 if $s\equiv \pm 1 \pmod 3$. In particular, all partial quotients of $\prod_{h=0}^{\infty}(1-X^{-3^h})$ are linear.

4. Other results and comments. The infinite products

$$\prod_{h=0}^{\infty}(1+X^{-k^h}), \quad k=5, 7, \ldots,$$

considered over a ground field $\boldsymbol{K}$ of characteristic zero (or of characteristic 3) have the property that two of any three of their partial quotients are of degree 1. The argument leading to that result is virtually identical to that of the theorem and surprisingly – given the special rôle played by the integer 3 in the enunciation of the theorem – these more general products must again be considered over a ground field of characteristic 3 to validate the transition process.

One needs little more than the remark that if f is of degree t then $\tilde{f}$ has degree $t-2$. The transition formulae of Lemma 1 then show that, if k is odd,

$$\prod_{h=0}^{\infty}(1+X^{-k^h})=[1, c_1, c_2, \ldots]$$

with $\deg c_h = 1$ if $h\equiv 0, 1 \pmod 3$ and otherwise

$$\deg c_h = k^r - 2k^{r-1} - \ldots - 2k - 2$$

if h is of the shape $h = 3^r n - 3^{r-1} - \ldots - 3 - 1$ for some $n = 1, 2, \ldots$ chosen minimally.

The cases $k > 2$ even are quite different essentially because $x^k \equiv x \pmod 3$ for all $x \in \mathbf{Z}$ only if k is odd. The argument of [4] is rather different, but its main result can also be obtained from transition formulae of the sort displayed at Lemma 1. Those formulae turn out to be somewhat more natural than those required here and we leave them as an exercise for the mildly energetic reader.

Cases where truncations of the product do not yield partial quotients seem more difficult.

Our indebtedness to an idea of Mills and Robbins will be evident to readers familiar with their paper [5].

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Jean-Paul Allouche and Michel Mendès France

CNRS AND UER MATHÉMATIQUES ET INFORMATIQUE
UNIVERSITÉ BORDEAUX I
351, cours de la Libération
F-33405 Talence cedex
France
allouche@frbdx11.bitnet

Alfred J. van der Poorten

SCHOOL OF MATHEMATICS, PHYSICS, COMPUTING AND ELECTRONICS
MACQUARIE UNIVERSITY NSW 2109
Australia
alf@mqcomp.mqcs.mq.oz.au

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On $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ and Rosser's sieve

by

J. W. Sander (Hannover)

1. Introduction. An old conjecture of Erdős and Straus says that for any given integer $n > 1$, the equation

(1) $$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

has a solution in positive integers x, y, z. For references to the huge amount of (partial) results concerning the conjecture as well as its generalizations by Sierpiński, Schinzel and others, we refer the reader to [1], problem D 11, and [9].

We just like to mention an outstanding result by Vaughan [11] which gives an upper bound for the exceptional set $E_m(N)$ of integers $n \leqslant N$ for which $m/n = 1/x + 1/y + 1/z$ has no solution ($m \geqslant 4$ is a fixed integer), namely

$$E_m(N) \ll N \exp(-c(\log N)^{2/3}),$$

where c may only depend on m.

In order to prove the conjecture it obviously suffices to solve (1) for all primes q (instead of n). Moreover, one can easily see that, if there is a solution of $4/q = 1/x + 1/y + 1/z$, then either exactly one of the numbers x, y, z is divisible by q, or exactly two of them have a divisor q. The second case, namely the equation

(2) $$\frac{4}{q} = \frac{1}{w} + \frac{1}{gq} + \frac{1}{hq}$$

for a given prime q, is equivalent to the solvability of

(3) $$(4g-1)(4h-1) = 4tq + 1, \qquad t \mid gh,$$

in positive integers g, h, t (see [9]). In [9] lower bounds for

$$V(x; k, l; t) = \operatorname{card}\{q \leqslant x\colon\ q \equiv l \bmod k,\ (2) \text{ unsolvable with } gh/w = t\}$$