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The homogeneous transfinite diameter of a compact subset of $\mathbb{C}^N$

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Abstract. Let K be a compact subset of $\mathbb{C}^N$. A sequence of nonnegative numbers defined by means of extremal points of K with respect to homogeneous polynomials is proved to be convergent. Its limit is called the homogeneous transfinite diameter of K. A few properties of this diameter are given and its value for some compact subsets of $\mathbb{C}^N$ is computed.

1. Introduction. Let K be a compact subset of $\mathbb{C}^N$. For a nonnegative integer s let

$$h_s := \binom{s+N-1}{N-1}.$$

Let $e_{s,1}(z), \ldots, e_{s,h_s}(z)$ be all monomials $z^\alpha := z_1^{\alpha_1} \ldots z_N^{\alpha_N}$ of degree s ordered lexicographically.

For an integer k ($1 \le k \le h_s$) let $x^{(k)} = \{x_1, \ldots, x_k\}$ be a system of k points in $\mathbb{C}^N$. Define the "homogeneous Vandermondian" $W_s(x^{(k)})$ of the system $x^{(k)}$ by

$$W_s(x^{(k)}) := \det[e_{s,i}(x_j)]_{i,j=1,\ldots,k}.$$

Then $W_s(x^{(k)})$ is a polynomial in $x_1, \ldots, x_k$ of degree sk. Let

$$W_{s,k} := \sup\{|W_s(x^{(k)})| : x^{(k)} \subset K\}.$$

A system $x^{(k)}$ of k points in K is called a *system of extremal points of K with respect to homogeneous polynomials* if

$$|W_s(x^{(k)})| = W_{s,k}.$$

In this paper we prove that for every compact subset K of $\mathbb{C}^N$ the limit

$$D(K) := \lim_{s\to\infty} (W_{s,h_s})^{1/(sh_s)}$$

exists. We call it the *homogeneous transfinite diameter* of K.

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This result gives a positive answer to a question put in [11] (see also [12], p. 93). It is obvious that the limit exists for $N = 1$. For $N = 2$ the convergence was proved by Leja [4] (see also [5], p. 261). The limit is then equal to $\sqrt{2\Delta(K)}$, where $\Delta(K)$ is the triangular ecart of K.

We also prove a few properties of $D(K)$ (e.g. comparison of $D(K)$ with some other constants connected with K). Using a characterization of $D(K)$ in terms of directional Chebyshev constants, we compute $D(K)$ for

$$K := \{(z_1, \ldots, z_N) \in \mathbb{C}^N : |z_1|^{p_1} + \ldots + |z_N|^{p_N} \le M\},$$

where $M, p_1, \ldots, p_N$ are real positive constants.

We also indicate another method for computing $D(K)$ without calculating W_{s,h_s}.

2. Preliminaries. Let K be a compact subset of $\mathbb{C}^N$. Let $\|f\|_K$ denote the supremum norm of a function $f : K \to \mathbb{C}$.

Definition 2.1. K is called *unisolvent with respect to homogeneous polynomials* if no nonzero homogeneous polynomial vanishes identically on K.

Definition 2.2. K is called *circled* if

$$\{(e^{i\theta} z_1, \ldots, e^{i\theta} z_N) : (z_1, \ldots, z_N) \in K, \quad \theta \in \mathbb{R}\} \subset K.$$

Definition 2.3. K is called *N-circular* if

$$\{(e^{i\theta_1} z_1, \ldots, e^{i\theta_N} z_N) : (z_1, \ldots, z_N) \in K, \theta_1, \ldots, \theta_N \in \mathbb{R}\} \subset K.$$

Definition 2.4. Let μ be a nonnegative Borel measure with $\operatorname{supp} \mu \subset K$. The pair (K, μ) is said to satisfy the *Bernstein–Markov property* if for every $\lambda > 1$ there exists an $M > 0$ such that for all polynomials p

$$\|p\|_K \le M \lambda^{\deg p} \|p\|_2, \qquad \text{where } \|p\|_2 := \Big(\int_K |p|^2 \, d\mu\Big)^{1/2}.$$

Remark. A few examples of pairs satisfying the Bernstein–Markov property can be found e.g. in [2], [7], [9], [13].

Let δ denote the Lebesgue surface area measure on the unit sphere

$$S := \{z \in \mathbb{C}^N : |z_1|^2 + \ldots + |z_N|^2 = 1\},$$

normalized so that $\int_S d\delta = 1$.

Definition 2.5 (see [1]). The *Alexander constant* $\gamma(K)$ is

$$\gamma(K) := \inf_{s \in \mathbb{N}} (\gamma_s(K))^{1/s} = \lim_{s \to \infty} (\gamma_s(K))^{1/s},$$

where $\gamma_s(K) := \inf\{\|Q\|_K\}$, the infimum being taken over all homogeneous polynomials Q of N complex variables of degree s, normalized so that

$$\int_S \log|Q|^{1/s}\,d\delta = \kappa_N := \int_S \log|z_N|\,d\delta.$$

It is known that

$$\kappa_N = -\frac{1}{2}\left(1+\frac{1}{2}+\ldots+\frac{1}{N-1}\right).$$

Definition 2.6 (see [12]). The *Chebyshev constant* $\varrho(K)$ is

$$\varrho(K) := \inf_{s\in\mathbb{N}}(\varrho_s(K))^{1/s} = \lim_{s\to\infty}(\varrho_s(K))^{1/s},$$

where $\varrho_s(K) := \inf\{\|Q\|_K\}$, the infimum being taken over all homogeneous polynomials Q of N complex variables of degree s, normalized so that $\|Q\|_S = 1$.

3. The transfinite diameter of a compact subset K of $\mathbb{C}^N$. For a nonnegative integer s put

$$m_s := \binom{s+N}{N}.$$

Let $e_1(z), e_2(z), \ldots$ be all monomials $z^\alpha := z_1^{\alpha_1}\ldots z_N^{\alpha_N}$ ordered so that the degrees of the $e_j(z)$ are nondecreasing and the monomials of a fixed degree are ordered lexicographically. It is easy to check that $e_{s+1,k} = e_{m_s+k}$.

For an integer k let $x^{(k)} = \{x_1,\ldots,x_k\}$ be a system of k points in $\mathbb{C}^N$. Define the "Vandermondian" $V(x^{(k)})$ of the system $x^{(k)}$ by

$$V(x^{(k)}) := \det[e_i(x_j)]_{i,j=1,\ldots,k}.$$

Then $V(x^{(m_s)})$ is a polynomial in $x_1,\ldots,x_{m_s}$ of degree

$$l_s := \sum_{j=1}^{m_s}\deg e_j = \sum_{k=0}^{s} kh_k.$$

It is easy to prove that $l_s = N\binom{s+N}{N+1}$. Put

$$V_k := \sup\{|V(x^{(k)})| : x^{(k)} \subset K\}.$$

Zakharyuta proved in [14] that for every compact subset K of $\mathbb{C}^N$ the limit

$$d(K) := \lim_{s\to\infty}(V_{m_s})^{1/l_s}$$

exists; it is called the *transfinite diameter* of K. This result gave a positive answer to a question put in [6]. For $N = 1$ the convergence was proved by Fekete [3] (see also [5]).

Zakharyuta also computed $d(K)$ in terms of the directional Chebyshev constants. Put

$$\Sigma = \Sigma^{N-1} := \Big\{\theta = (\theta_1, \ldots, \theta_N) \in \mathbb{R}^N : \sum_{j=1}^{N} \theta_j = 1, \theta_j \geq 0\Big\},$$

$$\Sigma_0 = \Sigma_0^{N-1} := \big\{\theta \in \Sigma^{N-1} : \theta_j > 0 \text{ for } j = 1, \ldots, N\big\}.$$

For an integer $j \geq 1$ let $\alpha(j) := (\alpha_1, \ldots, \alpha_N)$, where $z_1^{\alpha_1} \ldots z_N^{\alpha_N} = e_j(z)$. Let

$$M_j := \inf\Big\{\Big\|e_j(z) + \sum_{i<j} c_i e_i(z)\Big\|_K : c_i \in \mathbb{C}\Big\}$$

denote the Chebyshev constant of K associated to the monomial $e_j(z)$ and the given ordering. It is known that the infimum is attained for at least one polynomial $t_j(z) = e_j(z) + \sum_{i<j} c_i e_i(z)$. It is called the *Chebyshev polynomial* of K. Put

$$\tau_j := M_j^{1/|\alpha(j)|},$$

where, as usual, $|\alpha(j)| = \alpha_1 + \ldots + \alpha_N$ is the length of the multiindex $\alpha(j)$.

For $\theta \in \Sigma$ let $\tau(K,\theta)$ and $\tau_-(K,\theta)$ denote the "Chebyshev constants in the direction θ", i.e.

$$\tau(K,\theta) := \limsup\{\tau_j : j \to \infty, \ \ \alpha(j)/|\alpha(j)| \to \theta\},$$
$$\tau_-(K,\theta) := \liminf\{\tau_j : j \to \infty, \ \ \alpha(j)/|\alpha(j)| \to \theta\}.$$

Zakharyuta proved that $\tau(K,\theta) = \tau_-(K,\theta)$ for each $\theta \in \Sigma_0$ and that $\log \tau(K,\theta)$ is a convex function on Σ_0. Let

$$\tau(K) := \exp\Big\{\frac{1}{\operatorname{mes} \Sigma} \int_{\Sigma} \log \tau(K,\theta)\, d\omega(\theta)\Big\},$$

where $\operatorname{mes} \Sigma := \int_\Sigma d\omega(\theta)$ and ω denotes the Lebesgue surface area measure on the hyperplane $\{\theta_1 + \ldots + \theta_N = 1\}$ in $\mathbb{R}^N$. Zakharyuta proved that $d(K) = \tau(K)$.

4. The homogeneous transfinite diameter. For two integers s, k ($s \geq 0, 1 \leq k \leq h_s$) put

$$M_{s,k} := \inf\Big\{\Big\|e_{s,k}(z) + \sum_{i<k} c_i e_{s,i}(z)\Big\|_K : c_i \in \mathbb{C}\Big\}.$$

It is easy to check that there exists at least one homogeneous polynomial $t_{s,k}(z) = e_{s,k}(z) + \sum_{i<k} c_i e_{s,i}(z)$ attaining the infimum. It is called the *Chebyshev polynomial* of K.

Let $\beta(s,k) := \alpha(m_{s-1} + k)$, where $m_{-1} := 0$. Hence $\beta(s,k) = (\beta_1, \ldots, \beta_N)$, where $z_1^{\beta_1} \ldots z_N^{\beta_N} = e_{s,k}(z)$. It is obvious that $|\beta(s,k)| = s$.

Put

$$\tau_{s,k} := M_{s,k}^{1/s}.$$

For $\theta \in \Sigma$ let

$$\widetilde{\tau}(K,\theta) := \limsup\{\tau_{s,k} : s \to \infty,\ \beta(s,k)/s \to \theta\},$$
$$\widetilde{\tau}_-(K,\theta) := \liminf\{\tau_{s,k} : s \to \infty,\ \beta(s,k)/s \to \theta\}.$$

It is clear that $\widetilde{\tau}(K,\theta) \le C$ if

$$K \subset \{z \in \mathbb{C}^N : |z_1| \le C, \ldots, |z_N| \le C\}.$$

The following lemmas can be proved in the same manner as the similar results in [14] (it suffices to replace the polynomials $e_j(z) + \sum_{i<j} c_i e_i(z)$ by $e_{s,k}(z) + \sum_{i<k} b_i e_{s,i}(z)$, where $e_{s,k} = e_j$):

LEMMA 4.1. *For each* $\theta \in \Sigma_0$, $\widetilde{\tau}(K,\theta) = \widetilde{\tau}_-(K,\theta)$.

LEMMA 4.2. *The function* $\log \widetilde{\tau}(K,\theta)$ *is convex in* Σ_0.

COROLLARY 4.3. *If* $\widetilde{\tau}(K,\theta') = 0$ *for some* $\theta' \in \Sigma_0$, *then* $\widetilde{\tau}(K,\theta) \equiv 0$ *in* Σ_0.

COROLLARY 4.4. *The function* $\log \widetilde{\tau}(K,\theta)$ *is continuous in* Σ_0.

LEMMA 4.5. *If* $\theta \in \Sigma \setminus \Sigma_0$, *then*

$$\widetilde{\tau}_-(K,\theta) = \liminf\{\widetilde{\tau}(K,\theta') : \theta' \to \theta, \theta' \in \Sigma_0\}.$$

COROLLARY 4.6.

$$\limsup_{s\to\infty} \tau_{s,k} = \sup\{\widetilde{\tau}(K,\theta) : \theta \in \Sigma\},$$
$$\liminf_{s\to\infty} \tau_{s,k} = \inf\{\widetilde{\tau}(K,\theta) : \theta \in \Sigma\}$$
$$= \inf\{\widetilde{\tau}(K,\theta) : \theta \in \Sigma_0\} = \inf\{\widetilde{\tau}_-(K,\theta) : \theta \in \Sigma\}.$$

COROLLARY 4.7. *If* $\widetilde{\tau}(K,\theta) \not\equiv 0$ *in* Σ_0, *then* $\inf\{\widetilde{\tau}(K,\theta) : \theta \in \Sigma\} > 0$.

DEFINITION 4.8. The *Chebyshev constant* $\widetilde{\tau}(K)$ is

$$\widetilde{\tau}(K) := \exp\left\{\frac{1}{\operatorname{mes}\Sigma} \int_{\Sigma} \log \widetilde{\tau}(K,\theta)\, d\omega(\theta)\right\}.$$

If $\widetilde{\tau}(K,\theta) \equiv 0$ in Σ_0, then $\widetilde{\tau}(K) = 0$. Assume that $\widetilde{\tau}(K,\theta) \not\equiv 0$ in Σ_0. Then $\log \widetilde{\tau}(K,\theta)$ is continuous in Σ_0 and bounded on Σ (see Corollaries 4.4 and 4.7). Therefore the integral above exists and is finite. Hence $0 < \widetilde{\tau}(K) < \infty$ in this case.

LEMMA 4.9. $\lim_{s\to\infty} \widetilde{\tau}_s^0(K) = \widetilde{\tau}(K)$, *where*

$$\widetilde{\tau}_s^0(K) := \Big(\prod_{k=1}^{h_s} \tau_{s,k}\Big)^{1/h_s}.$$

Lemma 4.10. *Let s, k be nonnegative integers such that $1 \le k \le h_s$. Then*

$$\tau_{s,k}^{s} W_{s,k-1} \le W_{s,k} \le k\tau_{s,k}^{s} W_{s,k-1},$$

where $W_{s,0} := 1$.

Corollary 4.11. *If $W_{s,k} > 0$ for each $k \in \{1, \ldots, h_s\}$, then*

$$(\widetilde{\tau}_s^0(K))^{sh_s} \le W_{s,h_s} \le h_s!(\widetilde{\tau}_s^0(K))^{sh_s}.$$

Theorem 4.12. *For every compact subset K of $\mathbb{C}^N$ the limit*

$$D(K) := \lim_{s\to\infty} (W_{s,h_s})^{1/(sh_s)}$$

exists and is equal to $\widetilde{\tau}(K)$.

We call this limit the *homogeneous transfinite diameter* of K.

Proof. If K is not unisolvent with respect to homogeneous polynomials then $Q \equiv 0$ on K, where $Q = e_{s,k} + \sum_{i<k} c_i e_{s,i}$. Hence for each positive integer j

$$z_1^j \ldots z_N^j Q(z_1, \ldots, z_N) \equiv 0 \quad \text{on } K.$$

Letting $j \to \infty$ we obtain $\widetilde{\tau}(K, \theta') = 0$, where $\theta' = (1/N, \ldots, 1/N)$. By Corollary 4.3, $\widetilde{\tau}(K, \theta) \equiv 0$ on Σ_0. On the other hand, one sees immediately that $W_{r,h_r} = 0$ for $r \ge s$, which completes the proof in this case.

Assume now that K is unisolvent with respect to homogeneous polynomials. Then $\tau_{s,k} > 0$ for $s \ge 0$ and $1 \le k \le h_s$. So $W_{s,k} > 0$ by Lemma 4.10. Applying Lemma 4.9 and Corollary 4.11 we get the desired conclusion.

Corollary 4.13. *If K is not unisolvent with respect to homogeneous polynomials, then $D(K) = 0$.*

5. Properties of the constant $D(K)$

Lemma 5.1. *For every compact subset K of $\mathbb{C}^N$, $d(K) \le D(K)$. If K is circled, then $d(K) = D(K)$.*

Proof. It is obvious that $\|t_j\|_K \le \|t_{s,k}\|_K$ if $\beta(s,k) = \alpha(j)$, i.e. $e_{s,k} = e_j$. By Theorem 4.12 and the equality $d(K) = \tau(K)$, it suffices to show that $\|t_{s,k}\|_K \le \|t_j\|_K$ if K is circled. By the Cauchy inequalities $\|t_j\|_K \ge \|q_j\|_K$, where $t_j = q_j + p_j$, q_j is homogeneous and $\deg p_j < \deg t_j$ (or $p_j \equiv 0$). Obviously, $\|q_j\|_K \ge \|t_{s,k}\|_K$, which proves the lemma.

Lemma 5.2. *If K is N-circular and $\theta \in \Sigma_0$, then*

$$\tau(K, \theta) = \widetilde{\tau}(K, \theta) = \sup\{|z_1|^{\theta_1} \ldots |z_N|^{\theta_N} : (z_1, \ldots, z_N) \in K\}.$$

Proof. Clearly, $\|t_j\|_K \le \|t_{s,k}\|_K \le \|e_j\|_K$, where $e_{s,k} = e_j$. Since K is N-circular, by the Cauchy inequalities $\|e_j\|_K \le \|t_j\|_K$. Hence for $\theta \in \Sigma_0$

$$\begin{aligned}\tau(K,\theta) = \widetilde{\tau}(K,\theta) &= \lim\{\|e_j\|_K^{1/|\alpha(j)|} : j \to \infty,\ \alpha(j)/|\alpha(j)| \to \theta\}\\ &= \sup\{|z_1|^{\theta_1} \ldots |z_N|^{\theta_N} : (z_1,\ldots,z_N) \in K\}.\end{aligned}$$

which is the desired conclusion.

LEMMA 5.3. $D(K) = D(\widehat{K})$, *where* $\widehat{K}$ *is the convex hull of* K *with respect to homogeneous polynomials*, *i.e.*

$$\widehat{K} := \{z \in \mathbb{C}^N : |Q(z)| \le \|Q\|_K \text{ for all homogeneous polynomials } Q\}.$$

Proof. It suffices to use Theorem 4.12 together with the obvious equality $\widetilde{\tau}(K,\theta) = \widetilde{\tau}(\widehat{K},\theta)$.

LEMMA 5.4. *Let* $K_1 = F(K_2)$, *where* $F(z_1,\ldots,z_N) := (c_1z_1,\ldots,c_Nz_N)$ *for* $(z_1,\ldots,z_N) \in \mathbb{C}^N$ *and* $c_1,\ldots,c_N \in \mathbb{C}$. *Then*

$$D(K_1) = |c_1 \ldots c_N|^{1/N} D(K_2).$$

Proof. It is sufficient to compare the constants W_{s,h_s} for K_1 with those for K_2. The details are left to the reader.

LEMMA 5.5. *If* $U : \mathbb{C}^N \to \mathbb{C}^N$ *is a unitary transformation*, *then* $D(U(K)) = D(K)$.

Proof. The lemma can be proved in the same way as the similar result $d(U(K)) = d(K)$ (see [8]).

COROLLARY 5.6. *If* $A : \mathbb{C}^N \to \mathbb{C}^N$ *is a linear mapping*, *then*

$$D(A(K)) = |\det A|^{1/N} D(K).$$

Proof. Combine Lemmas 5.4 and 5.5.

THEOREM 5.7. *If* K *is compact and* R *is a positive constant such that*

$$K \subset B_R := \{(z_1,\ldots,z_N) \in \mathbb{C}^N : |z_1|^2 + \ldots + |z_N|^2 \le R^2\},$$

then

$$\varrho(K)/\sqrt{N} \le D(K) \le R^{1-1/N} \varrho(K)^{1/N}.$$

Proof. The theorem can be proved in the same manner as Theorem 3 in [8] (it suffices to replace $e_j(z) + \sum_{i<j} c_i e_i(z)$ by $e_{s,k}(z) + \sum_{i<k} b_i e_{s,i}(z)$, where $e_{s,k} = e_j$).

COROLLARY 5.8. *If* K *is compact and* R *is a positive constant such that* $K \subset B_R$, *then*

$$\gamma(K)/\sqrt{N} \le D(K) \le R^{1-1/N} \exp(-\kappa_N/N) \gamma(K)^{1/N}.$$

Proof. It is known that $\gamma(K) \le \varrho(K) \le \gamma(K) \exp(-\kappa_N)$ (see [12], Proposition 12.1). Now apply Theorem 5.7.

THEOREM 5.9. *Let K be a compact subset of $\mathbb{C}^N$. Let μ be a nonnegative Borel measure with* $\operatorname{supp}\mu \subset K$. *If the pair (K,μ) satisfies the Bernstein–Markov property and $\mu(K) < \infty$, then*

$$D(K) = \lim_{s\to\infty} (G_{s,h_s})^{1/(2sh_s)},$$

where

$$G_{s,k} := \det\Big\{\Big[\int_K e_{s,i}(z)\overline{e_{s,j}(z)}\,d\mu(z)\Big]_{i,j=1,\ldots,k}\Big\},$$

for nonnegative integers s,k ($k \in \{1,\ldots,h_s\}$).

Proof. If K is not unisolvent with respect to homogeneous polynomials then $D(K) = 0$ (see Corollary 4.13). On the other hand, for all but a finite number of integers r there exists a nonzero homogeneous polynomial Q_r of degree r that vanishes identically on K, say

$$Q_r = \sum_{j=1}^{h_r} d_j e_{r,j} \qquad (d_j \in \mathbb{C}).$$

Obviously, $\|Q_r\|_K = 0$ implies $\|Q_r\|_2 = 0$. Therefore $G_{r,h_r} = 0$ for such r.

Assume that K is unisolvent with respect to homogeneous polynomials. Then none of the Gram determinants $G_{s,k}$ is zero. Indeed, if $G_{s,k} = 0$ for some s and k, we should have $\|Q\|_2 = 0$, where $Q = \sum_{j=1}^k d_j e_{s,j}$ ($d_j \in \mathbb{C}$) and $Q \not\equiv 0$. By the Bernstein–Markov property, $\|Q\|_K = 0$, which is impossible.

Analysis similar to that in the proof of Theorem 3.3 in [2] now yields our statement (upon replacing again $e_j(z) + \sum_{i<j} c_i e_i(z)$ by $e_{s,k}(z) + \sum_{i<k} b_i e_{s,i}(z)$, where $e_{s,k} = e_j$). Lemma 4.9 and Theorem 4.12 are used in the proof.

6. The value of $D(K)$ and $d(K)$ for some compact sets K. Consider the following compact N-circular set $K = K(p_1,\ldots,p_N,M)$:

$$K := \{(z_1,\ldots,z_N) \in \mathbb{C}^N : |z_1|^{p_1} + \ldots + |z_N|^{p_N} \le M\},$$

where M, p_1, …, p_N are real positive constants.

THEOREM 6.1. *If $K = K(p_1,\ldots,p_N,M)$ and $a_j = 1/p_j$ for $j = 1,\ldots,N$, then*

$$D(K)=d(K)=\exp\bigg\{\frac{1}{N}\bigg(\sum_{j=1}^N a_j\log(Ma_j) - \frac{1}{2\pi i}\int_C \frac{z^N \operatorname{Log} z\,dz}{(z-a_1)\ldots(z-a_N)}\bigg)\bigg\},$$

where C is any contour in the right half-plane $\{z\in\mathbb{C} : \operatorname{Re} z > 0\}$ enclosing all the points $a_1,\ldots,a_N$ and $\operatorname{Log} z$ is the principal branch of the logarithm.

In particular, if $p_j \neq p_k$ for $j \neq k$, then

$$D(K) = d(K) = \exp\biggl\{\frac{1}{N}\biggl(\sum_{j=1}^{N} a_j \log(Ma_j) - \sum_{j=1}^{N} \frac{a_j^N \log a_j}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)}\biggr)\biggr\}.$$

If $p_1 = \ldots = p_N = p$ and $M = R^p$ $(R > 0)$, then

$$D(K) = d(K) = R\exp\biggl(-\frac{1}{p}\sum_{k=2}^{N}\frac{1}{k}\biggr).$$

We first prove two lemmas.

LEMMA 6.2. *If $f(\theta_1, \ldots, \theta_N)$ is a continuous function on Σ^{N-1}, then*

$$\frac{1}{\operatorname{mes}\Sigma^{N-1}} \int_{\Sigma^{N-1}} f(\theta_1, \ldots, \theta_N)\, d\omega(\theta) = (N-1)\int_0^1 x^{N-2} H(x)\, dx,$$

where

$$H(x) := \frac{1}{\operatorname{mes}\Sigma^{N-2}} \int_{\Sigma^{N-2}} f(\xi_1 x, \xi_2 x, \ldots, \xi_{N-1} x, 1-x)\, d\omega(\xi).$$

Proof. Obviously,

$$\frac{1}{\operatorname{mes}\Sigma^{N-1}} \int_{\Sigma^{N-1}} f(\theta_1, \ldots, \theta_N)\, d\omega(\theta)$$
$$= \frac{1}{\operatorname{mes}\Sigma_*^{N-1}} \int_{\Sigma_*^{N-1}} f\Bigl(\theta_1, \ldots, \theta_{N-1}, 1 - \sum_{j=1}^{N-1}\theta_j\Bigr)\, d\theta_1 \ldots d\theta_{N-1},$$

where $\Sigma_*^{N-1} := \{(\theta_1, \ldots, \theta_{N-1}) \in \mathbb{R}^{N-1} : \sum_{j=1}^{N-1}\theta_j \leq 1,\ \theta_j \geq 0\}$. We change the variables:

$$\theta_j = \xi_j x \quad \text{for } j = 1, \ldots, N-2,$$
$$\theta_{N-1} = \Bigl(1 - \sum_{j=1}^{N-2}\xi_j\Bigr)x,$$

where $0 \leq x \leq 1$ and $(\xi_1, \ldots, \xi_{N-2}) \in \Sigma_*^{N-2}$. It is obvious that $d\theta_1 \ldots d\theta_{N-1} = x^{N-2}\, dx\, d\xi_1 \ldots d\xi_{N-2}$ and that

$$\operatorname{mes}\Sigma_*^{N-2} / \operatorname{mes}\Sigma_*^{N-1} = N-1.$$

This proves the lemma (the details are left to the reader).

LEMMA 6.3. *If $a_j \neq a_k$ for $j \neq k$, then*

$$\sum_{j=1}^{N} \frac{1}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)} = 0, \tag{6.1}$$

$$\sum_{j=1}^{N} \frac{a_j^N}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)} = \sum_{j=1}^{N} a_j. \tag{6.2}$$

Proof. Consider the polynomial

$$P(x) = \sum_{m=0}^{N-1} b_m x^m := -1 + \sum_{j=1}^{N} P_j(x),$$

where

$$P_j(x) := \prod_{\substack{k=1\\k\neq j}}^{N} \frac{x - a_k}{a_j - a_k}.$$

It is clear that $\deg P \le N-1$ and $P(a_j) = 0$ for $j = 1, \ldots, N$, which implies $P \equiv 0$. So $b_{N-1} = 0$, and (6.1) follows.

To prove (6.2), let

$$Q(x) = -x^N + \sum_{m=0}^{N-1} c_m x^m := -x^N + \sum_{j=1}^{N} a_j^N P_j(x).$$

Since $\deg Q = N$ and $Q(a_j) = 0$ for $j = 1, \ldots, N$, we have

$$Q(x) = -(x - a_1)(x - a_2)\ldots(x - a_N).$$

Therefore $c_{N-1} = \sum_{j=1}^{N} a_j$, which completes the proof.

Proof of Theorem 6.1. It is easy to check, applying Lemma 5.2, that for $K = K(p_1, \ldots, p_N, M)$ and $\theta \in \Sigma_0$

$$\begin{aligned}\log \tau(K,\theta) &= \log \widetilde{\tau}(K,\theta)\\ &= \sum_{j=1}^{N} a_j\theta_j \log(Ma_j) + \sum_{j=1}^{N} a_j\theta_j \log\theta_j\\ &\quad - \sum_{j=1}^{N} a_j\theta_j \log(a_1\theta_1 + \ldots + a_N\theta_N).\end{aligned}$$

Since $D(K) = d(K) = \widetilde{\tau}(K)$, it is sufficient to prove the following three formulas ($j = 1, \ldots, N$):

$$\frac{1}{\operatorname{mes} \Sigma^{N-1}} \int_{\Sigma^{N-1}} \theta_j \, d\omega(\theta) = \frac{1}{N}, \tag{6.3}$$

$$\frac{1}{\operatorname{mes} \Sigma^{N-1}} \int_{\Sigma^{N-1}} \theta_j \log \theta_j \, d\omega(\theta) = -\frac{1}{N} \sum_{k=2}^{N} \frac{1}{k}, \tag{6.4}$$

$$\frac{1}{\operatorname{mes} \Sigma^{N-1}} \int_{\Sigma^{N-1}} \Big(\sum_{j=1}^{N} a_j \theta_j\Big) \log\Big(\sum_{j=1}^{N} a_j \theta_j\Big) \, d\omega(\theta) \tag{6.5}$$
$$= -\frac{1}{N}\Big(\sum_{k=2}^{N} \frac{1}{k}\Big)\Big(\sum_{j=1}^{N} a_j\Big) + \frac{1}{N} \cdot \frac{1}{2\pi i} \int_C \frac{z^N \operatorname{Log} z \, dz}{(z-a_1)\ldots(z-a_N)}.$$

Observe that the particular cases

$$p_j \neq p_k \quad \text{for } j \neq k$$

and

$$p_1 = \ldots = p_N = p, \quad M = R^p$$

can be obtained from the main formula (it suffices to apply the Residue Theorem and observe that $f^{(N-1)}(z) = N!z(\operatorname{Log} z + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{N})$ if $f(z) = z^N \operatorname{Log} z$).

It suffices to prove (6.3) and (6.4) for $j = N-1$. Obviously $\operatorname{mes} \Sigma_*^{N-1} = 1/(N-1)!$ and

$$\frac{1}{\operatorname{mes} \Sigma^{N-1}} \int_{\Sigma^{N-1}} \theta_{N-1} \, d\omega(\theta) = \frac{1}{\operatorname{mes} \Sigma_*^{N-1}} \int_{\Sigma_*^{N-1}} \theta_{N-1} \, d\theta_1 \ldots d\theta_{N-1}.$$

So (6.3) follows immediately if we change the variables:

$$\begin{aligned}
\theta_1 &= (1-v_1)v_2 \ldots v_{N-1}, \\
\theta_2 &= (1-v_2)v_3 \ldots v_{N-1}, \\
&\vdots \\
\theta_{N-2} &= (1-v_{N-2})v_{N-1}, \\
\theta_{N-1} &= 1 - v_{N-1},
\end{aligned}$$

where $0 \leq v_j \leq 1$ for $j = 1, \ldots, N-1$.

Apply the same change of variables to compute

$$\frac{1}{\operatorname{mes} \Sigma_*^{N-1}} \int_{\Sigma_*^{N-1}} \theta_{N-1} \log \theta_{N-1} \, d\theta_1 \ldots d\theta_{N-1}.$$

Then it is sufficient to check that

$$\int_0^1 x^{N-2}(1-x)\log(1-x)\, dx = -\frac{1}{N(N-1)} \sum_{k=2}^{N} \frac{1}{k}.$$

Let $x = 1 - e^{-t}$. We obtain

$$
\begin{aligned}
-\int_0^\infty & te^{-2t}(1-e^{-t})^{N-2}\,dt \\
&= -\sum_{j=0}^{N-2}\binom{N-2}{j}(-1)^j \int_0^\infty te^{-(j+2)t}\,dt \\
&= -\sum_{j=0}^{N-2}\binom{N-2}{j}(-1)^j \frac{1}{(j+2)^2} \\
&= -\frac{1}{N(N-1)}\sum_{j=0}^{N-2}(-1)^j\binom{N}{j+2}\left(1-\frac{1}{j+2}\right) \\
&= -\frac{1}{N(N-1)}\left(\sum_{k=2}^{N}(-1)^k\binom{N}{k} + \sum_{k=2}^{N}(-1)^{k+1}\binom{N}{k}\frac{1}{k}\right) \\
&= -\frac{1}{N(N-1)}\left(\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{N}\right).
\end{aligned}
$$

We have applied the well-known formula

$$
\sum_{k=1}^{N}(-1)^{k+1}\binom{N}{k}\frac{1}{k} = \sum_{j=1}^{N}\frac{1}{j}
$$

and the obvious equality

$$
0 = (1-1)^N = 1 - N + \sum_{k=2}^{N}(-1)^k\binom{N}{k}.
$$

Let us prove (6.5). Both its sides are continuous functions of the parameters a_j. Therefore it suffices to show that the formula is true if $a_j \neq a_k$ for $j \neq k$. So we have to check that

$$
\begin{aligned}
(6.6)\qquad \frac{1}{\operatorname{mes}\Sigma^{N-1}} & \int_{\Sigma^{N-1}} \Big(\sum_{j=1}^{N} a_j\theta_j\Big)\log\Big(\sum_{j=1}^{N} a_j\theta_j\Big)\,d\omega(\theta) \\
&= -\frac{1}{N}\Big(\sum_{k=2}^{N}\frac{1}{k}\Big)\Big(\sum_{j=1}^{N} a_j\Big) + \frac{1}{N}\sum_{j=1}^{N}\frac{a_j^N \log a_j}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j-a_k)}.
\end{aligned}
$$

The proof is by induction on N. It is easy to check the case $N = 2$. Assuming (6.6) to hold for $N-1$ ($N \geq 3$), we will prove it for N. We are going to apply Lemma 6.2. We first compute

$$\operatorname{mes} \Sigma^{N-2} \cdot H(x) = \int_{\Sigma^{N-2}} \Big\{ a_N(1-x) + \sum_{j=1}^{N-1} a_j \xi_j x \Big\} \log \Big\{ a_N(1-x) + \sum_{j=1}^{N-1} a_j \xi_j x \Big\}\, d\omega(\xi).$$

We have $a_N(1-x) \equiv a_N(1-x)\sum_{j=1}^{N-1} \xi_j$ on $\Sigma^{N-2} := \{\sum_{j=1}^{N-1} \xi_j = 1\}$. Therefore

$$\operatorname{mes} \Sigma^{N-2} \cdot H(x) = \int_{\Sigma^{N-2}} \Big(\sum_{j=1}^{N-1} A_j \xi_j \Big) \log \Big(\sum_{j=1}^{N-1} A_j \xi_j \Big)\, d\omega(\xi),$$

where $A_j = A_j(x) := a_N + (a_j - a_N)x$ for $j = 1, \ldots, N-1$. By assumption,

$$\begin{aligned} H(x) &= -\frac{1}{N-1} \Big(\sum_{k=2}^{N-1} \frac{1}{k} \Big) \Big(\sum_{j=1}^{N-1} A_j \Big) + \frac{1}{N-1} \sum_{j=1}^{N-1} \frac{A_j^{N-1} \log A_j}{\prod_{\substack{k=1 \\ k\neq j}}^{N-1} (A_j - A_k)} \\ &= -\frac{1}{N-1} \Big(\sum_{k=2}^{N-1} \frac{1}{k} \Big) \Big\{ (N-1)a_N + \Big(\sum_{j=1}^{N-1} a_j - (N-1)a_N \Big) x \Big\} \\ &\quad + \frac{1}{N-1} \sum_{j=1}^{N-1} \frac{(a_N + (a_j - a_N)x)^{N-1} \log(a_N + (a_j - a_N)x)}{\prod_{\substack{k=1 \\ k\neq j}}^{N-1} (a_j - a_k)x}. \end{aligned}$$

Applying Lemma 6.2 we obtain

$$\begin{aligned} &\frac{1}{\operatorname{mes} \Sigma^{N-1}} \int_{\Sigma^{N-1}} \Big(\sum_{j=1}^{N} a_j \theta_j \Big) \log \Big(\sum_{j=1}^{N} a_j \theta_j \Big)\, d\omega(\theta) \\ &\qquad = (N-1) \int_0^1 x^{N-2} H(x)\, dx = B_1 + B_2 + B_3, \end{aligned}$$

where

$$\begin{aligned} B_1 &= -(N-1) \Big(\sum_{k=2}^{N-1} \frac{1}{k} \Big) a_N \int_0^1 x^{N-2}\, dx, \\ B_2 &= -\Big(\sum_{k=2}^{N-1} \frac{1}{k} \Big) \Big(\sum_{j=1}^{N-1} a_j - (N-1)a_N \Big) \int_0^1 x^{N-1}\, dx, \\ B_3 &= \sum_{j=1}^{N-1} \frac{\int_0^1 (a_N + (a_j - a_N)x)^{N-1} \log(a_N + (a_j - a_N)x)\, dx}{\prod_{\substack{k=1 \\ k\neq j}}^{N-1} (a_j - a_k)}. \end{aligned}$$

It is easy to check that

$$B_1 + B_2 = -\frac{1}{N}\Big(\sum_{k=2}^{N-1}\frac{1}{k}\Big)\Big(\sum_{j=1}^{N} a_j\Big).$$

Integrating by parts the integral in B_3, we obtain $B_3 = C_1 + C_2$, where

$$C_1 = -\frac{1}{N^2}\sum_{j=1}^{N-1}\frac{a_j^N - a_N^N}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)},$$

$$C_2 = \frac{1}{N}\sum_{j=1}^{N-1}\frac{a_j^N\log a_j - a_N^N\log a_N}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)}.$$

Applying (6.1) and (6.2) we get

$$C_1 = -\frac{1}{N^2}\Big(\sum_{j=1}^{N-1}\frac{a_j^N}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)} + \frac{a_N^N}{\prod_{k=1}^{N-1}(a_N - a_k)}\Big) = -\frac{1}{N^2}\sum_{j=1}^{N} a_j.$$

Therefore

$$B_1 + B_2 + C_1 = -\frac{1}{N}\Big(\sum_{k=2}^{N}\frac{1}{k}\Big)\Big(\sum_{j=1}^{N} a_j\Big).$$

By (6.1),

$$C_2 = \frac{1}{N}\Big(\sum_{j=1}^{N-1}\frac{a_j^N\log a_j}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)} - a_N^N\log a_N\sum_{j=1}^{N-1}\frac{1}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)}\Big)$$
$$= \frac{1}{N}\sum_{j=1}^{N}\frac{a_j^N\log a_j}{\prod_{\substack{k=1\\k\neq j}}^{N}(a_j - a_k)}.$$

Thus $B_1+B_2+C_1+C_2$ is equal to the right-hand side of (6.6), which proves the theorem.

Corollary 6.4 (see [10]). *If*

$$K := \{(z_1,\ldots,z_N)\in\mathbb{C}^N : |z_1|\le R_1,\ldots,|z_N|\le R_N\},$$

where $R_j > 0$ *for* $j = 1,\ldots,N$, *then*

$$D(K) = d(K) = \Big(\prod_{j=1}^{N} R_j\Big)^{1/N}.$$

Proof. It is easy to check, applying Lemma 5.2, that for $\theta \in \Sigma_0$

$$\log \tau(K, \theta) = \log \widetilde{\tau}(K, \theta) = \sum_{j=1}^{N} \theta_j \log R_j .$$

Applying Theorem 4.12 and (6.3) we obtain the desired conclusion.

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