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Some criteria for the injectivity of holomorphic mappings

by Stanisław Spodzieja (Łódź)

Abstract. We prove some criteria for the injectivity of holomorphic mappings.

Let $K \subset \mathbb{C}^n$ be a bounded and closed domain such that

(1) no closed proper subset of ∂K disconnects $\mathbb{C}^n$.

Theorem 1. *If a mapping* $f : K \to \mathbb{C}^n$ *is continuous*, *the restriction* $f_{|\operatorname{Int} K}$ *is holomorphic and* $f_{|\partial K}$ *is injective*, *then* f *is injective.*

Proof. The proof will be carried out in three steps:

1. $f_{|\operatorname{Int} K} : \operatorname{Int} K \to \mathbb{C}^n$ *is an open mapping.* By the assumption, for any $y \in \mathbb{C}^n$, $f^{-1}(y) \cap \partial K$ has at most one point. Consequently, from the Remmert–Stein theorem on removable singularities, f has isolated fibres. So, by Remmert's theorem on open mappings, $f_{|\operatorname{Int} K}$ is an open mapping.

2. $f(\partial K) \cap f(\operatorname{Int} K) = \emptyset$. It is known (see [1], Cor. in Sec. 12, p. 248) that if $A, B \subset \mathbb{R}^m$ are compact and homeomorphic, and A disconnects R^m, then so does B. Hence and from (1), $f(\partial K)$ disconnects $\mathbb{C}^n$, but no closed proper subset of $f(\partial K)$ does. Since $f(\operatorname{Int} K)$ is open, $f(\operatorname{Int} K) \subset \operatorname{Int} f(K)$. Consequently, $\partial f(K) \subset f(\partial K)$. Since $\partial f(K)$ disconnects $\mathbb{C}^n$, we get $\partial f(K) = f(\partial K)$, and so $f(\partial K) \cap f(\operatorname{Int} K) = \emptyset$.

3. f *is injective.* Let $V = \{(x, y) \in K \times K : f(x) = f(y)\}$. Then each irreducible component of $V \cap \operatorname{Int}(K \times K)$ has a positive dimension. Define

$$g_i : V \ni (x_1, \ldots, x_n, y_1, \ldots, y_n) \mapsto x_i - y_i \in \mathbb{C}, \quad i = 1, \ldots, n.$$

By the maximum principle for holomorphic functions on analytic sets, there exist $(x_0^i, y_0^i) \in \partial(K \times K) \cap V$, $i = 1, \ldots, n$, such that

$$|g_i(x_0^i, y_0^i)| = \max_{(x,y) \in V} |g_i(x, y)|, \quad i = 1, \ldots, n.$$

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From the definition of V we have $f(x_0^i) = f(y_0^i)$, thus, by step 2, $x_0^i, y_0^i \in \partial K$. Hence and from the injectivity of $f_{|\partial K}$ we have $x_0^i = y_0^i$, that is, $g_i(x_0^i, y_0^i) = 0$, and thus $g_i \equiv 0$ for $i = 1, \ldots, n$. Hence $V = \{(x,x) : x \in K\}$, therefore f is injective.

The proof is complete.

Remark. In the case $n = 1$, this theorem is known (see [2], §11, Ch. IV, p. 209).

We shall now give another criterion in which we weaken the assumption on the boundary of the domain at the cost of strengthening the assumption on the mapping.

Let $D \subset \mathbb{C}^n$ be a bounded and closed domain with connected boundary.

THEOREM 2. *If $f : D \to \mathbb{C}^n$ is a continuous mapping, $f_{|\operatorname{Int} D}$ is holomorphic, $f_{|\partial D}$ is injective, and*

(2) *each $x \in \partial D$ has a neighbourhood $U \subset \mathbb{C}$ such that $f_{|U \cap D}$ is injective,*

then f is injective.

Proof. The proof will be carried out in three steps:

1. $f_{|\operatorname{Int} D} : \operatorname{Int} D \to \mathbb{C}^n$ *is an open mapping.* This is proved in the same way as step 1 in the proof of Theorem 1.

2. $f(\partial D) \cap f(\operatorname{Int} D) = \emptyset$. Assume to the contrary that $f(\partial D) \cap f(\operatorname{Int} D) \neq \emptyset$. By step 1, $f(\partial D) \cap f(\operatorname{Int} D)$ is open in $f(\partial D)$. Take any sequence $y_n \in f(\partial D) \cap f(\operatorname{Int} D)$ such that $\lim y_n = y_0$. Then there exist sequences $z_n \in \partial D$, $x_n \in \operatorname{Int} D$ such that $f(z_n) = y_n$, $f(x_n) = y_n$. Passing to subsequences if necessary, we may assume that $\lim x_n = x_0$, $\lim z_n = z_0$. From (2) we have $z_0 \neq x_0$. So, from the injectivity of $f_{|\partial D}$ we get $z_0 \in \partial D$, $x_0 \in \operatorname{Int} D$. In consequence, $y_0 \in f(\partial D) \cap f(\operatorname{Int} D)$. Thus $f(\partial D) \cap f(\operatorname{Int} D)$ is closed in $f(\partial D)$, that is, by the connectedness of ∂D, $f(\partial D) = f(\partial D) \cap f(\operatorname{Int} D)$. To sum up, $f(D) = f(\operatorname{Int} D)$, which is impossible because $f(D)$ is compact and $f(\operatorname{Int} D)$ open.

3. f *is injective.* This is proved in the same way as step 3 in the proof of Theorem 1.

The proof is complete.

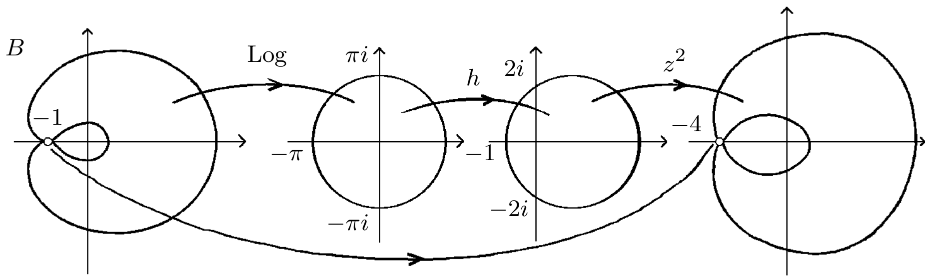
COROLLARY. *If $f : D \to \mathbb{C}^n$ is a holomorphic mapping such that $f_{|\partial D}$ is injective and the Jacobian of f does not vanish anywhere in D, then f is injective.*

We shall now give an example illustrating the fact that the assumptions (1) in Theorem 1 and (2) in Theorem 2 cannot be omitted.

EXAMPLE 1. Let $B = \exp(\{z \in \mathbb{C} : |z| < \pi\})$. Take a homography h such that $h(\pi i) = 2i$, $h(-\pi i) = -2i$, $h(-\pi) = -1$, and a function $f : B \to \mathbb{C}$

defined by

$$f(z) = \begin{cases} [h(\operatorname{Log} z)]^2, & z \in B \setminus \{-1\}, \\ -4, & z = -1. \end{cases}$$



Then f and B have the following properties:

1) f is injective on ∂B,
2) ∂B does not satisfy (1),
3) f does not satisfy (2) at the point -1,
4) f is not injective in B.

It is easy to show, using the Osgood–Brown theorem, that we need not assume the connectedness of the boundary of the domain in Theorem 2 for $n \geq 2$. In the case $n = 1$, this assumption is essential, which is shown by the following example.

EXAMPLE 2. Let $D = \{z \in \mathbb{C} : 1/5 \leq |z| \leq 4\}$ and $f : D \to \mathbb{C}$, $f(z) = z + 1/z$. It is easy to see that $f_{|\partial D}$ is injective. Since $f'(z) = 0$ for $z = 1$ and $z = -1$, condition (2) in Theorem 2 is also satisfied. But $f(3) = f(1/3)$, thus f is not injective.

References

[1] K. Borsuk, *Über Schnitte der n-dimensionalen Euklidischen Räume*, Math. Ann. 106 (1932), 239–248.

[2] S. Saks and A. Zygmund, *Analytic Functions*, PWN, Warszawa 1965.

INSTITUTE OF MATHEMATICS
ŁÓDŹ UNIVERSITY
BANACHA 22
90-238 ŁÓDŹ, POLAND

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