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A HELSON SET OF UNIQUENESS BUT NOT OF SYNTHESIS

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In [3] I showed that there are Helson sets on the circle $\mathbb{T}$ which are not of synthesis, by constructing a Helson set which was not of uniqueness and so automatically not of synthesis. In [2] Kaufman gave a substantially simpler construction of such a set; his construction is now standard. It is natural to ask whether there exist Helson sets which are of uniqueness but not of synthesis; this has circulated as an open question. The answer is "yes" and was also given in [3, pp. 87–92] but seems to have got lost in the depths of that rather long paper. Furthermore, the proof depends on the methods of [3], which few people would now wish to master. The object of this note is to give a proof using the methods of [2].

We begin by recalling two definitions.

DEFINITION 1. A closed set $F \subset \mathbb{T}$ is called *Helson* if, for every measure $\mu \in M(F)$, we have $\sup_{n\in\mathbb{Z}} |\widehat{\mu}(n)| = \|\mu\|$.

DEFINITION 2. A closed set $F \subset \mathbb{T}$ is called *Dirichlet* if

$$\liminf_{|n|\to\infty} \sup_{t\in F} |e^{int} - 1| = 0\,.$$

We are going to prove the following theorem.

THEOREM 1. *Any closed set of multiplicity in* $\mathbb{T}$ *contains a Helson* 1 *set which is of uniqueness but not of synthesis.*

(The set constructed is, in fact, weak Kronecker in the sense of [3].)

Theorem 1 follows at once from the next theorem.

THEOREM 2. *Any closed nowhere dense set of multiplicity in* $\mathbb{T}$ *contains a Helson* 1 *set which is Dirichlet but not of synthesis.*

Proof. We know that every Dirichlet set is of uniqueness (see e.g. [1, p. 97]). On the other hand, any closed set of multiplicity contains a nowhere dense set of multiplicity. ∎

The proof of Theorem 2 takes up the rest of this note.

Let F be a nowhere dense set of multiplicity. Then F supports a pseudo-function S, i.e., a distribution such that $\widehat{S}(n) \to 0$. Let $\chi_n(t) = e^{int}$. By considering $\lambda\chi_m S$ we may suppose that $\widehat{S}(0) = 1$ and $|\widehat{S}(n)| \le 1$ for every n. Also, we can find a sequence $f_1, f_2, \dots$ with $f_j : \mathbb{T} \to \mathbb{C}$ such that $|f_j(t)| = 1$ for all $t \in \mathbb{T}$ and such that given any $f : \mathbb{T} \to \mathbb{C}$ with $|f(t)| = 1$ for all $t \in \mathbb{T}$ and any $\epsilon > 0$ there is a j such that $|f_j(t) - f(t)| < \epsilon$ for all $t \in F$.

Our proof will mimic Kaufman's proof of the existence of Helson sets of multiplicity [2] but will have an extra twist. Let $S_0 = S$ and $F_0 = F$. We construct pseudofunctions $S_1, S_2, \dots$, closed sets $F_1, F_2, \dots$ and integers $M(0) = 1, M(1), \dots$, with $M(j) \ge M(j-1) + 1$, such that for all j,

(1) $\operatorname{supp} S_j \subset F_j$,

(2) $F_j \subset F_{j-1}$,

(3) $|\widehat{S}_j(r) - \widehat{S}_{j-1}(r)| \le 2^{-j-2}$ for all $|r| \le M(j-1)$,

(4) $|\widehat{S}_j(r)| \le 2 - 2^j$,

(5) there are $a_n(n), a_{n+1}(n), \dots, a_{M(2n)}(n) \ge 0$ such that

(a) $\displaystyle\sum_{k=n}^{M(2n)} a_k(n) = 1$,

(b) $\displaystyle\sup_{t \in F_{2n}} \Big| f_n(t) - \sum_{k=n}^{M(2n)} a_k(n)\chi_k(t) \Big| \le 2^{-n}$,

(c) if $a_k(n) \ne 0$ then $|\widehat{S}_{2n}(k)| \le 2^{-n-1}$,

(6) there exists an $L(n) > 1$ such that $\displaystyle\sup_{t \in F_{2n+1}} |1 - \chi_{L(n)}(t)| \le 2^{-n}$.

Let us see the consequences of these facts. By (3) and (4), S_j converges to a pseudomeasure T (in the sense that $\widehat{S}_j(r) \to \widehat{T}(r)$ for each r) with $\sup_{r \in \mathbb{Z}} |\widehat{T}(r) - \widehat{S}_j(r)| \le 2^{-1}$. Since $S_0 = S$ and $1 = \widehat{S}(0) \ge |\widehat{S}(r)|$ for all r, it follows that $|\widehat{T}(0)| \ge 2^{-1}$, so T is not zero.

By (1) and (2), $\operatorname{supp} T \subset F_j$ for each j. Thus by (5)(b)

$$\lim_{n\to\infty} \sup_{t \in \operatorname{supp} T} \Big| f_n(t) - \sum_{k=n}^{M(2n)} a_k(n)\chi_k(t) \Big| = 0$$

Hence by standard arguments (using the theorems of Radon–Nikodym and Lusin) $\operatorname{supp} T$ is Helson 1. We also note that (6) gives

$$\lim_{n\to\infty} \sup_{t \in \operatorname{supp} T} |1 - \chi_{L(n)}(t)| = 0$$

so, since $L(n)$ is never zero, $\operatorname{supp} T$ is Dirichlet.

To see that $\operatorname{supp} T$ is not of synthesis it is sufficient to show that T is not a measure. (If E is a Helson set which is of synthesis then $M(E) = C^*(E) = A^*(E) = PM(E)$ (see e.g. [1, p. 61]) and E cannot support a true pseudomeasure). We do this by using (3) together with (5)(b) and (c) to obtain

$$\text{(b}'\text{)} \qquad \sup_{t\in\operatorname{supp} T}\Big|f_n(t) - \sum_{k=n}^{M(2n)} a_k(n)\chi_k(t)\Big| \le 2^{-n}\,,$$

$$\text{(c}'\text{)} \qquad \text{if } a_k(n) \neq 0 \text{ then } |\widehat{T}(k)| \le 2^{-n}\,.$$

Thus, if $T = \mu$ is a measure we have from (5)(a) and (b′)

$$\Big|\int f_n\,d\mu - \sum_{k=n}^{M(2n)} a_k(n)\widehat{T}(k)\Big| \le 2^{-n}\|\mu\|,$$

so by (c′)

$$\Big|\int f_n\,d\mu\Big| \le 2^{-n} + 2^{-n}\|\mu\|.$$

Thus, if f is any continuous function $f : \mathbb{T} \to \mathbb{C}$ with $|f(t)| = 1$ for all $t \in \mathbb{T}$ we see, by considering a sequence $n(r) \to \infty$ with $\sup_{t\in F} |f_{n(r)}(t) - f(t)| \to 0$, that $\int f\,d\mu = 0$. Therefore $\mu = 0$, which is impossible since $|\widehat{\mu}(0)| \ge 2^{-1}$.

Our proof will thus be complete if we can show how to handle the inductive steps required to obtain conditions (1) to (6). We require two different constructions according as j is even (when we need to satisfy (1) to (5) but not (6)) or odd (when we need to satisfy (1) to (4) and (6) but not (5)). This will be done in the two lemmas that follow. ∎

If j is even we follow the standard Kaufman proof.

Lemma 1. *Suppose S is a pseudofunction, M a positive integer, $f : \mathbb{T} \to \mathbb{C}$ a continuous function with $|f(t)| = 1$ for all $t \in \mathbb{T}$ and $\epsilon \ge 0$. Then we can find a pseudofunction T such that*

(1′) $\operatorname{supp} T \subset \operatorname{supp} S$,

(3′) $|\widehat{S}(r) - \widehat{T}(r)| \le \epsilon$ *for all* r,

(4′) $|\widehat{T}(r)| \le \sup_{n\in\mathbb{Z}}|\widehat{S}(n)| + \epsilon$ *for all* r,

(5′) *there are* $a_M, a_{M+1}, \dots, a_P \ge 0$ *such that*

(a) $\displaystyle\sum_{k=M}^{P} a_k = 1$,

(b) $\displaystyle\sup_{t\in\operatorname{supp} T}\Big|f(t) - \sum_{k=M}^{P} a_k\chi_k(t)\Big| \le \epsilon$,

(c) *if* $a_k \neq 0$ *then* $|\widehat{T}(k)| \le \epsilon$.

Proof. Let $\delta \geq 0$ be a very small number, to be determined later. Then by Kaufman's fundamental construction we can find an $A(\delta)$ such that if $\eta > 0$ and N are chosen independently of δ we can find an $F : \mathbb{T} \to \mathbb{R}$ which is a C^∞ function, $\widehat{F}(0) = 1$, and

(A) $|\widehat{F}(r)| \leq \delta$ for all $|r| \geq 1$,

(B) if $r_1, r_2 \in \mathbb{Z}$ are such that $|r_1 - r_2| < 2N+2$ then $\min(|\widehat{F}(r_1)|, |\widehat{F}(r_2)|) \leq \eta$,

(C) $\sum_{r\in\mathbb{Z}} |\widehat{F}(r)| \leq A(\delta)$,

(D) there are $a_M, a_{M+1}, \ldots, a_P \geq 0$ such that

(a) $\sum_{k=M}^{P} a_k = 1$,

(b) $\sup_{t\in \operatorname{supp} F} \Big| f(t) - \sum_{k=M}^{P} a_k \chi_k(t) \Big| \leq \epsilon$.

Now let $T = FS$. Since $\widehat{F}(0) = 1$ we have, using (A), (B) and (C),

$$|\widehat{S}(r) - \widehat{T}(r)| = \Big| \sum_{k\neq 0} \widehat{F}(k)\widehat{S}(r-k) \Big| \leq \sum_{k\neq 0} |\widehat{S}(r-k)|\,|\widehat{F}(k)|$$
$$\leq \sum_{|n|\leq N} |\widehat{S}(n)|\eta + (\sup_{|n|\leq N} |\widehat{S}(n)|)\delta + (\sup_{|n|>N} |\widehat{S}(n)|)A(\delta)\,.$$

But $\widehat{S}(n) \to 0$ as $|n| \to \infty$, so, provided that N is large enough and η small enough,

$$|\widehat{S}(r) - \widehat{T}(r)| \leq 2\delta \sup_{n\in\mathbb{Z}} |\widehat{S}(n)| \quad \text{for all } r\,.$$

Hence, choosing δ sufficiently small we have

$$|\widehat{S}(r) - \widehat{T}(r)| < \epsilon \quad \text{for all } r\,.$$

Thus $(3')$ and $(4')$ are satisfied. Since $T = FS$, $\operatorname{supp} T \subset \operatorname{supp} S$ and $\operatorname{supp} T \subset \operatorname{supp} F$, so $(1')$ and $(5')$ are satisfied. Finally, since F is C^∞ and S is a pseudofunction it follows by simple estimates that $T = FS$ is a pseudofunction. ■

In the inductive step with $j = 2n$ we take $M = n$, $\epsilon = 2^{-n}$, $f = f_n$, $M(2n) = P$, $S = S_{j-1}$, $S_j = T$. Thus conditions (1) to (5) are satisfied.

If j is odd we use the following lemma.

Lemma 2. *Suppose S is a pseudofunction, M a positive integer and $1 > \epsilon > 0$. Then we can find a pseudofunction T such that*

$(1'')$ $\operatorname{supp} T \subset \operatorname{supp} S$, (3″) $|\widehat{S}(r) - \widehat{T}(r)| \le \epsilon$ *for all* $|r| < M$,

(4″) $|\widehat{T}(r)| \le \sup_{n\in\mathbb{Z}} |\widehat{S}(n)| + \epsilon$ *for all* r,

(5″) *there exists an* $L > 0$ *such that* $\sup_{t\in \operatorname{supp} T} |1 - \chi_L(t)| \le \epsilon$.

Proof. Let g be a C^∞ function such that $g(t) \ge 0$ for all $t \in \mathbb{T}, g(t) = 0$ for $|t| \ge \epsilon/10$ and $\widehat{g}(0) = 1$. Let $F_L(t) = g(Lt)$ and $T = F_L S$. If $t \in \operatorname{supp} T$ then $t \in \operatorname{supp} F_L$, so that $|1-\chi_L(t)| \le \epsilon$. Thus (1″) and (5″) are automatic. Now

$$\widehat{T}(r) = \sum_k \widehat{F}_L(k)\widehat{S}(r-k) = \sum_k \widehat{g}(k)\widehat{S}(r-kL)\,,$$

$$\sum_k |\widehat{g}(k)| < \infty, \qquad \lim_{|n|\to\infty} \widehat{S}(n) = 0, \quad \widehat{g}(0) = 1\,.$$

Thus, provided L is large enough, we have $|\widehat{S}(r) - \widehat{T}(r)| \le \epsilon$ for $|r| < M$ and in general $|\widehat{T}(r)| \le \sup_{n\in\mathbb{Z}} |\widehat{S}(n)| + \epsilon$. Thus (3″) and (4″) are satisfied. Finally, we observe that $\widehat{T}(r) \to 0$ as $|r| \to \infty$, so T is a pseudofunction. ∎

In the inductive step with $j = 2n+1$ we take $\epsilon = 2^{-n}$, $S = S_{j-1}$, $S_j = T$. Thus conditions (1) to (4) and (6) are satisfied. This completes the proof of Theorem 2 and concludes the note.

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