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HOLOMORPHIC LIPSCHITZ FUNCTIONS AND APPLICATION TO THE $\overline{\partial}$-PROBLEM

BY

DER-CHEN E. CHANG (COLLEGE PARK, MARYLAND)
AND STEVEN G. KRANTZ (ST. LOUIS, MISSOURI)

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1. Introduction. In his fundamental paper [22], Stein announced that a Lipschitz α holomorphic function on a C^2 domain in $\mathbb{C}^n, n > 1$, is actually Lipschitz 2α in complex tangential directions—the extra smoothness comes for free. Details of the proof appear, for instance, in Krantz [12].

It is apparent (see Krantz [12]) that Stein's result is optimal only in the strongly pseudoconvex case. Near boundary points where the Levi form degenerates, one expects even greater tangential smoothness. And near strongly pseudoconcave points the Hartogs extension phenomenon tells us that *any* holomorphic function will continue analytically past the boundary, and hence be in *every* Lipschitz class. In the paper [16] Krantz uses Kobayashi metric language to find a general version of Stein's theorem which contains all the aforementioned phenomena as special cases.

In the present paper we use results of Catlin [4] and Nagel–Stein–Wainger [19] to work out what the theorem of Krantz [16] says in the case of finite type domains in $\mathbb{C}^2$. In view of recent development concerning estimates for the $\overline{\partial}$-problem on such domains (see Fefferman–Kohn [8], Christ [6], Nagel–Rosay–Stein–Wainger [20], Chang [5], Belanger [2], Range [21], it is important to have detailed information about the relevant function spaces. This paper is a first contribution to the theory.

Section 2 contains basic definitions and recalls the result of Krantz [16]. It also contains the statements of our main results. Section 3 contains a few technical facts about the non-isotropic geometry of domains in $\mathbb{C}^2$. Section 4 contains the proofs of the main results. Section 5 has some applications to the regularity properties for the solutions of the $\overline{\partial}$-equations and some concluding remarks.

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2. Definitions, terminology, basic properties of finite type domains in $\mathbb{C}^2$ and statement of theorems. If $\Omega \subset \mathbb{R}^n$ is a bounded, connected open set (a *domain*), define

$$\mathrm{Lip}_\alpha(\Omega) = \{f \text{ continuous on } \Omega : \sup_{x,x+h\in\Omega,h\neq 0} |f(x+h)-f(x)|/|h|^\alpha + \|f\|_{L^\infty} \equiv \|f\|_{\mathrm{Lip}_\alpha} < \infty\}$$

for $0<\alpha<1$ and

$$\mathrm{Lip}_1(\Omega) = \{f \text{ continuous on } \Omega : \sup_{x,x+h,x-h\in\Omega,h\neq 0} |f(x+h)+f(x-h)-2f(x)|/|h|^1 + \|f\|_{L^\infty} \equiv \|f\|_{\mathrm{Lip}_1} < \infty\}\,;$$

for $\alpha > 1$ we say that $f \in \mathrm{Lip}_\alpha(\Omega)$ if $f \in C^1(\Omega), f \in \mathrm{Lip}_{\alpha-1}(\Omega)$, and $\nabla f \in \mathrm{Lip}_{\alpha-1}(\Omega)$ (with an obvious norm). See Krantz [13] for detailed discussion and motivation concerning these spaces. This reference also contains a thorough discussion of the finite difference operator Δ_h^k, whose definition we now recall:

If f is a function on a domain $\Omega \subset \mathbb{R}^n$, $x \in \Omega$, and h is sufficiently small then

$$\Delta_h^1 f(x) \equiv f(x+h) - f(x-h)\,,$$
$$\Delta_h^k f(x) \equiv \Delta_h^1(\Delta_h^{k-1} f)(x), \qquad k \geq 2\,.$$

For a continuous function f on a domain $\Omega \subset \mathbb{R}^n$ with C^2 boundary and $0<\alpha<1$, $f \in \mathrm{Lip}_\alpha(\Omega)$ if and only if

$$\sup_\Omega |f(x)| + \sup_{x,h} |\Delta_h^k f(x)|/|h|^\alpha \leq C\,.$$

(Of course the latter supremum is taken over x, h such that $\Delta_h^k f(x)$ is well defined on Ω.) Again see Krantz [13] for a proof that this finite difference characterization of Lip_α is equivalent to the original definition.

Now fix a C^2 bounded domain $\Omega \subset \mathbb{C}^n$, i.e., assume that there is a real valued C^2 function r on $\mathbb{C}^n$ such that $\nabla r \neq 0$ on $\partial\Omega$ and $\Omega = \{z \in \mathbb{C}^n : r(z) < 0\}$. Let $B \subset \mathbb{C}^n$ be the unit ball and let (Ω, B) denote the collection of all holomorphic maps from B to Ω. For $z \in \Omega$ we define the *Eisenman–Kobayashi volume form* (see Krantz [12]) to be

$$M_K^\Omega(z) = \inf\{1/|\det \mathrm{Jac}_{\mathbb{C}}\, \Phi(0)| : \Phi \in (\Omega, B),\ \Phi(0) = z\}\,.$$

Once Ω is fixed and $z \in \Omega$ chosen there is (by a normal families argument) a function $\Phi_z \in (\overline{\Omega}, B)$ such that

$$M_K^{\Omega}(z,\xi) = 1/|\det \mathrm{Jac}_{\mathbb{C}}\, \Phi_z(0)|\,.$$

The function Φ_z is not necessarily unique. Nevertheless, another normal families argument shows that the maps

$$z \to \|\Phi_z(\zeta)\| \quad \text{and} \quad z \to \|(\partial/\partial z)^{\beta}\Phi_z(0)\|\,,$$

for any multi-index β, can be taken to be upper semicontinuous. (Here $\|\cdot\|$ is the standard Euclidean norm.) By the Cauchy estimates, they are bounded on compact sets. In particular, we may compose these functions with curves in B and integrate.

Let $D \subset \mathbb{C}$ be the unit disc. If $z \in \Omega$ and $\xi \in \mathbb{C}^n$ then define the *infinitesimal Kobayashi metric* to be

$$F_K^{\Omega}(z,\xi) = \inf\{|\xi|/\|\varphi'(0)\| : \varphi \in (\Omega, B)\,,\ \varphi(0) = z\,,\\ \varphi'(0) \text{ is a multiple of } \xi\}\,.$$

Notice that the formulas for $M_K^{\Omega}(z)$, $F_K^{\Omega}(z,\xi)$ are analogous, but $F_K^{\Omega}(z,\xi)$ treats one direction at z at a time while $M_K^{\Omega}(z)$ treats all directions simultaneously. Because in many examples it is very difficult to compare the two quantities, we will mandate their comparability by replacing $M_K^{\Omega}(z)$ with

$$\widetilde{M}_K^{\Omega}(z) = \inf\{1/|\det \mathrm{Jac}_{\mathbb{C}}\, \widetilde{\Phi}(0)| : \widetilde{\Phi} \in (\Omega, B),\ \widetilde{\Phi}(0) = z\,,\\ |(\mathrm{Jac}_{\mathbb{C}}\, \widetilde{\Phi}^{-1}(z))(v(z))| \le 2F_K^{\Omega}(z, v(z))\}\,,$$

where $v(z)$ is the unit outward normal which will be defined in a moment. In short, we calculate $\widetilde{M}_K^{\Omega}(z)$ by restricting attention to $\widetilde{\Phi}$ with the property that $|\,\mathrm{Jac}_{\mathbb{C}}\, \widetilde{\Phi}^{-1}|$ is, in the normal direction, comparable to $F_K^{\Omega}(z, v(z))$. Given a domain Ω, we assume that a semicontinuous assignment $z \to \widetilde{\Phi}_z$ of optimal $\widetilde{\Phi}$'s has been selected once and for all.

Now we describe our new Lipschitz classes. Fix a bounded domain $\Omega \subset \mathbb{C}^n$ with C^2 boundary and associated functions $\widetilde{\Phi}_z$. For $z \in \Omega$, let $\delta(z)$ denote the Euclidean distance of z to $\partial\Omega$. Choose $\varepsilon = \varepsilon(\Omega) > 0$ such that $U = \{z \in \mathbb{C}^n : \delta(z) < 2\varepsilon\}$ is a tubular neighborhood of $\partial\Omega$. Let $v : \overline{\Omega} \to \mathbb{C}^n$ be a C^1 function which satisfies the condition: For points $z \in \Omega \cap \{z \in \mathbb{C}^n : \delta(z) < 2\varepsilon\}$, $v(z)$ is the (well-defined) outward unit normal at z.

Define $\mathcal{C}^k(\Omega) = \mathcal{C}^k(\Omega, C_0)$ to be the class of all C^∞ curves

$$\gamma : [0,1] \to \Omega \cap \{z \in \mathbb{C}^n : \delta(z) < 2\varepsilon\}$$

such that

$$|d\gamma(t)/dt| \le 1, \quad \forall t \in [0,1]\,,$$

and

$$|(d/dt)^j\gamma(t)| \le C_0, \qquad \forall t \in [0,1],\ 2 \le j \le k\,.$$

Here C_0 is a positive constant, fixed in advance.

We now attach a number $\beta_0(\gamma)$, its *smoothness index*, to each $\gamma \in \mathcal{C}^k(\Omega)$. We will see in what follows that $\beta_0(\gamma) \ge 1$ always. Generically, a holomorphic Lipschitz α function on Ω will turn out to be Lipschitz of order $\beta_0(\gamma)\cdot\alpha$ along a curve γ. The Lipschitz norm will of course depend on the constant C_0 in the definition of $\mathcal{C}^k(\Omega, C_0)$.

The following notation will simplify formulas in the sequel: once a domain Ω, a semicontinuous assignment of functions $\widetilde{\Phi}_z$, and a curve $\gamma \in \mathcal{C}^k(\Omega)$ are fixed, we set

$$D_N\widetilde{\Phi}_{\gamma(t)} \equiv \frac{1}{\|[\mathrm{Jac}_{\mathbb{C}}\,\widetilde{\Phi}^{-1}_{\gamma(t)}]v(\gamma(t))\|}\,,$$

$$D_T\widetilde{\Phi}_{\gamma(t)} \equiv \frac{1}{\left\|[\mathrm{Jac}_{\mathbb{C}}\,\widetilde{\Phi}_{\gamma(t)}]\left(\dfrac{d\gamma(t)}{dt}\right)\right\|}\,.$$

Note that when γ is complex normal with $|d\gamma(t)/dt| = 1$ then of course $D_N\widetilde{\Phi}_{\gamma(t)} = D_T\widetilde{\Phi}_{\gamma(t)}$.

For any curve $\gamma \in \mathcal{C}^k$ define

$$\lambda_0(\gamma) = \sup_{t\in[0,h_0]} \log_{\delta(\gamma(t))} |D_N\widetilde{\Phi}_{\gamma(t)}|\,.$$

Elementary estimates (see Krantz [16]) show that $1/2 \le \lambda_0(\gamma) \le 1$. For a given curve γ and $0 < \zeta$ small we define

$$\gamma_\zeta(t) = \gamma(t) - \zeta v(\gamma(t))\,.$$

When we study the Lipschitz smoothness of a function f along a curve $\gamma \in \mathcal{C}^k$, we need only consider $\Delta_h^k(f\circ\gamma)$ for h small. Thus we restrict attention to $0 < h < \varepsilon = \varepsilon(\Omega)$. For $\gamma \in \mathcal{C}^k(\Omega)$ and h fixed, define

$$\sigma_\beta(t) = \log_{[D_N\widetilde{\Phi}_{\gamma_{h^\beta}(t)}]}[D_T\widetilde{\Phi}_{\gamma_{h^\beta}(t)}]\,,$$

$$\varphi(\beta) = \frac{1}{\beta} - 1 - \lambda_0(\gamma)\left[\frac{1}{h}\int_0^h \sigma_\beta(t)dt - 1\right], \qquad 1 \le \beta < \infty\,.$$

Elementary estimates on the Kobayashi metric (see [16]) show that $0 < \sigma_\beta(t) \le 1,\ \forall t \in [0, h_0]$. It follows that $\varphi(1) \ge 0$ and, for β large enough, $\varphi(\beta) < 0$. We select

$$\beta_0(\gamma) = \inf\{\beta \in [1,\infty) : \varphi(\beta) < 0\}\,.$$

When no confusion is possible we write $\lambda_0 = \lambda_0(\gamma),\ \beta_0 = \beta_0(\gamma)$.

DEFINITION 2.1. Let $0 < \alpha < \infty$. A continuous function $f : \Omega \to \mathbb{C}$ is said to be in the space $\mathcal{L}_\alpha(\Omega)$ if

(i) For any $\gamma \in \mathcal{C}^k(\Omega)$, $t \in (0, h_0)$, h sufficiently small, and k sufficiently large,

$$|\Delta_h^k(f \circ \gamma(t))| \le C|h|^{\beta_0\lambda_0(\alpha-1)+\beta_0}.$$

(Here $k > \beta_0\lambda_0(\alpha-1)+\beta_0$, $|h| < \min\{|t|, |h_0 - t|\}/(2k)$ will do.)

(ii) If $\gamma : [0,1] \to \Omega$ satisfies $|d\gamma(t)/dt| \equiv 1, |(d/dt)^j\gamma(t)| \le C_0, 2 \le j \le k$, $\forall t \in [0, h_0]$, then

$$|\Delta_h^k(f \circ \gamma(t))| \le C|h|^\alpha.$$

Krantz's principal result in [16] is:

THEOREM 2.2. *If $\Omega \subset \mathbb{C}^n$ is a bounded domain with C^2 boundary and $f : \Omega \to \mathbb{C}$ is holomorphic and in $\mathrm{Lip}_\alpha(\Omega)$ then $f \in \mathcal{L}_\alpha(\Omega)$.*

Now we turn to domains of finite type in $\mathbb{C}^2$. We begin by recalling some notions connected with the complex structure in $\partial\Omega$. If $\Omega = \{(z_1, z_2) \in \mathbb{C}^2 : r(z) < 0\}$, $P \in \partial\Omega$, U is a small neighborhood of P, and $\partial r/\partial z_2 \ne 0$ on U then we define the vector fields

$$Z_1 = \frac{\partial r}{\partial z_2}\frac{\partial}{\partial z_1} - \frac{\partial r}{\partial z_1}\frac{\partial}{\partial z_2} \quad \text{and} \quad Z_2 = 2\left(\frac{\partial r}{\partial \overline{z}_1}\frac{\partial}{\partial z_1} + \frac{\partial r}{\partial \overline{z}_2}\frac{\partial}{\partial z_2}\right).$$

Set $Z_j = X_j + iY_j$. It is easy to see that X_1, Y_1, Y_2 span the three dimensional real tangent space $T_P(\partial\Omega)$ at each point $P \in \overline{U} \cap \partial\Omega$. The vectors $\{X_1, Y_1\}$, equivalently $\{Z_1, \overline{Z}_1\}$, span the maximal complex subspace $T_P^{\mathbb{C}}(\partial\Omega) \subset T_P(\partial\Omega)$ at each $P \in U \cap \partial\Omega$. Following convention, we set $Y_2 = T$, $X_2 = N$.

For each $P \in \partial\Omega$, we define the *Levi form*

$$\lambda(P) = \langle \partial r, [Z_1, \overline{Z}_1]\rangle(P),$$

where $[Z_1, \overline{Z}_1] = Z_1\overline{Z}_1 - \overline{Z}_1 Z_1$ denotes the Lie bracket.

Let $\mathfrak{L}_1$ be the module spanned by Z_1 and $\overline{Z}_1$ over the C^∞ functions, and for $k \ge 2$ let $\mathfrak{L}_k$ be the module spanned by elements of $\mathfrak{L}_{k-1}$ and elements of the form $[X, Z_1]$, $[X, \overline{Z}_1]$ with $X \in \mathfrak{L}_{k-1}$.

DEFINITION 2.3. A point $P \in \partial\Omega$ is said to be *of type* κ ($\kappa \ge 2$) if

$$\langle \partial r(P), X(P)\rangle = 0, \quad \forall X \in \mathfrak{L}_{k-1},$$

while

$$\langle \partial r(P), X(P)\rangle \ne 0, \quad \text{for some } X \in \mathfrak{L}_k.$$

Remark. In some references our points of type κ are called points of type $\kappa - 1$.

DEFINITION 2.4. If $\Omega \Subset \mathbb{C}^2$ is a domain and $P \in \partial\Omega$ is of type κ, then we say that $\partial\Omega$ is *of type* κ *at* P. We say that Ω is *of type* κ if there exists at least one point in $\partial\Omega$ of type κ and all other points in $\partial\Omega$ are of type not greater than κ.

Remarks. (1) It can be proved (see Kohn [9]) that the type of a given point P must be an even integer if the boundary of Ω is pseudoconvex near P.

(2) If $P \in \partial\Omega$ is a strongly pseudoconvex point, then P is of type 2. Among pseudoconvex points, the converse is true as well.

For convenience, let $L_1 = Z_1, L_2 = \overline{Z}_1$. Now, for $z \in \partial\Omega$ near P and $i_1, \ldots, i_j = 1$ or 2, define $\lambda_{i_1,\ldots,i_j}(P)$ by the equation

$$[L_{i_j}, [\ldots[L_{i_2}, L_{i_1}]\ldots]]_P = \lambda_{i_1,\ldots,i_j}(P) T_P \mod T^{\mathbb{C}}(U)\,.$$

We define

$$\Lambda_j(P) = \sum |\lambda_{i_1,\ldots,i_j}(P)|\,,$$

where $j \geq 2$ and the sum ranges over all $i_1, \ldots, i_k$ with $k \leq j$ and $i_1, \ldots, i_k = 1$ or 2. Note than when $j = 2$, the function $\lambda_{12}(P)$ is the usual Levi form of $\partial\Omega$ at P.

Remark. Here we give another definition for finite type. A point $P \in \partial\Omega$ has type κ if for every choice of vector fields $Z_1, \overline{Z}_1$ and T, all commutators of Z_1 and $\overline{Z}_1$ at P of length less than κ have zero T component, yet there is some commutator among Z_1 and $\overline{Z}_1$ at P of length κ which has a non-zero T component. This definition was first given by Kohn [9] and is equivalent to Definition 2.3. We note (see Kohn [9], Bloom–Graham [3]) that in $\mathbb{C}^2$ this definition of type is equivalent to a definition in terms of order of contact of one-dimensional non-singular complex varieties. The situation in $\mathbb{C}^n$ is much more complicated, and there is no such simple description of points of finite type (see D'Angelo [7]).

Following the results of Nagel–Stein–Wainger [19], [20], we now define the "higher Levi invariant" $\Lambda_{\partial\Omega}(z,\delta)$ for $z \in \partial\Omega$ near P and $\delta > 0$ by

$$\Lambda_{\partial\Omega}(z,\delta) = \sum_{j=2}^{\kappa} \Lambda_j(z)\delta^j\,, \tag{2.5}$$

where κ is the maximum type of any point on $\partial\Omega$.

We also define another version of the "higher Levi invariant" $\mu_{\partial\Omega}(z,\delta)$ for $z \in \partial\Omega$ near P and $\delta > 0$ by

$$\mu_{\partial\Omega}(z,\delta) \approx \min_{2\leq j\leq\kappa} (\delta/\Lambda_j(z))^{1/j}\,. \tag{2.6}$$

Remark. The functions $\Lambda_{\partial\Omega}$, $\mu_{\partial\Omega}$ depend on the choice of vector fields $Z_1, \overline{Z}_1$ and T. A different choice of r and hence of $Z_1, \overline{Z}_1$ and T will result in new functions $\widetilde{\Lambda}_{\partial\Omega}$, $\widetilde{\mu}_{\partial\Omega}$. However, elementary considerations show that the ratios $\widetilde{\Lambda}_{\partial\Omega}/\Lambda_{\partial\Omega}$ and $\widetilde{\mu}_{\partial\Omega}/\mu_{\partial\Omega}$ will be bounded and bounded away from zero near P. Thus our results will be independent of the choices that have been made. Although we will generally work locally near a fixed point $P \in \partial\Omega$, we can, if necessary, patch the locally defined functions together to obtain global functions $\Lambda_{\partial\Omega}(z,\delta)$ and $\mu_{\partial\Omega}(z,\delta)$ defined for all $z \in \partial\Omega$ and $\delta > 0$.

The finite type hypothesis implies that for every compact set K in $\partial\Omega$ there are constants C_1 and C_2 so that for $z \in K$ and $0 \le \delta \le 1$

$$C_1\delta^\kappa \le \Lambda_{\partial\Omega}(z,\delta) \le C_2\delta^2\,, \qquad C_1\sqrt{\delta} \le \mu_{\partial\Omega}(z,\delta) \le C_2\sqrt[\kappa]{\delta}\,.$$

From this remark we see that at points of type 2, $\Lambda_{\partial\Omega}(z,\delta) \approx \delta^2$ for small δ while at a point of maximum type $\kappa, \Lambda_{\partial\Omega}(z,\delta) \approx \delta^\kappa$ for small δ. For larger δ, the quantity $\Lambda_{\partial\Omega}(z,\delta)$ provides a transition between points of type 2 and points of higher type. On the other hand, if we consider the function $\mu_{\partial\Omega}(z,\delta)$, then at points of type 2, $\mu_{\partial\Omega}(z,\delta) \approx \sqrt{\delta}$ for small δ. And at points of type κ, $\mu_{\partial\Omega}(z,\delta) \approx \sqrt[\kappa]{\delta}$ for small δ. This explains why, in the proof of Stein's theorem, we can embed a polydisc which has "size" δ in the normal direction and "size" $\sqrt{\delta}$ in the tangential directions. We can also see that in the example $|z_1|^2 + |z_2|^{2\kappa} < 1$, along the curve of points of type 2κ on $\partial\Omega$ (i.e., the equator $(e^{i\theta},0)$, $\theta \in [0,2\pi]$), we can embed a polydisc which has "size" δ in the normal direction but has "size" $\sqrt[2\kappa]{\delta}$ in the tangential direction.

Observe that the vector fields X_1, Y_1, T are well defined in $\overline{U} \cap \Omega$. We define the subspace $\mathcal{C}_1^k(\Omega) = \mathcal{C}_1^k(\Omega, C_0) = \mathcal{C}_1^k(\Omega \cap \overline{U}, C_0) \subset \mathcal{C}^k(\Omega, C_0)$ by the condition

$$d\gamma(t)/dt \in \mathrm{Span}_{\mathbb{R}}\{X_1(\gamma(t)), Y_1(\gamma(t)), T(\gamma(t))\}\,, \qquad 0 \le t \le 1\,.$$

(Curves supported away from $\partial\Omega$ are of no interest so we content ourselves with defining $\mathcal{C}_1^k$ only on $\overline{U} \cap \Omega$.)

Now we recall the definition of a certain non-isotropic metric on $\partial\Omega$ and metric space constructs which are associated to it. First, let $\widetilde{\mathcal{C}}_1^k$ denote continuous curves which are composed of the union of finitely many elements of $\mathcal{C}_1^k$ (that is, "piecewise" $\mathcal{C}_1^k$ curves).

Definition 2.7. (i) For $P, Q \in \partial\Omega$ set

$$\begin{aligned}\widetilde{\varrho}(P,Q) = \inf\{\delta > 0 : \exists\gamma \in \widetilde{\mathcal{C}}_1^k(\Omega) \text{ with } \gamma(0) = P,\ \gamma(1) = Q\,, \text{ and}\\ \frac{d\gamma(t)}{dt} = a(t)X_1 + b(t)Y_1 + c(t)T \text{ where}\\ |a(t)| < \delta\,,\ |b(t)| < \delta,\ |c(t)| < \Lambda_{\partial\Omega}(P,\delta)\}\,.\end{aligned}$$

(ii) For $P \in \partial\Omega$ and $\delta > 0$ set

$$\begin{aligned}\widetilde{B}_1(P,\delta) = \{Q \in \partial\Omega : Q = \exp(aX_1 + bY_1 + cT)(P)\\ \text{with } |a| < \delta\,,\ |b| < \delta,\ |c| < \Lambda_{\partial\Omega}(P,d)\}\,,\end{aligned}$$

$$\widetilde{B}(P,\delta) = \{Q \in \partial\Omega : \widetilde{\varrho}(P,Q) < \delta\}\,.$$

(i′) For $P, Q \in \partial\Omega$ set

$$\begin{aligned}\varrho(P,Q) = \inf\{\delta > 0 : \exists\gamma \in \widetilde{\mathcal{C}}_1^k(\Omega) \text{ with } \gamma(0) = P,\ \gamma(1) = Q \text{ and}\\ \frac{d\gamma(t)}{dt} = a(t)X_1 + b(t)Y_1 + c(t)T \text{ where}\\ |a(t)| < \mu_{\partial\Omega}(P,\delta),\ |b(t)| < \mu_{\partial\Omega}(P,\delta),\ |c(t)| < \delta\}\,.\end{aligned}$$

(ii″) For $P \in \partial\Omega$ and $\delta > 0$ set

$$\begin{aligned}B_1(P,\delta) = \{Q \in \partial\Omega : Q = \exp(aX_1 + bY_1 + cT)(P) \text{ where}\\ |a| < \mu_{\partial\Omega}(P,\delta), |b| < \mu_{\partial\Omega}(P,\delta), |c| < \delta\}\,,\end{aligned}$$

$$B(P,\delta) = \{Q \in \partial\Omega : \varrho(P,Q) < \delta\}\,.$$

We have the following facts about the functions $\widetilde{\varrho}, \varrho$ and $\Lambda_{\partial\Omega}$ and $\mu_{\partial\Omega}$ and about the families of "balls" $\{\widetilde{B}(P,\delta)\}$ and $\{B(P,\delta)\}$:

(2.8) There are positive constants C_1, C_2 so that for all $P \in \partial\Omega$ and all $\delta > 0$

$$\widetilde{B}_1(P, C_1\delta) \subset \widetilde{B}(P,\delta) \subset \widetilde{B}_1(P, C_2\delta)\,,$$
$$B_1(P, C_1\delta) \subset B(P,\delta) \subset B_1(P, C_2\delta)\,.$$

We shall also need to consider certain non-isotropic subsets of the domain Ω. First recall that a *projection* $\pi : U \to \partial\Omega$ is a smooth mapping such that for every $P \in \partial\Omega$, $\pi(P) = P$ and $\pi^{-1}(P)$ is a smooth curve in U which intersects $\partial\Omega$ transversely at P. Projections always exist if $\partial\Omega$ is C^2 and the open set U is sufficiently small. For a projection π there are positive c_1 and c_2 such that if $z \in U$ then

$$c_1|r(z)| \le |\pi(z) - z| \le c_2|r(z)|\,.$$

If π_1 and π_2 are two projections there is a positive constant c such that for $z \in U$

$$|\pi_1(z) - \pi_2(z)| \le c|r(z)|\,.$$

Now fix a projection $\pi : U \to \partial\Omega$. We make the following

Definition 2.9. For $z, w \in \overline{\Omega} \cap U$ set

$$\varrho(z,w) = \inf\{\tau > 0 : \varrho(\pi(z), \pi(w)) < \tau\,,\ |r(z)| \le \tau,\ |r(w)| < \tau\}\,.$$

Note that if $z, w \in \partial\Omega$ then this definition of ϱ agrees with our earlier Definition 2.8. Now our theorem is

Theorem 2.10. *Let $\Omega \subset \mathbb{C}^2$ be a domain with $C^{\kappa+1}$ boundary. If Ω is pseudoconvex and of finite type κ and $f : \Omega \to \mathbb{C}$ is holomorphic and in* $\mathrm{Lip}_\alpha(\Omega)$ *with $0 < \alpha < 1/\kappa$, then*

$$|f(\gamma)(0) - f(\gamma)(h)| \le C\Lambda^{\alpha}_{\partial\Omega}(\pi(\gamma(0)), \widetilde{\varrho}(\gamma(0), \gamma(h)))\,,$$

$$\forall \gamma \in \mathcal{C}_1^\kappa(\Omega, C_0)\,,\ \forall h \in [0, h_0/(2\kappa)]\,.$$

We can also prove a dual result:

THEOREM 2.11. *Let $\Omega \subset \mathbb{C}^2$ be a domain with $C^{\kappa+1}$ boundary. Assume that Ω is pseudoconvex and of finite type κ and $f : \Omega \to \mathbb{C}$ is holomorphic and satisfies*

$$|f(z) - f(w)| \le C\varrho^\alpha(z, w)$$

for some $\alpha \in (0,1)$, and all $z, w \in \overline{\Omega} \cap U$. Then

$$|f(z) - f(w)| \le C\mu^\alpha_{\partial\Omega}(\pi(z)\,, \|z - w\|)\,.$$

Remark. It is a straightforward exercise, using the metric calculations of Catlin [4] (see also Aladro [1]), to see that the spaces described in Theorems 2.10, 2.11 are precisely the spaces $\mathcal{L}_\alpha$ of Krantz [16]. (More will be said about this in Section 4, where the theorems are proved.) The philosophy of [16] is to do function theory in an abstract metric setting; in specific situations one calculates the relevant metric to arrive at an integration of the theorems. In the present paper we use the important machinery of Nagel–Stein–Wainger [19] and Nagel–Rosay–Stein–Wainger [20] to describe the $\mathcal{L}_\alpha$ for domains of finite type in $\mathbb{C}^2$. We restrict ourselves to considering small α for the sake of both brevity and clarity. The treatment of $\alpha \ge 1/\kappa$ would entail the development of a rather elaborate calculus, which we defer to another time.

3. Some technical facts and notation. We note here some technical information, all of which is derived from [20]. We refer to [20] for further details.

Set $B^\#(P,\delta) = \{z \in \overline{\Omega} : \pi(z) \in B(P,\delta),\ |r(z)| \le \delta\}$. Observe that

$$B^\#(P,\delta) \approx \{z \in \overline{\Omega} \cap U : \varrho(z,P) < \delta\}, \qquad \mathrm{Vol}(B^\#(P,\delta)) \approx \delta^2\mu^2_{\partial\Omega}(P,\delta)\,.$$

Next, there are positive constants $\delta_0, \varepsilon_0, c_1, c_2$ such that for $P \in \partial\Omega$ there is a biholomorphic $H_P : \mathbb{C}^2 \to \mathbb{C}^2$ with $H_P(0) = P$ and $H_P(\{z : |z| < \varepsilon_0\}) \subseteq U$ such that H_P is the composition of a translation operator T_P, a unitary operator U_P and the mapping

$$\mathcal{P}_P(\varrho_1, \varrho_2) = \Big(\varrho_1, \varrho_2 + \sum_{j=2}^{\kappa} d_j(P)\varrho_1^j\Big)\,,$$

where $d_j : \partial\Omega \to \mathbb{C}$, $j = 2, \ldots, \kappa$, are smooth. In particular, we have $|\det \mathrm{Jac}_{\mathbb{C}}\, H_P(P)| = 1$.

For each $P \in \partial\Omega$ there is a smooth $h^P : \mathbb{C} \times \mathbb{R} \to \mathbb{R}$ such that

(3.1) $\{z \in \mathbb{C}^2 : |z| < \varepsilon_0 \text{ and } H_P(z) \in \Omega\} = \{z \in \mathbb{C}^2 : |z| < \varepsilon_0 \text{ and } \Im z_2 > h^P(z_1, \Re z_2)\}$.

(3.2) $h^P(0,0) = 0$, $\nabla h^P(0,0) = 0$, $(\partial^j h^P/\partial z_1^j)(0,0) = (\partial^j h^P/\partial \bar{z}_1^j)(0,0) = 0$ for $2 \le j \le \kappa$.

(3.3) The set of functions $\{h^P\}_{P\in\partial\Omega}$ is a bounded subset of the space $C^\infty(\{(z,t) : |z| < 2\varepsilon_0\,,\ |t| < 2\varepsilon_0\})$.

Now h^P has the following properties: For every $P \in \partial\Omega$ and $2 \le j \le \kappa$

$$c_1\Lambda_j(P) \le \sum_{\alpha+\beta\le j}\left|\frac{\partial^{\alpha+\beta}h^P(0,0)}{\partial z_1^\alpha \partial \bar{z}_1^\beta}\right| \le c_2\Lambda_j(P)\,.$$

For every $P \in \partial\Omega$ and all $\delta \ge 0$

$$c_1\mu_{\partial\Omega}(P,\delta) \le \min_{2\le m\le\kappa}\left(\left\{\sum_{\alpha+\beta\le m}\left|\frac{\partial^{\alpha+\beta}h^P(0,0)}{\partial z_1^\alpha \partial \bar{z}_1^\beta}\right|\right\}^{-1}\delta\right)^{1/m} \le c_2\mu_{\partial\Omega}(P,\delta)\,.$$

For every $P \in \partial\Omega$ and all δ with $0 \le \delta \le \delta_0$

$$B(P,c_1\delta) \subset H_P(\{(z_1,t+ih^P(z_1,t)) : |z_1| < \mu_{\partial\Omega}(P,\delta), |t| < \delta\}) \subset B(P,c_2\delta)\,.$$

Once a projection $\pi : U \to \partial\Omega$ is fixed we have: There are constants δ_0, c_1, and c_2 so that for every $P \in \partial\Omega$ and all δ with $0 < \delta \le \delta_0$

(3.4) $$B^\#(P,c_1\delta) \subset H_P(\{(z_1,t+iy+ih^P(z_1,t)) : y \ge 0,\ |z_1| < \mu_{\partial\Omega}(P,\delta),\ |t+iy| < \delta\}) \subset B(P,c_2\delta)\,.$$

If we normalize π so that $\pi(z) = P$ if and only if $(H_P)^{-1}(z) = (0,it)$ for some $t \in \mathbb{R}$, then we have: There are positive constants δ_0, c_1, and c_2 with the following properties: let $z, w \in \overline{\Omega} \cap U$ with $\varrho(z,w) \le \delta_0$ and assume $|r(z)| \le |r(w)|$. Write $P = \pi(z)$ and suppose

(3.5) $$(H_P)^{-1}(z) = (0,it)\,, \qquad (H_P)^{-1}(w) = (w_1, s+iy+ih^P(w_1,s))$$

with $t, y \ge 0, s \in \mathbb{R}, w_1 \in \mathbb{C}$. Then

(3.6) $$c_1 t \le |r(z)| \le c_2 t\,;$$

(3.7) $$c_1 y \le |r(w)| \le c_2 y\,;$$

(3.8) $$c_1\Lambda_{\partial\Omega}(P,\varrho(z,w)) \le |s+iy| + \Lambda_{\partial\Omega}(P,|w_1|) \le c_2\Lambda_{\partial\Omega}(P,\varrho(z,w))\,.$$

If we define

$$B_P^\#(\delta) = \{(z_1,t+iy+ih^P(z_1,t)) : y \ge 0, |z_1| < \mu_{\partial\Omega}(P,\delta), |t+iy| < \delta\}$$

then, by (3.4) and (3.5), $(H_P)^{-1}(B^\#(P,\delta)) \approx B_P^\#(\delta)$.

The biholomorphic map H_P allows us to pull back the vector fields $Z_1, \overline{Z}_1, Z_2$, and $\overline{Z}_2$ defined on U to vector fields $Z_1^P, \overline{Z}_1^P, Z_2^P$, and $\overline{Z}_2^P$ defined on $\{z \in \mathbb{C}^2 : |z| < \varepsilon_0\}$ by the formulas

$$Z_1(f\circ H_P)(z) = (Z_1 f)(H_P(z)), \qquad Z_2^P(f\circ H_P)(z) = (Z_2 f)(H_P(z)), \quad \text{etc.}$$

4. Proofs of the theorems. In Stein's work, the key fact (for small α) is that, for a holomorphic Lipschitz α function f,

$$|\nabla f(z)| \leq C\delta(z)^{\alpha-1},$$

while for a tangential derivative D_τ

$$|D_\tau f(z)| \leq C\delta(z)^{\alpha-1/2}.$$

For us it is more convenient to formulate these matters differently.

LEMMA 4.1. *If f is a holomorphic Lipschitz α function on Ω and $(\partial/\partial\zeta)^\beta$ is any differential monomial then*

$$|(\partial/\partial\zeta)^\beta(f \circ \widetilde{\Phi}_z)(0)| \leq C|D_N\widetilde{\Phi}_z(0)|^\alpha.$$

Remark. It is a bit surprising that the estimate on the right is independent of β. But examination of the chain rule on the left alleviates the surprise.

Proof. This is contained in Krantz [16].

Now we turn to the proof of Theorem 2.10. Before we prove the theorem, we first state a result of Nagel–Rosay–Stein–Wainger [20] which we need to use in our proof:

LEMMA 4.2. *Suppose Ω is a smooth, pseudoconvex, finite type domain in $\mathbb{C}^2$ of type κ. There are constants C_1 and C_2 depending only on κ such that if $\widetilde{\varrho}(z,w) < \delta$, then*

$$C_1 \leq \Lambda_{\partial\Omega}(z,\delta)/\Lambda_{\partial\Omega}(w,\delta) \leq C_2,$$

where $\widetilde{\varrho}(z,w)$ denotes the non-isotropic distance between z and w defined in Definition 2.7.

Proof of Theorem 2.10. We can use the same techniques as used in the proof of Theorem 2.2 (see Krantz [16]). But this time we need to plug in the non-isotropic distance function $\Lambda_{\partial\Omega}$. Fix a holomorphic function f in $\mathrm{Lip}_\alpha(\Omega)$ with $0 < \alpha < 1/\kappa$. Here κ is the type of Ω. Suppose $\gamma \in \widetilde{\mathcal{C}}_1^{\kappa+1}(\Omega)$ is a curve satisfying the property in Definition 2.7(i), with domain $[0,h]$, $0 < h < \varepsilon = \varepsilon(\Omega)$. Since $0 < \alpha < \kappa^{-1}$ is small, we just need to estimate $|\Delta_h f(\gamma(0))|$. We define three auxiliary curves as follows: For $0 \leq t \leq h$,

$$\begin{aligned}
\gamma_1(t) &= \gamma(0) - \Lambda_{\partial\Omega}(\pi(\gamma(0)), t)v(\gamma(0)),\\
\gamma_2(t) &= \gamma(h) - \Lambda_{\partial\Omega}(\pi(\gamma(h)), t)v(\gamma(h)),\\
\gamma_3(t) &= \gamma(t) - \Lambda_{\partial\Omega}(\pi(\gamma(t)), h)v(\gamma(t)).
\end{aligned}$$

Then, as usual,

$$\begin{aligned}|\Delta_h f(\gamma(0))| &= |f(\gamma(h)) - f(\gamma(0))| \\ &\le |f(\gamma_1(h)) - f(\gamma_1(0))| + |f(\gamma_2(h)) - f(\gamma_2(0))| + |\Delta_h f(\gamma_3(0))| \\ &\equiv \mathrm{I} + \mathrm{II} + \mathrm{III}\,.\end{aligned}$$

Terms I and II are just the estimation in the normal direction, hence we may apply directly the property that $f \in \mathrm{Lip}_\alpha(\Omega)$:

$$\begin{aligned}|f(\gamma_1(h)) - f(\gamma_1(0))| &\le C\Lambda^\alpha_{\partial\Omega}(\pi(\gamma_1(0)), h)\,; \\ |f(\gamma_2(h)) - f(\gamma_2(0))| &\le C\Lambda^\alpha_{\partial\Omega}(\pi(\gamma_2(0)), h)\,.\end{aligned}$$

By Lemma 4.2, we know that

$$C_1 \le \Lambda_{\partial\Omega}(\pi(\gamma(0)), h)/\Lambda_{\partial\Omega}(\pi(\gamma(h)), h) \le C_2\,.$$

Hence our last estimate is

$$\le \widetilde{C}\Lambda^\alpha_{\partial\Omega}(\pi(\gamma(0)), h)\,,$$

where $\widetilde{C} = C\max\{C_1, C_2\}$.

To get an estimate for III we write

$$\Delta_h f(\gamma_3(0)) = \int_0^h (f(\gamma_3))'(h-s)\,ds \tag{4.3}$$

and estimate the integrand pointwise by

$$\begin{aligned}|(f(\gamma_3))'(t)| &= |(\nabla f \cdot \nabla\widetilde{\Phi}_{\gamma_3}(t) \cdot (\nabla\widetilde{\Phi}_{\gamma_3(t)})^{-1} \cdot \gamma_3')(t)| \\ &= |(f\widetilde{\Phi}_{\gamma_3(t)})\,((\nabla\widetilde{\Phi}_{\gamma_3(t)})^{-1} \cdot \gamma_3')(t)| = C|D_N\widetilde{\Phi}_{\gamma_3(t)}|^\alpha |D_T\widetilde{\Phi}_{\gamma_3(t)}|^{-1}\,.\end{aligned}$$

By the remark after Definition 2.4, the problem reduces to knowing the size of the largest polydisc with center $\gamma_3(t)$ which can be embedded in Ω. In our case, it is easy to see that (see also Nagel–Stein–Wainger [19])

$$\begin{aligned}|D_N\widetilde{\Phi}_{\gamma_3(t)}| &\approx C'\Lambda_{\partial\Omega}(\pi(\gamma_3(t)), h) \approx C^*\Lambda_{\partial\Omega}(\pi(\gamma_3(0)), h) \\ &\approx C^*\Lambda_{\partial\Omega}(\pi(\gamma(0)), \widetilde{\varrho}(\gamma(0), \gamma(h)))\,,\end{aligned}$$

and

$$|D_T\widetilde{\Phi}_{\gamma_3(t)}| \approx C''h \approx C^*\widetilde{\varrho}(\gamma(0), \gamma(h))\,.$$

Plugging all this information into (4.3), we obtain Theorem 2.10.

Remark. As noted before, it is the shape of the polydisc that can be imbedded in the domain near a given boundary point that controls the geometry *and* the function theory. By the calculations in [1] and [4], this polydisc is comparable to a Kobayashi metric ball. These observations are the key to passing back and forth between the language of [16] and the language of the present paper.

In order to prove Theorem 2.11, we need to apply the biholomorphic mapping H_P defined in Nagel–Rosay–Stein–Wainger [20] and another version of "higher Levi invariant" $\mu_{\partial\Omega}(P,\delta)$.

Proof of Theorem 2.11. Let us consider a small boundary neighborhood $V \subset \overline{\Omega} \cap U$. We shrink V if necessary so that x_1, x_2, x_3, r is a coordinate system on a neighborhood W of V with $r(Q)$ being the "height" of a point Q in W, i.e., $r(Q) > 0$ if $Q \in V$, $r(Q) = 0$ if $Q \in V \cap \partial\Omega$ and $r(Q) < 0$ if $Q \in W \setminus V$. Consider $z, w \in V$ and $\varrho(z,w) < \delta$. We also set $\delta = c\|z-w\|$ where c is a fixed small constant. Since $\varrho(z,w) < \delta$, there are two different cases: (i) $\|\pi(z) - \pi(w)\| \approx \mu_{\partial\Omega}(\pi(z),\delta) \gg \delta$, and (ii) $\|\pi(z) - \pi(w)\| \le \delta$. To handle case (i) is easy:

$$\begin{aligned}|f(z) - f(w)| &\le |f(z) - f(\pi(z))| + |f(\pi(z)) - f(\pi(w))| + |f(\pi(w)) - f(w)|\\ &= \mathrm{I} + \mathrm{II} + \mathrm{III}\,.\end{aligned}$$

Since $\varrho(z,w) < \delta$, we therefore have $\varrho(z,\pi(z)) \approx |r(z)| \le \delta \approx \|z-w\|$ and $\varrho(w,\pi(w)) \approx |r(w)| \le \delta \approx \|z-w\|$. Then

$$\begin{aligned}\mathrm{I} &= |f(z) - f(\pi(z))| \le C\widetilde{\varrho}^{\alpha}(z,\pi(z)) \le C\|z-w\|^{\alpha}\\ &\le C\mu^{\alpha}_{\partial\Omega}(\pi(z),\|z-w\|)\,,\\ \mathrm{III} &= |f(w) - f(\pi(w))| \le C\widetilde{\varrho}^{\alpha}(w,\pi(w)) \le C\|z-w\|^{\alpha}\\ &\le C\mu^{\alpha}_{\partial\Omega}(\pi(w),\|z-w\|) \approx C\mu^{\alpha}_{\partial\Omega}(\pi(z),\|z-w\|)\,,\\ \mathrm{II} &= |f(\pi(z)) - f(\pi(w))| \le C\widetilde{\varrho}^{\alpha}(\pi(z),\pi(w)) \approx C\mu^{\alpha}_{\partial\Omega}(\pi(z),\delta)\,.\end{aligned}$$

To handle case (ii), we need to use the biholomorphic map defined in Section 3. Let $H_{\pi(z)}$ be the biholomorphic map in (3.5). Then we have

$$(H_{\pi(z)})^{-1}(z) = (0,it), \qquad (H_{\pi(z)})^{-1}(w) = (w_1, s + iy + ih^{\pi(z)}(w_1,s))\,.$$

Then

$$\begin{aligned}|f(z) - f(w)| &= |(f\circ H_{\pi(z)})\circ(H_{\pi(z)})^{-1}(z) - (f\circ H_{\pi(z)})\circ(H_{\pi(z)})^{-1}(w)|\\ &= |(f\circ H_{\pi(z)})(0,it) - (f\circ H_{\pi(z)})(w_1, s+iy+ih^{\pi(z)}(w_1,s))|\\ &\le |(f\circ H_{\pi(z)})(0,it) - (f\circ H_{\pi(z)})(0,iy)|\\ &\quad + |(f\circ H_{\pi(z)})(0,iy)\\ &\qquad - (f\circ H_{\pi(z)})(0, 0+iy+ih^{\pi(z)}(w_1,s))|\\ &\quad + |(f\circ H_{\pi(z)})(0, s+iy+ih^{\pi(z)}(w_1,s)\\ &\qquad - (f\circ H_{\pi(z)})(0, 0+iy+ih^{\pi(z)}(w_1,s))|\\ &\quad + |(f\circ H_{\pi(z)})(0, s+iy+ih^{\pi(z)}(w_1,s)\\ &\qquad - (f\circ H_{\pi(z)})(w_1, s+iy+ih^{\pi(z)}(w_1,s))|\\ &\equiv \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\,.\end{aligned}$$

Using the fact that $H_{\pi(z)}(0,it) = z \in V \cap B^{\#}(\pi(z),\delta)$, $H_{\pi(z)}(w_1, s+iy+ih^{\pi(z)}(w_1,s)) = w \in V \cap B^{\#}(\pi(z),\delta)$ and the relation

$$B^{\#}_{\pi(z)}(\delta) \approx H^{-1}_{\pi(z)}(B^{\#}(\pi(z),\delta))\,,$$

it is easy to see that $H_{\pi(z)}(0, 0+iy)$, $H_{\pi(z)}(0, 0+iy+ih^{\pi(z)}(w_1,s))$, and $H_{\pi(z)}0, s+iy+ih^{\pi(z)}(w_1,s))$ are points in $V \cap B^{\#}(\pi(z),\delta)$. Now we estimate the terms I to IV.

To estimate I, we just need to use the fact that if f is holomorphic in Ω, then

$$\begin{aligned}
\mathrm{I} &= |(f\circ H_{\pi(z)})(0,it) - (f\circ H_{\pi(z)})(0,iy)| \\
&\le \max_V \left|\frac{\partial}{\partial\zeta_2}(f\circ H_{\pi(z)})\right| |t-y| \le C|c'|r(z)| - c''|r(w)|\,| \\
&\le C^*|r(z)-r(w)| \le C^*|r(z)-r(w)|^{\alpha} \\
&\le c^*\mu^{\alpha}_{\partial\Omega}(\pi(z), |r(z)-r(w)|)\,.
\end{aligned}$$

Here we have applied the properties (3.6) with $t \approx c'|r(z)|$ and (3.7) with $y \approx c''|r(w)|$. The last inequality holds for $|r(z)|$, $|r(w)|$ small and $0<\alpha<1$. The constant C^* depends on the function f but is independent of $H_{\pi(z)}$. For the second term we write

$$\begin{aligned}
\mathrm{II} &= |(f\circ H_{\pi(z)})(0,iy) - (f\circ H_{\pi(z)})(0, iy+ih^{\pi(z)}(w_1,s))| \\
&= |(f\circ H_{\pi(z)})(0,iy+ih^{\pi(z)}(0,0)) - (f\circ H_{\pi(z)})(0, iy+ih^{\pi(z)}(w_1,s))| \\
&\le \max_V \left|\frac{\partial}{\partial\zeta_2}(f\circ H_{\pi(z)})\right| \max_V |\nabla h^{\pi(z)}|\, \|(w_1,s)-(0,0)\| \\
&\le C^*\mu_{\partial\Omega}(\pi(z),\delta)\,.
\end{aligned}$$

Here we have applied property (3.3) together with the fact that $\{h^P\}_{P\in\partial\Omega}$ is a bounded subset of the space $C^\infty(\{(z,t) : |z| < 2\varepsilon_0,\ |t| < 2\varepsilon_0\})$. The last inequality holds because $z, w \in V \cap B^{\#}(\pi(z),\delta)$ and $(0,iy)$, $(w_1, s+iy+ih^{\pi(z)}(w_1,s)) \in B^{\#}_{\pi(z)}(\delta)$.

To estimate III and IV, we need to use the fact f is Lipschitz in the non-isotropic sense. Thus

$$\begin{aligned}
\mathrm{III} &= |(f\circ H_{\pi(z)})(0, s+iy+ih^{\pi(z)}(w_1,s)) \\
&\qquad - (f\circ H_{\pi(z)})(0, 0+iy+ih^{\pi(z)}(w_1,s))| \\
&\le C\varrho^{\alpha}(H_{\pi(z)}(0, s+iy+ih^{\pi(z)}(w_1,s))\,, \\
&\qquad\qquad H_{\pi(z)}(0, 0+iy+ih^{\pi(z)}(w_1,s)))\,, \\
\mathrm{IV} &= |(f\circ H_{\pi(z)})(0, s+iy+ih^{\pi(z)}(w_1,s)) \\
&\qquad - (f\circ H_{\pi(z)})(w_1, s+iy+ih^{\pi(z)}(w_1,s))| \\
&\le C\varrho^{\alpha}(H_{\pi(z)}(0, s+iy+ih^{\pi(z)}(w_1,s))\,,
\end{aligned}$$

$$H_{\pi(z)}(w_1, s + iy + ih^{\pi(z)}(w_1, s)))\,.$$

Once again we apply the fact that $H_{\pi(z)}(B^{\#}_{\pi(z)}(\delta)) \approx B^{\#}(\pi(z), \delta)$, so that

$$\begin{aligned} \varrho^\alpha(H_{\pi(z)}(0, s + iy + ih^{\pi(z)}(w_1, s)), H_{\pi(z)}(0, 0 + iy + ih^{\pi(z)}(w_1, s))) \\ \approx \delta^\alpha \le \mu^\alpha_{\partial\Omega}(\pi(z), \delta)\,, \\ \varrho^\alpha(H_{\pi(z)}(0, s + iy + ih^{\pi(z)}(w_1, s)), H_{\pi(z)}(w_1, s + iy + ih^{\pi(z)}(w_1, s))) \\ \approx \mu^\alpha_{\partial\Omega}(\pi(z), \delta)\,. \end{aligned}$$

Combining the estimates of I to IV, we have

$$\mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} \le C^*(\mu^\alpha_{\partial\Omega}(\pi(z), |r(z) - r(w)|) + \mu^\alpha_{\partial\Omega}(\pi(z), \delta))\,. \tag{4.4}$$

Now we set $\delta = c\|\pi(z) - \pi(w)\|$ to obtain

$$|f(z) - f(w)| \le C\mu^\alpha_{\partial\Omega}(\pi(z), \|z - w\|)\,;$$

the constant C depends on the function f but it can be absorbed into the Lipschitz norm of f. This completes the proof.

5. Application to the regularity properties for the solutions of the Cauchy–Riemann equation. In this section we apply the techniques of non-isotropic geometry and the results in Section 2 to look at the Hölder estimates for the solutions of the $\overline{\partial}$-equation on finite type domains. Theorem 5.1 below and its corollary give estimates on the non-isotropic Lipschitz smoothness of both the Henkin and Kohn solution of $\overline{\partial}u = f$ when f has coefficients which are L^p, p sufficiently large. Examples (see below) show that the estimates are sharp.

In [5], the first-named author studied the Henkin solutions $\mathbb{H}$ and the Kohn solution $\overline{\partial}^*\mathbb{N} = \mathbb{H} - \mathbb{P}_0\mathbb{H}$ for the Cauchy–Riemann equations on the domain

$$H_\kappa = \Big\{(z', z_{n+1}) \in \mathbb{C}^{n+1} : \Im z_{n+1} > \Big(\sum_{1\le j\le n} |z_j|^2\Big)^\kappa\Big\}\,, \qquad \kappa \in \mathbb{N}\,.$$

These domains, while more complex than strongly pseudoconvex domains, have the property that they are no more difficult to study for $n > 1$ than for $n = 1$. In particular, the non-isotropic geometry is the same in all directions. Thus, while Theorems 2.11 and 2.12 have been proved in full generality only on domains in $\mathbb{C}^2$, their proofs apply without change to the domains H_κ. We will apply these observations systematically as we proceed.

We computed in [5] that the kernel for the Henkin solution $\mathbb{H}$ has the form $\sum_{k,\ell\ge 0} E_k F_\ell$, where E_k is a homogeneous kernel of degree $-k$ in the Euclidean sense, i.e.,

$$\begin{aligned} E_k(\lambda(z', w', t - s; \varrho + \mu)) &= E_k(\lambda z', \lambda w', \lambda(s - t); \lambda(\varrho + \mu)) \\ &= \lambda^{-k} E_k(z', w', s - t; \varrho + \mu)\,, \end{aligned}$$

and the kernel F_ℓ is a homogeneous kernel of degree $-\ell$ in the "finite type" sense, i.e.,

$$\begin{aligned}F_\ell(\lambda(z'w', t-s;\varrho+\mu)) &= F_\ell(\lambda z', \lambda w', \lambda^{2\kappa}(s-t); \lambda^{2\kappa}(\varrho+\mu)) \\ &= \lambda^{-\ell} F_\ell(z', w', s-t; \varrho+\mu)\,.\end{aligned}$$

To simplify the computations, we just assume $n = 1$ and $\kappa = 2$. Since $\mathbb{H}$ is not a convolution operator, we now need to go through many tedious calculations to look at the Hölder estimates for $\mathbb{H}$ on the domains H_κ.

Once again, we need to deal with the non-isotropic geometry on H_κ. Let $z = (\pi(z), \upsilon) = (z', t; \upsilon)$, $w = (\pi(w); \mu) = (w', s; \mu) \in \overline{H}_\kappa$ with $\pi(z) = (z', t)$, $\pi(w) = (w', s) \in \partial H_\kappa$. Here $\upsilon = \Im z_{n+1} - (\sum_{1\le j\le n} |z_j|^2)^\kappa$ and $\mu = \Im w_{n+1} - (\sum_{1\le j\le n} |w_j|^2)^\kappa$ are the "height functions" defined on $\overline{H}_\kappa$. Following the results in Chang [5], we define the "quasi-metric" $\widetilde{\varrho}(\pi(z), \pi(w))$ on the boundary ∂H_κ as follows: $\widetilde{\varrho}(\pi(z), \pi(w)) < \delta$ if and only if

$$\begin{aligned}&\left|\frac{1}{2}(t-s) - \Im\Big[\kappa\Big(\sum_{m=1}^n |w_m|^2\Big)^{\kappa-1}\Big(\sum_{j=1}^n \overline{w}_j z_j\Big)\Big]\right| \\ &\qquad\qquad < C\Big\{\Big(\sum_{m=1}^n |w_m|^2\Big)^{\kappa-1}\delta^2 + \delta^{2\kappa}\Big\}\end{aligned}$$

and $(\sum_{j=1}^n |z_j - w_j|^2)^{1/2} < \delta$.

Now the main result of this section is

THEOREM 5.1. *For $\kappa \in \mathbb{N}$, let $H_\kappa = \{(z', z_{n+1}) \in \mathbb{C}^{n+1} : \Im z_{n+1} > (\sum_{1\le j\le n} |z_j|^2)^\kappa\}$. Let U be a boundary neighborhood of any type 2κ points. If f is a $\overline{\partial}$-closed $(0,1)$ form on H_κ with $L^p(\overline{U})$ coefficients, then the Henkin solutions $\mathbb{H}(f)$ for the $\overline{\partial}$-equation satisfy*

$$\begin{aligned}&|\mathbb{H}(f)(\gamma(0)) - \mathbb{H}(f)(\gamma(h))| \\ &\quad\le \begin{cases} C\|f\|_{L^p(U)}(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-(2n+4\kappa)/p} & \text{if } 2n+4\kappa < p < \infty, \\ C\|f\|_{L^\infty(U)}\widetilde{\varrho}(\gamma(0),\gamma(h))\log(\widetilde{\varrho}(\gamma(0),\gamma(h))) & \text{if } p = \infty,\end{cases}\end{aligned}$$

where $\gamma \in \widetilde{\mathcal{C}}_1^\kappa(\overline{U})$.

Since the Bergman projection preserves the non-isotropic Lipschitz spaces (see Chang [5] for the case H_κ and Nagel–Rosay–Stein–Wainger [20] for finite type domains in $\mathbb{C}^2$), we can get the following corollary immediately:

COROLLARY. *With the same hypotheses as in Theorem 5.1, the Kohn solution $\overline{\partial}^*\mathbb{N} = \mathbb{H} - \mathbb{P}_0\mathbb{H}$ for the $\overline{\partial}$-equation satisfies*

$$|\overline{\partial}^*\mathbb{N}(f)(\gamma(0)) - \overline{\partial}^*\mathbb{N}(f)(\gamma(h))|$$

$$\leq \begin{cases} C\|f\|_{L^p(U)}(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-(2n+4\kappa)/p} & \text{if } 2n+4\kappa<p<\infty, \\ C\|f\|_{L^\infty(U)}\widetilde{\varrho}(\gamma(0),\gamma(h))\log(\widetilde{\varrho}(\gamma(0),\gamma(h))) & \text{if } p=\infty. \end{cases}$$

Remarks. (1) When we restrict our attention to the case $n = 1$, then H_κ is a finite type domain in $\mathbb{C}^2$. From the results of Christ [6], Fefferman–Kohn [8], and Nagel–Rosay–Stein–Wainger [20], we know that the Kohn solution for the Cauchy–Riemann equation maps $L^\infty(U)$ into the intersection of the standard Lipschitz space $\mathrm{Lip}_{1/(2\kappa)}(U)$ (see [8]) and the non-isotropic Lipschitz space $\Gamma_1(U)$ (see [20]). From Theorem 2.10, it is easy to see the Henkin solutions $\mathbb{H}$ satisfy

$$|\mathbb{H}(f)(\gamma(0))-\mathbb{H}(f)(\gamma(h))| \leq C\Lambda^1_{\partial\Omega}(\pi(\gamma(0)),\widetilde{\varrho}(\gamma(0),\gamma(h)))\,, \qquad \forall \gamma\in\widetilde{\mathcal{C}}^\kappa_1(\overline{U}),\ \forall h\in[0,h_0/(2\kappa)]\,,$$

for $f\in L^\infty(\overline{U})$.

(2) Theorem 5.1 is not only true for the case $n = 1$ but also true for general n, which in some sense is more general than the results in [6], [8], and [20]. Also our theorem not only deals with the case $p = \infty$ but also with $2n + 4\kappa < p < \infty$.

(3) Theorem 5.1 deals with the non-isotropic Lipschitz spaces which describe the correct geometric phenomena on these domains H_κ (hence on the domains $E_\kappa = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^{2\kappa} + |z_2|^2 < 1\}$, $\kappa \in \mathbb{N}$). Therefore in some sense our result is more general than the results of Belanger [2] and Range [21].

(4) It should be noted that the paper [17] gives methods for constructing examples to show that estimates for the $\overline{\partial}$ problem which are presented here are sharp.

Proof of the Theorem. As we discussed in Sections 2 and 3, we consider the problem only on the tubular neighborhood $\{z \in \mathbb{C}^2 : \delta(z) < 2\varepsilon\} \cap \overline{H}_\kappa$. Suppose that U is a small boundary neighborhood of $0 \in \partial H_\kappa$ and that $U \subset \{z \in \mathbb{C}^2 : \delta(z) < 2\varepsilon\} \cap \overline{H}_\kappa$. Now, given $f \in C^\infty_{(0,1)}(U)$ which satisfies $\overline{\partial} f = 0$ and $\gamma \in \widetilde{\mathcal{C}}^\kappa_1(U)$, we need to study

$$\mathbb{H}(f)(\gamma(h))-\mathbb{H}(f)(\gamma(0)), \qquad \forall h\in[0,h_0/4]\,. \tag{5.2}$$

As we have seen in [5], the crucial term for $\mathbb{H} = (K_1, K_2)$ + elliptic term is E_1F_8. Let $\mathrm{Op}(\mathbb{H})$ be the standard integral operator defined by the kernel $\mathbb{H}$. To simplify notation, set

$$\psi_{a,b} = \psi(\gamma(a)-bv_{\gamma(a)},w)\,.$$

We are just considering the "dominated" kernel as follows:

$$\mathrm{Op}(\mathbb{H})(f)(\gamma(h))-\mathrm{Op}(\mathbb{H})(f)(\gamma(0))$$

$$
\begin{aligned}
&= \int_U \Bigg\{ \frac{1}{\overline{\varphi}^2(\gamma(h),w)\psi(\gamma(h),w)} \\
&\quad - \frac{1}{\overline{\varphi}^2(\gamma(h) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(h)},w)\psi_{h,h}} \Bigg\} f(w)\,dV(w) \\
&\quad + \Bigg\{ \int_U \frac{f(w)\,dV}{\overline{\varphi}^2(\gamma(h) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(h)},w)\psi_{h,h}} \\
&\quad - \int_U \frac{f(w)\,dV}{\overline{\varphi}^2(\gamma(0) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(h)},w)\psi_{h,h}} \Bigg\} \\
&\quad + \int_U \frac{f(w)\,dV}{\overline{\varphi}^2(\gamma(0) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(0)},w)} \\
&\qquad\qquad \times [\psi(\gamma(h) - hv_{\gamma(0)},w) - \psi_{0,h}] \\
&\quad + \int_U \Bigg\{ \frac{f(w)}{\overline{\varphi}^2(\gamma(0) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(0)},w)\psi_{0,h}} \\
&\qquad\qquad - \frac{f(w)}{\overline{\varphi}^2(\gamma(0),w)\psi(\gamma(0),w)} \Bigg\} dV \\
&\equiv I_1 + I_2 + I_3 + I_4 \,.
\end{aligned}
$$

Here v_z is the unit outward normal to $\partial\Omega$ at z as defined in Section 2 and $dV = g dw_1\, d\overline{w}_1\, ds\, d\mu$ with $g \in C^\infty(\overline{U})$ the Euclidean volume form on $\overline{U}$. We have

$$
\begin{aligned}
|I_1| &= \Bigg| \int_0^h \frac{d}{d\theta} \Bigg\{ \int_U \frac{f(w)dV(w)}{\overline{\varphi}^2(\gamma(h) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),\theta)v_{\gamma(h)},w)\psi_{h,\theta}} \Bigg\} d\theta \Bigg| \\
&\le C \int_0^h \int_U \frac{\Lambda'_{\partial\Omega}(\pi(\gamma(0)),\theta)|f(w)|\,dV(w)\,d\theta}{|\overline{\varphi}^3(\gamma(h) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),\theta)v_{\gamma(h)},w)|\psi_{h,\theta}} \\
&\quad + C \int_0^h \int_U \frac{|f(w)|\,dV(w)\,d\theta}{|\overline{\varphi}^2(\gamma(h) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),\theta)v_{\gamma(h)},w)|\psi^2_{h,\theta}} \\
&\le C\|f\|_{L^p(U)} \\
&\quad \times \int_0^h \Bigg\{ \int_U \frac{\Lambda'_{\partial\Omega}(\pi(\gamma(0),\theta))^{p'}\,dV(w)}{|\overline{\varphi}^3(\gamma(h) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),\theta)v_{\gamma(h)},w)|\psi^{p'}_{h,\theta}} \Bigg\}^{1/p'} d\theta \\
&\quad + C\|f\|_{L^p(U)} \\
&\quad \times \int_0^h \Bigg\{ \int_U \frac{dV(w)}{|\overline{\varphi}^2(\gamma(h) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),\theta)v_{\gamma(h)},w)|\psi^{2p'}_{h,\theta}} \Bigg\}^{1/p'} d\theta
\end{aligned}
$$

$$\equiv I_{11} + I_{12}\,,$$

where $p' = p/(p-1)$ with $2 + 4\kappa = 10 < p \le \infty$. Thus estimating I_1 amounts to estimating I_{11} and I_{12}, which we do in a moment. It is also easy to see that the estimates for the term I_4 are just the same as those for I_1.

On the other hand, the term I_2 is dominated by

$$(5.3)\qquad |I_2| = \bigg| \int_0^h \frac{d}{d\theta}\bigg\{ \int_U \frac{f(w)dV(w)}{|\overline{\varphi}^2(\gamma(\theta) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(\theta)},w)|\psi_{h,h}} \bigg\}\, d\theta \bigg|\,.$$

Now the curve $\theta \to \gamma(\theta) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(\theta)}$ is in $\widetilde{\mathcal{C}}_1^\kappa(U)$. We use this, together with the fact that a complex tangential derivative of φ is not greater than $C|z-w|$, to majorize (5.2) by

$$C \int_0^h \int_U \frac{\{\Lambda_{\partial\Omega}(\pi(\gamma(0)),h) + |\gamma(\theta) - w|\}|f(w)|\,dV(w)}{|\overline{\varphi}^3(\gamma(\theta) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(\theta)},w)|\psi_{h,h}}\, d\theta\,.$$

Since $\Lambda_{\partial\Omega}(\pi(\gamma(0)),h) \approx C|(\pi(\gamma(0)))_1|^2h^2 + Ch^4 \le Ch$, it is easy to see that

$$\Lambda_{\partial\Omega}(\pi(\gamma(0)),h) + |\gamma(\theta) - w| \le C(h + |\gamma(\theta) - w|) \le C\psi(\gamma(h) - hv_{\gamma(h)}, w)\,.$$

So (5.2) is not greater than (we omit again the trivial term $C\|f\|_{L^p}h \approx C\|f\|_{L^p}\widetilde{\varrho}(\pi(\gamma(0)),\pi(\gamma(h))))$

$$(5.4)\qquad C\|f\|_{L^p(U)} \int_0^h \bigg\{ \int_U \frac{dV(w)}{|\overline{\varphi}^{3p'}(\gamma(\theta) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(\theta)},w)|} \bigg\}^{1/p'} d\theta$$

Thus to estimate I_2 it is enough to control (5.4). Finally, the term I_3 is dominated by

$$(5.5)\qquad |I_3| = \bigg| \int_0^h \frac{d}{d\theta}\bigg\{ \int_U \frac{f(w)dV(w)}{|\overline{\varphi}^2(\gamma(0) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(0)},w)\psi_{\theta,h}} \bigg\}\, d\theta \bigg|\,.$$

Once again the curve $\theta \to \gamma(\theta) - hv_{\gamma(\theta)}$ is a curve in $\widetilde{\mathcal{C}}_1^\kappa(U) \subset \widetilde{\mathcal{C}}^\kappa(U)$, so the complex derivatives of γ are dominated by a constant C. Thus the term (5.5) is dominated by

$$(5.6)\qquad C\|f\|_{L^p(U)} \times \int_0^h \bigg\{ \int_U \frac{dV(w)}{|\overline{\varphi}^{2p'}(\gamma(0) - \Lambda_{\partial\Omega}(\pi(\gamma(0)),h)v_{\gamma(0)},w)\psi_{\theta,h}^{2p'}} \bigg\}^{1/p'} d\theta\,.$$

Estimating (5.6) is essentially the same as estimating I_{12}.

In conclusion, all terms I_1, I_2, I_3, and I_4 are controlled once we have estimated I_{11}, I_{12}, and (5.4). We proceed to this task. If we look at the Henkin kernel for the $\overline{\partial}$-equation on the domains H_κ (see [5]), we can choose a local coordinate system so that we may rewrite I_{11} and I_{12} as follows

$$I_{11} \leq C\|f\|_{L^p(U)}h + C\|f\|_{L^p(U)} \int_0^h \left\{ \int_0^R d\mu \int_0^{\sqrt{R^2-\mu^2}} r^2\, dr \right.$$

$$\left. \times \int_{-1}^1 \frac{ds}{[(\varrho+\mu+\mu^2+\theta^4+r^4)^2+r^2s^2]^{3p'/2}(r^2+\theta^2+\varrho^2+\mu^2)^{p'/2}} \right\}^{1/p'} \theta d\theta\,,$$

$$I_{12} \leq C\|f\|_{L^p(U)}h + C\|f\|_{L^p(U)} \int_0^h \left\{ \int_0^R d\mu \int_0^{\sqrt{R^2-\mu^2}} r^2\, dr \right.$$

$$\left. \times \int_{-1}^1 \frac{ds}{[(\varrho+\mu+\mu^2+\theta^4+r^4)^2+r^2s^2]^{p'}(r^2+\theta^2+\varrho^2+\mu^2)^{p'}} \right\}^{1/p'} \theta d\theta$$

with $\varrho = \mathrm{dist}(r(h), \partial H_\kappa)$, $0 < R \leq 1$ is a constant depending on H_κ. Here we use the obvious estimates

$$\Lambda_{\partial\Omega}(\pi(\gamma(0)), \theta) \leq c\theta^2 \quad \text{and} \quad \frac{d(\Lambda_{\partial\Omega}(\pi(\gamma(0)), \theta))}{d\theta} \leq c\theta^1\,.$$

Now we calculate the term I_{11} first:

$$\|f\|_{L^p(U)} \int_0^h \left\{ \int_0^R d\mu \int_0^{\sqrt{R^2-\mu^2}} r^2\, dr \right.$$

$$\left. \times \int_{-1}^1 \frac{ds}{[(\varrho+\mu+\mu^2+\theta^4+r^4)^2+r^2s^2]^{3p'/2}(r^2+\theta^2+\varrho^2+\mu^2)^{p'/2}} \right\}^{1/p'} \theta d\theta$$

$$\begin{aligned} &\leq \int_0^h \left\{ \int_0^R d\mu \int_0^\mu r^2\, dr \int_{-1}^1 * \, ds \right\}^{1/p'} \theta d\theta \\ &\quad + \int_0^h \left\{ \int_0^R d\mu \int_\mu^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 * \, ds \right\}^{1/p'} \theta d\theta \\ &\quad + \int_0^h \left\{ \int_0^R d\mu \int_{\sqrt{\mu}}^R r^2\, dr \int_{-1}^1 * \, ds \right\}^{1/p'} \theta d\theta \\ &\equiv A_1 + A_2 + A_3\,. \end{aligned}$$

Write A_1 as follows:

$$A_1 = \int_0^h \left\{ \int_0^{\theta^4} d\mu \int_0^\mu r^2\, dr \int_{-1}^1 * \, ds \right\}^{1/p'} \theta d\theta$$

$$
\begin{aligned}
&+ \int_0^h \Big\{ \int_{\theta^4}^{2\theta} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 \ * \ ds \Big\}^{1/p'} \theta d\theta \\
&+ \int_0^h \Big\{ \int_{2\theta}^{R} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 \ * \ ds \Big\}^{1/p'} \theta d\theta \\
&\equiv A_{11} + A_{12} + A_{13}\,.
\end{aligned}
$$

Then

$$
\begin{aligned}
\text{(5.7)}\quad A_{11} &\le \int_0^h \Big\{ \int_0^{\theta^4} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 \frac{ds}{(r^2s^2+\theta^2+\mu^2)^{3p'/2}(\theta^2)^{p'/2}} \Big\}^{1/p'} \theta d\theta \\
&\le C \int_0^h \Big\{ (\theta^2)^{-3p'}\theta^{-p'} \Big\}^{1/p'} \Big\{ \int_0^{\theta^4} \mu^3\,d\mu \Big\}^{1/p'} \theta d\theta \\
&\le Ch^{-4+12/p'} \le Ch^{-4+12(1-1/p)} \le Ch^{1-10/p} \\
&\approx C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \qquad \text{if } 10 < p \le \infty\,,
\end{aligned}
$$

$$
\begin{aligned}
\text{(5.8)}\quad A_{12} &\le \int_0^h \Big\{ \int_{\theta^4}^{2\theta} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 \frac{ds}{(r^8+r^2s^2+\mu^2)^{3p'}\theta^{p'}} \Big\}^{1/p'} \theta d\theta \\
&\le C \int_0^h \Big\{ \int_{\theta^4}^{2\theta} \mu^{3-3p'}\,d\mu \Big\}^{1/p'} \theta^{-1}\theta d\theta \le Ch^{-2+3/p'} \\
&= Ch^{-2+3(1-1/p)} \le Ch^{1-10/p} \\
&\approx C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \qquad \text{if } 10 < p \le \infty\,,
\end{aligned}
$$

$$
\begin{aligned}
\text{(5.9)}\quad A_{13} &\le \int_0^h \Big\{ \int_{2\theta}^{R} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 \frac{ds}{(\theta^4+r^4+\mu)^{3p'}\mu^{p'}} \Big\}^{1/p'} \theta d\theta \\
&\le C \int_0^h \Big\{ \int_{2\theta}^{R} \mu^{3-4p'}\,d\mu \Big\}^{1/p'} \theta d\theta \\
&\le C \int_0^h \theta^{-4+3/p'}\,\theta d\theta \le Ch^{-2+4/p'} \\
&= Ch^{-2+4(1-1/p)} \le Ch^{1-10/p} \\
&\begin{cases} \approx C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} & \text{if } 10 < p < \infty, \\ \le Ch\log h \approx C\widetilde{\varrho}(\gamma(0),\gamma(h))\log(\widetilde{\varrho}(\gamma(0),\gamma(h))) & \text{if } p = \infty\,. \end{cases}
\end{aligned}
$$

Next we estimate the term A_2:

$$A_2 = \Big| \int_0^h \Big\{ \int_0^R d\mu \int_\mu^{\sqrt{\mu}} r^2 dr$$

$$\times \int_{-1}^1 \frac{ds}{[(\varrho+\mu+\mu^2+\theta^4+r^4)^2+r^2s^2]^{3p'/2}(r^2+\theta^2+\varrho^2+\mu^2)^{p'/2}} \Big\}^{1/p'} \theta d\theta \Big|$$

$$\le \int_0^h \Big\{ \int_0^\theta d\mu \int_\mu^{\sqrt{\mu}} r^2 dr \int_{-1}^1 \frac{ds}{[\mu^2+r^2s^2]^{3p'/2}\theta^{p'}} \Big\}^{1/p'} \theta d\theta$$

$$+ \int_0^h \Big\{ \int_\theta^R d\mu \int_\mu^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \frac{ds}{[\mu^2+r^2s^2]^{3p'/2}\theta^{p'}} \Big\}^{1/p'} \theta d\theta$$

$$\equiv A_{21} + A_{22}\,.$$

We have

$$\begin{aligned} (5.10)\qquad A_{21} &= C \int_0^h \Big\{ \int_0^\theta d\mu \int_\mu^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \frac{ds}{[\mu^2+r^2s^2]^{3p'/2}} \Big\}^{1/p'} \theta^{-1}\theta^1\, d\theta \\ &\le C \int_0^h \Big\{ \int_0^\theta \mu^{3/2-3p'/2}\, d\mu \Big\}^{1/p'} d\theta \\ &\le C \int_0^h \theta^{5/(2p')-3/2}\, d\theta \le C h^{-1/2+5(1-1/p)/2} \\ &= Ch^{2-5/(2p)} \le Ch^{1-10/p} \\ &\approx C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \quad \text{if } 10 < p \le \infty\,, \end{aligned}$$

$$\begin{aligned} (5.11)\qquad A_{22} &\le \Big| \int_0^h \Big\{ \int_\theta^R d\mu \int_\mu^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \frac{ds}{[\mu^2+r^2s^2]^{3p'/2}\theta^{p'}} \Big\}^{1/p'} \theta d\theta \Big| \\ &\le C \Big| \int_0^h \Big\{ \int_\theta^R \mu^{3/2-3p'/2}\, d\mu \Big\}^{1/p'} d\theta \Big| \\ &\le C \Big| \int_0^h \{ R^{5/(2p')-3/2} + \theta^{5/(2p')-3/2} \}\, d\theta \Big| \\ &\le Ch + \widetilde{C} h^{-1/2+5(1-1/p)/2} \\ &\le C'h \approx C\widetilde{\varrho}(\gamma(0),\gamma(h)) \quad \text{if } 10 < p \le \infty\,. \end{aligned}$$

Now we represent the term A_3 as follows:

$$\begin{aligned} A_3 = & \int_0^h \Big\{ \int_0^\theta d\mu \int_{\sqrt{\mu}}^R r^2\, dr \int_{-1}^1 \ * \ ds \Big\}^{1/p'} \theta d\theta \\ & + \int_0^h \Big\{ \int_\theta^R d\mu \int_{\sqrt{\mu}}^R r^2\, dr \int_{-1}^1 \ * \ ds \Big\}^{1/p'} \theta d\theta \\ \equiv & \ A_{31} + A_{32}\,. \end{aligned}$$

We have

$$\begin{aligned} A_{31} & \le \Big| \int_0^h \Big\{ \int_0^\theta d\mu \int_{\sqrt{\mu}}^R r^2\, dr \int_{-1}^1 \frac{ds}{[\mu^2 + r^2 s^2]^{3p'/2} \theta^{p'}} \Big\}^{1/p'} \theta d\theta \Big| \\ & \le \Big| \int_0^h \Big\{ \int_0^\theta d\mu \int_{\sqrt{\mu}}^R r^2\, dr \int_{-1}^1 \frac{ds}{[\mu^2 + r^2 s^2]^{3p'/2} \theta^{p'}} \Big\}^{1/p'} \theta d\theta \Big| \\ & \le C \Big| \int_0^h \Big\{ \int_0^\theta d\mu \int_{\sqrt{\mu}}^R r^{2-3p'}\, dr \int_{-1}^1 \frac{ds}{[s^2 + v^2]^{3p'/2}} \Big\}^{1/p'} d\theta \Big| \\ & \le C \Big| \int_0^h \Big\{ \int_0^\theta (R^{3-3p'} + \mu^{3/2-3p'/2})\, d\mu \Big\}^{1/p'} d\theta \Big| \\ & \le C \Big| \int_0^h (R^{3/p'-3} \theta^{1/p'} + \theta^{5/(2p')-3/2})\, d\theta \Big| \\ & \le C' h^{1+1/p'} + \widetilde{C} h^{5/(2p')-1/2} = C' h^{1+1/p'} + \widetilde{C} h^{2-5/(2p)} \\ & \le Ch \approx C'' \widetilde{\varrho}(\gamma(0), \gamma(h)) \qquad \text{if } 10 < p < \infty\,. \end{aligned} \tag{5.12}$$

When $p = \infty$, i.e., $p' = 1$, then

$$\begin{aligned} & C \int_0^h \Big\{ \int_0^\theta d\mu \int_{\sqrt{\mu}}^R r^{2-3}\, dr \Big\}^1 d\theta \\ & \qquad \le C' \int_0^h \int_0^\theta (\log R + \log \mu)\, d\mu\, d\theta \\ & \qquad \le C' \int_0^h \log R \cdot \theta d\theta + C' \int_0^h (\theta \log \theta + \theta)\, d\theta \\ & \qquad \le \widetilde{C} h + \widetilde{C} h^2 \log h \le \widetilde{C} h + \widetilde{C} h \log h \\ & \qquad \approx C'' \widetilde{\varrho}(\gamma(0), \gamma(h)) + C'' \widetilde{\varrho}(\gamma(0), \gamma(h)) \log(\widetilde{\varrho}(\gamma(0), \gamma(h)))\,. \end{aligned}$$

Next,

$$
\begin{aligned}
(5.13)\quad A_{32} &\le \Big| \int_0^h \Big\{ \int_\theta^R d\mu \int_{\sqrt{\mu}}^R r^2\,dr \int_{-1}^1 \frac{ds}{[\mu^2+r^2s^2]^{3p'/2}\theta^{p'}} \Big\}^{1/p'} \theta d\theta \Big| \\
&\le \Big| \int_0^h \Big\{ \int_\theta^R d\mu \int_{\sqrt{\mu}}^R r^{2-3p'}\,dr \int_{-1}^1 \frac{ds}{[s^2+v^2]^{3p'/2}} \Big\}^{1/p'} \theta^{-1}\theta^1\,d\theta \Big| \\
&\le C \Big| \int_0^h \Big\{ \int_\theta^R (R^{3-3p'} + \mu^{3/2-3p'/2})\,d\mu \Big\}^{1/p'} d\theta \Big| \\
&\le C \Big| \int_0^h \{R^{4-3p'}(R+\theta) + R^{5/2-3p'/2} + \theta^{5/2-3p'/2}\}^{1/p'}\,d\theta \Big| \\
&\le C'h + \widetilde{C}h^{-1/2+5/(2p')} \\
&\le C''h + \widetilde{C}h^{2-5/(2p)} \approx C\widetilde{\varrho}(\gamma(0),\gamma(h)) \quad \text{ if } 10 < p \le \infty\,.
\end{aligned}
$$

When we calculate the term I_{12}, we also need to consider the integral in three parts:

$$
\begin{aligned}
C\|f\|_{L^p(U)} &\int_0^h \Big\{ \int_0^R d\mu \int_0^{\sqrt{R^2-\mu^2}} r^2\,dr \\
&\times \int_{-1}^1 \frac{ds}{[(\varrho+\mu+\mu^2+\theta^4+r^4)^2+r^2s^2]^{p'}(r^2+\theta^2+\varrho^2+\mu^2)^{p'}} \Big\}^{1/p'} d\theta \\
&\le C\Big\{ \int_0^h \Big\{ \int_0^R d\mu \int_0^\mu r^2\,dr \int_{-1}^1 *\,ds \Big\}^{1/p'} d\theta \\
&\quad + \int_0^h \Big\{ \int_0^R d\mu \int_\mu^{\sqrt{\mu}} r^2\,dr \int_{-1}^1 *\,ds \Big\}^{1/p'} d\theta \\
&\quad + \int_0^h \Big\{ \int_0^R d\mu \int_{\sqrt{\mu}}^R r^2\,dr \int_{-1}^1 *\,ds \Big\}^{1/p'} d\theta \Big\} \\
&\equiv B_1 + B_2 + B_3\,.
\end{aligned}
$$

Write B_1 as follows:

$$
B_1 = C \int_0^h \Big\{ \int_0^{\theta^2} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 *\,ds \Big\}^{1/p'} d\theta
$$

$$+ C \int_0^h \Big\{ \int_{\theta^2}^{2\theta} d\mu \int_0^\mu r^2\, dr \int_{-1}^1 \ * \, ds \Big\}^{1/p'} d\theta$$

$$+ C \int_0^h \Big\{ \int_{2\theta}^{R} d\mu \int_0^\mu r^2\, dr \int_{-1}^1 \ * \, ds \Big\}^{1/p'} d\theta$$

$$\equiv B_{11} + B_{12} + B_{13} \,.$$

We have

$$B_{11} \le \int_0^h \bigg\{ \int_0^{\theta^2} d\mu \int_0^\mu r^2\, dr \int_{-1}^1 \frac{ds}{[(\mu^2+\mu)+\varrho]^{2p'}\mu^{2p'}} \bigg\}^{1/p'} d\theta \tag{5.14}$$

$$\le C \int_0^h \Big\{ \int_0^{\theta^2} \mu^{-2p'} \mu^{-2p'+3}\, d\mu \Big\}^{1/p'} d\theta$$

$$\le C \int_0^h \theta^{-8+8/p'}\, d\theta \le C h^{1-8/p}$$

$$\le C h^{1-10/p} \approx C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \qquad \text{if } 10 < p < \infty\,;$$

if $p = \infty$ ($p' = 1$), then

$$B_{11} \le C \int_0^h \log\theta\, d\theta \le C h \log h \tag{5.14$'$}$$

$$\approx C\widetilde{\varrho}(\gamma(0),\gamma(h)) \log(\widetilde{\varrho}(\gamma(0),\gamma(h)))\,.$$

Estimating the terms B_{12} and B_{13} is almost the same, so we just look at B_{12}:

$$B_{12} \le \int_0^h \bigg\{ \int_{\theta^2}^{2\theta} d\mu \int_0^\mu r^2\, dr \int_{-1}^1 \frac{ds}{[(\mu^2+\mu)+\varrho]^{2p'}} \bigg\}^{1/p'} d\theta \tag{5.15}$$

$$\le C \int_0^h \Big\{ \int_{\theta^2}^{2\theta} \mu^{-2p'} \mu^{-2p'+3}\, d\mu \Big\}^{1/p'} d\theta$$

$$\le C \int_0^h \theta^{-4+4/p'}\, d\theta \le C h^{1-4/p}$$

$$\le \begin{cases} C h^{1-10/p} & \text{if } 10 < p < \infty, \\ C \displaystyle\int_0^h \log\theta\, d\theta \le C h \log h \approx C\widetilde{\varrho}(\gamma(0),\gamma(h)) \log(\widetilde{\varrho}(\gamma(0),\gamma(h))) & \\ & \text{if } p = \infty. \end{cases}$$

It is easy to see that the estimation of the terms B_2 and B_3 is almost the same, so we just look at the term B_2:

$$\begin{aligned}B_2 = C \int_0^h \Big\{ \int_0^{2\theta} d\mu \int_\mu^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \; * \; ds\Big\}^{1/p'} d\theta \\ + C \int_0^h \Big\{ \int_{2\theta}^{R} d\mu \int_\mu^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \; * \; ds\Big\}^{1/p'} d\theta \equiv B_{21} + B_{22}\,.\end{aligned}$$

We have

$$\begin{aligned}B_{21} &\le C \int_0^h \left\{ \int_0^{2\theta} d\mu \int_\mu^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \frac{ds}{(\mu^2 + r^2 s^2)^{p'} \theta^{2p'}} \right\}^{1/p'} d\theta \\ &\le C \int_0^h \left\{ \int_0^{2\theta} d\mu \int_\mu^{\sqrt{\mu}} r^{2-2p'}\, dr \cdot \frac{\mu}{r} \int_{-\infty}^{\infty} \frac{ds}{(1+s^2)^{p'}} \right\} \theta^{-2}\, d\theta \\ &\le C' \int_0^h \Big\{ \int_0^{2\theta} \mu(\mu^{2-2p'} + \mu^{1-p'})\, d\mu\Big\}^{1/p'} \theta^{-2}\, d\theta \\ &\le C' \int_0^h (\theta^{4/p'-2} + \theta^{3/p'-1})\theta^{-2}\, d\theta \le \widetilde{C}(h^{-3+4/p'} + h^{-2+3/p'}) \\ &\le \widetilde{C}(h^{-3+4(1-1/p)} + h^{-2+3(1-1/p)}) \\ &\le \widetilde{C}h^{1-1/p} \approx \widetilde{C}(\widetilde{\varrho}(\gamma(0), \gamma(h)))^{1-10/p} \qquad \text{if } 10 < p \le \infty\,.\end{aligned} \tag{5.16}$$

For the term B_{22}, we have

$$\begin{aligned}B_{22} &\le C \int_0^h \left\{ \int_{2\theta}^{R} d\mu \int_\mu^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \frac{ds}{(\mu^2 + r^2 s^2)^{p'} r^{2p'}} \right\}^{1/p'} d\theta \\ &\le C \int_0^h \left\{ \int_{2\theta}^{R} d\mu \int_\mu^{\sqrt{\mu}} r^{2-2p'-2p'}\, dr \cdot \frac{\mu}{r} \int_{-\infty}^{\infty} \frac{ds}{1+s^2} \right\}^{1/p'} d\theta \\ &\le C \int_0^h \Big\{ \int_{2\theta}^{R} \mu(\mu^{2-4p'} + \mu^{1-2p'})\, d\mu\Big\}^{1/p'} d\theta \\ &\le C' \int_0^h (R^{4/p'-4} + R^{3/p'-2} + \theta^{4/p'-4} + \theta^{3/p'-2})\, d\theta \\ &\le \widetilde{C}h^{1-10/p} \approx \widetilde{C}(\widetilde{\varrho}(\gamma(0), \gamma(h)))^{1-10/p} \qquad \text{if } 10 < p \le \infty\,.\end{aligned} \tag{5.17}$$

Finally, we turn our attention to the term (5.4). In fact, (5.4) is dominated by

$$(5.18)\quad C\|f\|_{L^p(U)} \int_0^h \Big\{ \int_U \frac{dV(w)}{|\overline{\varphi}^{3p'}(\gamma(\theta) - \Lambda_{\partial\Omega}(\pi(\gamma(0),h)v_{\gamma(\theta)},w)|} \Big\}^{1/p'} d\theta$$

$$\le C\|f\|_{L^p(U)} \int_0^h \Big\{ \int_0^R d\mu \int_0^{\sqrt{R^2-\mu^2}} r^2\,dr$$

$$\times \int_{-1}^1 \frac{ds}{[(\varrho+\mu+\mu^2+\theta^4+r^4)^2+r^2s^2]^{3p'/2}} \Big\}^{1/p'} d\theta$$

$$= C\|f\|_{L^p(U)} \Big\{ \int_0^h \Big\{ \int_0^R d\mu \int_0^\mu r^2\,dr \int_{-1}^1 * \,ds \Big\}^{1/p'} d\theta$$

$$+ \int_0^h \Big\{ \int_0^R d\mu \int_\mu^{\sqrt{\mu}} r^2\,dr \int_{-1}^1 * \,ds \Big\}^{1/p'} d\theta$$

$$+ \int_0^h \Big\{ \int_0^R d\mu \int_{\sqrt{\mu}}^R r^2\,dr \int_{-1}^1 * \,ds \Big\}^{1/p'} d\theta \Big\}$$

$$\equiv D_1 + D_2 + D_3\,.$$

Write D_1 as follows:

$$D_1 = C\Big\{ \int_0^h \Big\{ \int_0^{\theta^2} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 * \,ds \Big\}^{1/p'} d\theta$$

$$+ \int_0^h \Big\{ \int_{\theta^2}^{2\theta} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 * \,ds \Big\}^{1/p'} d\theta$$

$$+ \int_0^h \Big\{ \int_{2\theta}^R d\mu \int_0^\mu r^2\,dr \int_{-1}^1 * \,ds \Big\}^{1/p'} d\theta \Big\}$$

$$\equiv D_{11} + D_{12} + D_{13}\,.$$

We have

$$(5.19)\quad D_{11} \le C \int_0^h \Big\{ \int_0^{\theta^2} d\mu \int_0^\mu r^2\,dr \int_{-1}^1 \frac{ds}{[(\mu+\varrho)^2+r^2s^2]^{3p'/2}} \Big\}^{1/p'} d\theta$$

$$\le C \int_0^h \Big\{ \int_0^{\theta^2} \mu^3\mu^{-3p'}\,d\mu \Big\}^{1/p'} d\theta \le C \int_0^h \theta^{8/p'-6}\,d\theta$$

$$\le C'h^{-5+8/p'} \le \widetilde{C}h^{1-10/p}$$
$$\approx C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \quad \text{if } 10 < p \le \infty\,.$$

Estimating the terms D_{12} and D_{13} is almost the same, so we just look at D_{12}:

$$\text{(5.20)}\quad D_{12} \le C\int_0^h \left\{ \int_{\theta^2}^{2\theta} d\mu \int_0^{\mu} r^2\,dr \int_{-1}^{1} \frac{ds}{[(\mu+\varrho)^2 + r^2 s^2]^{3p'/2}} \right\}^{1/p'} d\theta$$
$$\le C\left| \int_0^h \left\{ \int_{\theta^2}^{2\theta} \mu^3 \mu^{-3p'}\,d\mu \right\}^{1/p'} d\theta \right| \le C' \int_0^h (\theta^{4/p'-3} + \theta^{8/p'-6})\,d\theta$$
$$\le C'h^{4/p'-2} + C''h^{8/p'-5} \le C'h^{-2+4(1-1/p)} + C''h^{-5+8(1-1/p)}$$
$$\le \widetilde{C}h^{1-10/p} \approx C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \quad \text{if } 10 < p \le \infty\,.$$

Next,

$$D_2 = C\Big\{ \int_0^h \Big\{ \int_0^{2\theta} d\mu \int_{\mu}^{\sqrt{\mu}} r^2\,dr \int_{-1}^{1} *\,ds \Big\}^{1/p'} d\theta$$
$$+ \int_0^h \Big\{ \int_{2\theta}^{R} d\mu \int_{\mu}^{\sqrt{\mu}} r^2\,dr \int_{-1}^{1} *\,ds \Big\}^{1/p'} d\theta \Big\}$$
$$\equiv D_{21} + D_{22}\,.$$

We have

$$\text{(5.21)}\qquad D_{21} \le C\int_0^h \left\{ \int_0^{2\theta} d\mu \int_{\mu}^{\sqrt{\mu}} r^2\,dr \int_{-1}^{1} \frac{ds}{[(\mu+\varrho)^2 + r^2 s^2]^{3p'/2}} \right\}^{1/p'} d\theta$$
$$\le C\int_0^h \left\{ \int_0^{2\theta} d\mu \int_{\mu}^{\sqrt{\mu}} r^{2-3p'}\,dr \int_{-1}^{1} \frac{ds}{[s^2 + v^2]^{3p'/2}} \right\}^{1/p'} d\theta$$
$$\le C\int_0^h \left\{ \int_0^{2\theta} \mu^{3/2-3p'/2}\,d\mu \right\}^{1/p'} d\theta$$
$$\le C'' \int_0^h \theta^{5/(2p')-3/2}\,d\theta \le C''h^{5/(2p')-1/2}$$
$$\le C''h^{1-10/p} \approx \widetilde{C}(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \qquad \text{if } 10 < p < \infty\,,$$

$$\text{(5.22)}\qquad D_{22} \le C\int_0^h \left\{ \int_{2\theta}^{R} d\mu \int_{\mu}^{\sqrt{\mu}} r^2\,dr \int_{-1}^{1} \frac{ds}{[(\mu+\varrho)^2 + r^2 s^2]^{3p'/2}} \right\}^{1/p'} d\theta$$

$$
\begin{aligned}
&\le C \int_0^h \Big\{ \int_{2\theta}^R d\mu \int_\mu^{\sqrt{\mu}} r^{2-3p'}\, dr \int_{-1}^1 \frac{ds}{[s^2+\mu^2]^{3p'/2}} \Big\}^{1/p'} d\theta \\
&\le C' \int_0^h \Big\{ \int_{2\theta}^R \mu^{3/2-3p'/2}\, d\mu \Big\}^{1/p'} d\theta \le C'' \int_0^h R^{5/(2p')-3/2}\, d\theta \\
&\le C'' h \le C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \qquad \text{if } 10 < p < \infty\,.
\end{aligned}
$$

Combining the results from (5.7) to (5.21), we can see that our estimates are complete.

Remark. In our calculations (5.7) to (5.21), it appears that our estimate for the Lipschitz class is always better than the critical index, i.e., $\alpha = 1 - 10/p$. But when we consider $\gamma \in \widetilde{\mathcal{C}}_1^k$ and parallel to the type 4 points (i.e., the $\Re z_2$-axis), then $d(\Lambda_{\partial\Omega}(\pi(\gamma(0)),\theta))/\,d\theta = \theta^3 \ne \theta^1$. Then the computations really give us the critical index. For example

$$
\begin{aligned}
A_{11} &= \int_0^h \Big\{ \int_0^{\theta^4} d\mu \int_0^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \frac{ds}{(r^4+\theta^4+\mu^4+\varrho)^{3p'}(\theta^2)^{p'/2}} \Big\}^{1/p'} \theta^3\, d\theta \\
&\ge C' \int_0^h \Big\{ \int_0^{\theta^4} d\mu \int_0^{\sqrt{\mu}} r^2\, dr \int_{-1}^1 \frac{ds}{(\theta^2+\theta^4+\theta^4+\varrho)^{3p'}(\theta^2)^{p'/2}} \Big\}^{1/p'} \theta^3\, d\theta\,.
\end{aligned}
$$

Let $\varrho = 0$ (i.e. consider the boundary ∂H_κ). Then

$$
\begin{aligned}
A_{11} &\ge C' \int_0^h \Big\{ \int_0^{\theta^4} \mu^{3/2}\, d\mu \Big\}^{1/p'} \theta^3 \theta^{-13}\, d\theta \\
&= C' \int_0^h \theta^{10/p'-10}\, d\theta = \widetilde{C} h^{10/p'-9} = \widetilde{C} h^{-9+10((1-1/p)} = \widetilde{C} h^{1-10/p} \\
&\approx C(\widetilde{\varrho}(\gamma(0),\gamma(h)))^{1-10/p} \qquad \text{if } 10 < p < \infty\,.
\end{aligned}
$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND 20742
U.S.A.

DEPARTMENT OF MATHEMATICS
WASHINGTON UNIVERSITY
ST. LOUIS, MISSOURI 63130
U.S.A.

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