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ON A COMPACTIFICATION OF THE HOMEOMORPHISM GROUP OF THE PSEUDO-ARC

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1. Introduction. A *continuum* means a compact connected metric space. For a continuum X, $H(X)$ denotes the space of all homeomorphisms of X with the compact-open topology. It is well known that $H(X)$ is a completely metrizable, separable topological group. J. Kennedy [8] considered a compactification of $H(X)$ and studied its properties when X has various types of homogeneity. In this paper we are concerned with the compactification G_P of the homeomorphism group of the pseudo-arc P, which is obtained by the method of Kennedy. We prove that G_P is homeomorphic to the Hilbert cube. This is an easy consequence of a combination of the results of [2], Corollary 2, and [9], Theorem 1, but here we give a direct proof. The author wishes to thank the referee for pointing out the above reference [2]. We also prove that the remainder of $H(P)$ in G_P contains many Hilbert cubes. It is known that $H(P)$ contains no nondegenerate continua ([10]).

NOTATION AND BASIC DEFINITIONS 1.1. Let X be a continuum. Let $f : X \to X$ be a map. The *graph* of $f = \{(x, f(x)) \mid x \in X\} \subset X \times X$ is denoted by $\mathrm{gr}\, f$.

A map $f : X \to X$ is called a *near-homeomorphism* if, for each $\varepsilon > 0$, there exists a homeomorphism $h : X \to X$ such that $d(f, h) = \sup\{d(f(x), h(x)) \mid x \in X\} < \varepsilon$.

The *hyperspace* $C(X)$ is the space of all nonempty subcontinua of X with the Hausdorff metric d_H. The ε-neighbourhood of $K \in C(X)$ is denoted by $N_\varepsilon(K)$. The map $\varphi : H(X) \to C(X \times X)$ defined by $\varphi(f) = \mathrm{gr}\, f$ is an imbedding ([8], p. 43).

A compactification G_X of $H(X)$ is defined by $\mathrm{cl}_{C(X\times X)} \mathrm{im}\, \varphi$.

The space $C_\pi(X \times X)$ is defined by $C_\pi(X \times X) = \{K \in C(X \times X) \mid \pi_1(K) = \pi_2(K) = X\}$, where π_i is the projection onto the ith factor $(i = 1, 2)$.

A surjective map $f : X \to Y$ induces $f^* : C_\pi(X \times X) \to C_\pi(Y \times Y)$

defined by $f^*(K) = (f \times f)(K)$.

A continuum is called *arc-like* if it is represented as the limit of an inverse sequence of arcs. A continuum X is called *hereditarily indecomposable* if, for each pair A, B of subcontinua of X such that $A \cap B \neq \emptyset$, either $A \subset B$ or $A \supset B$ holds.

A hereditarily indecomposable arc-like continuum is topologically unique and is called the *pseudo-arc* (denoted by P). It is known that P is the unique homogeneous arc-like continuum ([1]).

In what follows, the Hilbert cube is denoted by Q.

The following theorem is fundamental in this paper.

THEOREM 1.2 ([13]). $G_P = C_\pi(P \times P)$.

2. G_P is homeomorphic to Q.

First we prove the following result.

THEOREM 2.1. *Let X be an arc-like continuum. Then $C_\pi(X \times X)$ is homeomorphic to Q.*

Proof. Let $X = \varprojlim(I_n, p_{n,n+1})$, where each I_n is an arc and each $p_{n,n+1} : I_{n+1} \to I_n$ is surjective. Let $p_n : X_n \to I_n$ be the projection onto the nth factor.

Step 1. First we prove that $C_\pi(X \times X) = \varprojlim(C_\pi(I_n \times I_n), p^*_{n,n+1})$. Notice that $p^*_{n,n+1} \circ p^*_{n+1} = p^*_n$. So the limit of p^*_n's, $\varprojlim p^*_n : C_\pi(X \times X) \to \varprojlim(C_\pi(I_n \times I_n), p^*_{n,n+1})$, is defined.

By [6], Proposition 1.2, $p^*_{n,n+1} : C_\pi(I_{n+1} \times I_{n+1}) \to C_\pi(I_n \times I_n)$ and $p^*_n : C_\pi(X \times X) \to C_\pi(I_n \times I_n)$ are surjective for each n. Using this fact, we can see that $\varprojlim p^*_n$ is a homeomorphism.

Step 2. Next we show that if I is an arc, then $C_\pi(I \times I)$ is homeomorphic to Q. It is clear that $C_\pi(I \times I)$ has the following property:

(1) If K and L are subcontinua of $I \times I$ such that $K \subset L$ and $K \in C_\pi(I \times I)$, then $L \in C_\pi(I \times I)$.

Using (1), we can see that $C_\pi(I \times I)$ is an AR in the same way as in [7], Theorem 4.4 (see also Remark, p. 29 of [7]). Using the method of [5], Lemma 4.4, we have

(2) for each $\varepsilon > 0$, there exists a map $g : C_\pi(I \times I) \to C_\pi(I \times I)$ such that $d(g, \mathrm{id}) < \varepsilon$ and $\operatorname{im} g$ is a Z-set in $C_\pi(I \times I)$.

Hence $C_\pi(I \times I)$ has the disjoint n-cell property for each n, so by Toruńczyk's characterization theorem [14], $C_\pi(I \times I)$ is homeomorphic to Q.

Step 3. $p^*_{n,n+1} : C_\pi(I_{n+1} \times I_{n+1}) \to C_\pi(I_n \times I_n)$ is a cell-like map. To show this, first we prove that

(3) $p^*_{n,n+1}$ is a monotone map.

Take $K \in C_\pi(I_n \times I_n)$ and let $\Lambda_K = p^{*-1}_{n,n+1}(K)$. For each $A, B \in \Lambda_K$, $A \cap B \neq \emptyset$, because $\pi_1(A) = \pi_1(B) = \pi_2(A) = \pi_2(B) = I_{n+1}$. So there exist order arcs α_A, β_B from A to $A \cup B$ and from B to $A \cup B$ respectively. It is easy to see that $\alpha_A \cup \alpha_B \subset \Lambda_K$. Hence Λ_K is an arcwise connected continuum.

Consider the hyperspace $C(\Lambda_K)$ (note that $C(\Lambda_K) \subset C(C(I_n \times I_n))$). Since Λ_K is a continuum, $C(\Lambda_K)$ has the trivial shape ([11], p. 180). Let $\sigma : C(C(I_n \times I_n)) \to C(I_n \times I_n)$ be the union function defined by $\sigma(\mathcal{A}) = \bigcup \mathcal{A}$ for each $\mathcal{A} \in C(C(I_n \times I_n))$.

Take any $\mathcal{A} \in C(\Lambda_K)$. Then $p_{n,n+1}(A) = K$ for each $A \in \mathcal{A}$, and hence $p^*_{n,n+1}(\sigma(\mathcal{A})) = p_{n,n+1}(\bigcup \mathcal{A}) = K$. This means $\sigma(C(\Lambda_K)) \subset \Lambda_K$, and it is easy to see that $\sigma(\{A\}) = A$ for each $A \in \Lambda_K$. Hence $\sigma(C(\Lambda_K)) = \Lambda_K$ and $\sigma|C(\Lambda_K)$ is a retraction onto Λ_K. The trivial shape is preserved under any retraction, so Λ_K has the trivial shape. (See [12], Lemma 2.1, for that argument.)

Remark. In fact, Λ_K is locally connected, and so $C(\Lambda_K)$ and Λ_K are AR's.

By Steps 2 and 3, each $p^*_{n,n+1}$ is a near-homeomorphism (see [4], pp. 105–106). Hence by [3] and Step 1, $C_\pi(X \times X)$ is homeomorphic to Q.

Combining Theorem 1.2 and Theorem 2.1, we have

COROLLARY 2.2. *G_P is homeomorphic to Q.*

3. The remainder of G_P

DEFINITION 3.1. Let X be a continuum. A continuous map $\mu : C(X) \to [0,1]$ is called a *Whitney map* if it satisfies the following conditions:

1) $\mu(X) = 1$ and $\mu(\{x\}) = 0$ for each $x \in X$.
2) If $A, B \in C(X)$ satisfy $A \subsetneq B$, then $\mu(A) < \mu(B)$.

DEFINITION 3.2. Let X be a hereditarily indecomposable continuum, and fix a Whitney map $\mu : C(X) \to [0,1]$.

1) Let p be a point of X. The order arc $\alpha_p : [0,1] \to C(X)$ is defined by $\alpha_p(0) = \{p\}$ and $\mu(\alpha_p(t)) = t$ for each $0 \le t \le 1$. By the hereditary indecomposability of X, α_p is uniquely determined ([7], (8.4), or [11], (1.61)).

2) Let $\alpha : X \times [0,1] \to C(X)$ be the map defined by $\alpha(p,t) = \alpha_p(t)$ for $(p,t) \in X \times [0,1]$. Then α is continuous ([11], (1.63), pp. 113–114).

LEMMA 3.3. *Let $\varphi : H(P) \to C(P \times P)$ be the map defined in* 1.1. *Then*

$\operatorname{im}\varphi = \{K \in C_\pi(P \times P) \mid \textit{for each } p \in P, \#(P \times p \cap K) = \#(p \times P \cap K) = 1\}$,

where $\#A$ denotes the cardinality of a set A.

Proof. It is clear that for each $f \in H(P)$ and for each $p \in P$, $\#(P \times p \cap \operatorname{gr} f) = \#(p \times P \cap \operatorname{gr} f) = 1$. Conversely, take any $K \in C_\pi(P \times P)$ such that for each $p \in P$, $\#(P \times p \cap K) = \#(p \times P \cap K) = 1$. By Theorem 1.2, $C_\pi(P \times P) = G_P$, hence there exists a sequence $(f_n) \subset H(P)$ such that $\operatorname{gr} f_n \to K$ (convergence in the Hausdorff metric). We claim that

(1) (f_n) is equicontinuous.

Suppose not. Then there exists an $\varepsilon_0 > 0$ such that for each $n \ge 1$, there exist $x_n, y_n \in P$ and a subsequence (f_{k_n}) such that $d(x_n, y_n) < 1/n$ and $d(f_{k_n}(x_n), f_{k_n}(y_n)) \ge \varepsilon_0$. We may assume that $\lim x_n = \lim y_n = p$ and $\lim f_{k_n}(x_n) = x$, $\lim f_{k_n}(y_n) = y$. Then $(p, x) = \lim(x_n, f_{k_n}(x_n)) \in K$ and similarly $(p, y) \in K$. But $x \ne y$, which contradicts the hypothesis.

By (1) and the Ascoli–Arzelà theorem, the sequence (f_n) converges uniformly to a continuous map f. So $K = \operatorname{gr} f$. Since $\#(P \times p \cap K) = 1$, we have $f \in H(P)$. This completes the proof.

THEOREM 3.4. *For each* $\varepsilon > 0$, *there exists a homotopy* $H : G_P \times [0,1] \to G_P$ *which satisfies the following conditions.*

(1) H *is an* ε*-homotopy and* $H_0 = \mathrm{id}$.

(2) $H(G_P \times (0,1]) \subset G_P - H(P)$.

Proof. Fix a Whitney map $\mu : C(P) \to [0,1]$. Take a small $t_0 > 0$ such that

(3) $0 < \operatorname{diam} A < \varepsilon$ for each $A \in \mu^{-1}(t_0)$.

Then $H : G_P \times [0,1] \to G_P$ is defined by

$$H(K,t) = \bigcup\{x \times \alpha_y(t \cdot t_0) \mid (x,y) \in K\}.$$

We prove that $H(K,t) \in G_P$ for each (K,t). Take $(x_n, z_n) \in H(K,t)$ and assume that $(x_n, z_n) \to (x,z)$. There exist $(x_n, y_n) \in K$ such that $(x_n, z_n) \in x_n \times \alpha_{y_n}(t \cdot t_0)$. We may assume that $y_n \to y$. Then $(x_n, y_n) \to (x,y)$ and by the continuity of α, $(x,z) \in x \times \alpha_y(t \cdot t_0) \subset H(K,t)$. Hence $H(K,t)$ is compact. It is clear that $H(K,t)$ is connected and contains K. So $H(K,t) \in C_\pi(P \times P) = G_P$. Using the continuity of α again, we see that H is continuous. By (3), H is an ε-homotopy, and by Lemma 3.3, condition (2) is satisfied.

THEOREM 3.5. *For each open subset* U *of* $G_P - H(P)$, *there exists an imbedding* $i : Q \to U$ *of* Q *into* U.

Proof. Let V be any open subset of $G_P - H(P)$. There exists an open subset V of G_P such that $V \cap (G_P - H(P)) = U$. Since $H(P)$ is dense in G_P, we can find $f \in H(P) \cap V$. Take $\varepsilon > 0$ sufficiently small so that $N_\varepsilon(\operatorname{gr} f) \subset V$.

Let (p_n) be a sequence in P such that $p_n \to p \in P$. Take a sequence (K_n) of subcontinua of P such that

$$f(p_n) \in K_n \quad \text{and} \quad \operatorname{diam} K_n \to 0 \text{ as } n \to \infty. \tag{1}$$

For each $n \geq 0$, let $\alpha_n : [0,1] \to C(K_n)$ be the order arc such that

$$\alpha_n(0) = \{f(p_n)\} \quad \text{and} \quad \alpha_n(1) = K_n. \tag{2}$$

Let $Q' = I^\infty$. We define a map $i : Q' \to V$ by

$$i((t_n)) = \operatorname{gr} f \cup \bigcup_{n \geq 0} \{p_n\} \times \alpha_n(t_n) \quad \text{for } (t_n)_{n \geq 0} \in Q'.$$

Then in the same way as in [7], Theorem 5.1, i is an imbedding. But $\operatorname{im} i \cap H(P) = \{\operatorname{gr} f\}$ by Lemma 3.3, and we can take a Hilbert cube $Q \subset Q'$ such that $i(Q) \subset V \cap (G_P - H(P)) = U$. This completes the proof.

Remark 3.6. $H(P)$ has no interior points in G_P by Theorem 3.4. Therefore $G_P - H(P)$ is not completely metrizable, and hence is not a Q-manifold.

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