On expansiveness of shift homeomorphisms of inverse limits of graphs

by

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Abstract. In this paper, we study expansiveness of shift homeomorphisms of inverse limits of graphs.

1. Introduction. All spaces under consideration are assumed to be metric. By a *continuum* we mean a compact connected nondegenerate space. Let X be a compact metric space with metric d. A homeomorphism $h\colon X \to X$ of X is called *expansive* if there exists a positive number c (called an *expansive constant* for h) such that if x and y are different points of X, there is an integer $n \in \mathbf{Z}$ such that

$$d\big(h^n(x), h^n(y)\big) > c.$$

Expansiveness does not depend on the choice of the metric d of X. This notion is important in topological dynamics and ergodic theory (see [7, 8 and 26]).

In [18], R. Mañé proved that if $h\colon X \to X$ is an expansive homeomorphism of a compact metric space X, then $\dim X < \infty$ and every minimal set is 0-dimensional. This result shows that there are restrictions on spaces which admit expansive homeomorphisms. We are interested in the following problem (see [8, (7), p. 349]): What kinds of continua admit expansive homeomorphisms? In general, for a given homeomorphism h, it is difficult to determine whether h is expansive or not. In [27], R. F. Williams showed that the shift homeomorphism of the dyadic solenoid is expansive. From continua theory in topology, we know that inverse limit spaces yield powerful techniques for constructing complicated spaces and maps from simple ones. Naturally the following problem will be interesting: What kinds of shift homeomorphisms are expansive? It is well known that the shift homeomorphisms of positively expansive maps are always expansive (see the proof of [27]). In [10], Jacobson and Utz stated that

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the shift homeomorphism of the inverse limit of every surjective map of an arc is not expansive (see [6] for the complete proof).

In this paper, we study expansiveness of shift homeomorphisms of inverse limits of graphs. In Section 2, we introduce a new notion of positively pseudo-expansive map. By definitions, every positively expansive map is positively pseudo-expansive.

In Section 3, we prove that if $f\colon G\to G$ is any map of a tree G, then the shift homeomorphism $\tilde{f}$ of f is not expansive whenever (G,f) is nondegenerate. Moreover, if $f\colon X\to X$ is a null-homotopic map of a compact connected 1-dimensional ANR X, then the shift homeomorphism $\tilde{f}$ of f is not expansive whenever (X,f) is nondegenerate.

In Section 4, we investigate the relations between positively pseudo-expansive maps and expansiveness of shift homeomorphisms of inverse limits of graphs.

In Section 5, we give some remarks which imply that the notion of positively pseudo-expansive map is useful and important for constructing various types of expansive homeomorphisms.

2. Definitions and preliminaries. Let X be a compact metric space with metric d. For a map $f\colon X\to X$ let

$$(X,f)=\{(x_i)_{i=0}^{\infty}\mid x_i\in X,\ f(x_{i+1})=x_i,\ i\geqslant 0\}.$$

Define a metric $\tilde{d}$ for (X,f) by

$$\tilde{d}(\tilde{x},\tilde{y})=\sum_{i=0}^{\infty} d(x_i,y_i)/2^i,\quad \text{where } \tilde{x}=(x_i)_{i=0}^{\infty},\ \tilde{y}=(y_i)_{i=0}^{\infty}\in(X,f).$$

Then the space (X,f) is called the *inverse limit* of the map $f\colon X\to X$. Note that (X,f) is a compact metric space. Also, define a map $\tilde{f}\colon (X,f)\to(X,f)$ by

$$\tilde{f}\big((x_i)_{i=0}^{\infty}\big)=\big(f(x_i)\big)_{i=0}^{\infty}\quad (=(x_{i-1})_{i=0}^{\infty}).$$

The map $\tilde{f}$ is a homeomorphism and it is called the *shift homeomorphism* of f. Let $p_n\colon (X,f)\to X_n=X$ be the projection defined by $p_n\big((x_i)_{i=0}^{\infty}\big)=x_n$.

Let X be a compact metric space with metric d and let A be a closed subset of X. A map $f\colon X\to X$ is called *positively expansive* on A if there is a positive number $c>0$ such that if $x,\ y\in A$ and $x\neq y$, then there is a natural number $n\geqslant 0$ such that

$$d\big(f^n(x),f^n(y)\big)>c.$$

Such a positive number c is called a *positively expansive constant* for $f\mid A$. If $f\colon X\to X$ is a positively expansive map on X, then f is called *positively expansive*.

Obviously, this notion is independent of the choice of the metric d of X. An onto map $f\colon X\to X$ is called a *local expansion* if for each point $x\in X$, there is an open set U containing x and a real number $M>1$ so that if $y,z\in U$, then

$$d\big(f(y),f(z)\big)\geqslant M\,d(y,z).$$

The notion of local expansion depends on the metric d of X. But the following is known:

(2.1) (W. Reddy [25, Theorem 1]). *Let $f\colon X\to X$ be an onto map. Then f is positively expansive if and only if f is a local expansion with respect to some metric of X.*

A map $f\colon X\to X$ is called *positively pseudo-expansive* if there exists a finite closed covering $\mathscr{A}=\{A_1,\ldots,A_k\}$ of X such that

(P_1) f is positively expansive on A for each $A\in\mathscr{A}$, and

(P_2) if $A_i,\ A_j\in\mathscr{A}$ and $A_i\cap A_j\neq\emptyset$, one of the following two conditions holds:

($*$) f is positively expansive on $A_i\cup A_j$.

($**$) If f is not positively expansive on $A_i\cup A_j$, then there is a natural number n such that for any $A',\ A''\in\mathscr{A}$ with $A'\cap A''\neq\emptyset$,

$$f^n(A'\cup A'')\cap(A_i-A_j)=\emptyset\quad\text{or}\quad f^n(A'\cup A'')\cap(A_j-A_i)=\emptyset.$$

By the definitions, we can easily see that each positively expansive map is positively pseudo-expansive.

By a *graph* we mean a space which is homeomorphic to a finite 1-dimensional connected polyhedron. Let $p\in G$. Then p is *of order less than or equal to n*, in writing $\mathrm{ord}_p G\leqslant n$, if for any $\varepsilon>0$, there is an open neighborhood U of p in G such that $\operatorname{diam} U<\varepsilon$ and the boundary $\mathrm{Cl}(U)-U$ consists of at most n points. Also, p is *of order n*, in writing $\mathrm{ord}_p G=n$, if $\mathrm{ord}_p G\leqslant n$ but p is not of order less than or equal to $n-1$. Let $\mathrm{ord}\, G=\max\{\mathrm{ord}_p G\mid p\in G\}$. If $p\in G$ and $\mathrm{ord}_p G=1$, p is called an *end point* of G.

(2.2) (J. J. Charatonik and S. Miklos [5, Theorem 1 and Theorem 2]). *Let G be a graph. Then G admits a positively expansive map of G onto itself if and only if there is a point $c\in G$ of maximal order in G (i.e., $\mathrm{ord}_c G=\mathrm{ord}\, G$) such that for every component of $G-\{c\}$ its closure contains a simple closed curve.*

(2.3) Example. Let $G=\{(x,y)\in\boldsymbol{R}^2\mid x^2+y^2=1\}\cup\{(x,0)\mid 1\leqslant x\leqslant 2\}$, where $\boldsymbol{R}$ is the set of real numbers. Then G admits no positively expansive maps, but it admits positively pseudo-expansive maps (see (4.3)).

3. Shift homeomorphisms of tree-like continua are not expansive. A *tree* is a graph containing no simple closed curves. By an ANR (resp. AR) we mean an absolute neighborhood retract (resp. absolute retract). Every 1-dimensional compact AR is a locally connected continuum containing no simple closed curves, and every 1-dimensional compact ANR is locally connected and contains no infinite simple closed curves. Let Y be a continuum. A continuum X is *Y-like* if for any $\varepsilon>0$ there is an onto map $f\colon X\to Y$ such that $\operatorname{diam} f^{-1}(y)<\varepsilon$ for each $y\in Y$. A continuum X is *tree-like* if for any $\varepsilon>0$ there is an onto map $f\colon X\to T$ such that T is a tree and $\operatorname{diam} f^{-1}(t)<\varepsilon$ for any $t\in T$.

(3.1) Remark. Let X be a 1-dimensional compact connected ANR and let $f\colon X\to X$ be a map. Then (X,f) is tree-like if and only if f^n is null-homotopic for some $n\geqslant 1$ (see [4]).

Let $h\colon X\to X$ be an expansive homeomorphism of a continuum X and let $c>0$ be an expansive constant for h. Fix $0<\varepsilon<c/2$. Let

$$W^s_\varepsilon(x) = \{\, y \in X \mid d(h^n(x), h^n(y)) \leqslant \varepsilon \quad \text{for all } n \geqslant 0\},$$
$$W^u_\varepsilon(x) = \{\, y \in X \mid d(h^{-n}(x), h^{-n}(y)) \leqslant \varepsilon \quad \text{for all } n \geqslant 0\}.$$

Then we need the following

(3.2) Lemma ([18, p. 315, Lemma I]). *For all $\gamma > 0$, there exists a natural number $N > 0$ such that*

$$h^n(W^s_\varepsilon(x)) \subset W^s_\gamma(h^n(x)) \quad \text{and} \quad h^{-n}(W^u_\varepsilon(x)) \subset W^u_\gamma(h^{-n}(x))$$

for all $x \in X$ and $n \geqslant N$.

(3.3) Lemma ([18, p. 315, Lemma III]). *There is a nondegenerate subcontinuum A of X such that for some $a \in A$,*

$$A \subset W^s_\varepsilon(a) \quad \text{or} \quad A \subset W^u_\varepsilon(a).$$

(3.4) Lemma ([13, (2.2)]). *There is $\delta > 0$ such that for each nondegenerate subcontinuum B of X, there is a natural number n_0 such that one of the following conditions holds:*

$$(*) \qquad \operatorname{diam} h^n(B) \geqslant \delta \quad \text{for all } n \geqslant n_0.$$

$$(**) \qquad \operatorname{diam} h^{-n}(B) \geqslant \delta \quad \text{for all } n \geqslant n_0.$$

Then we have the following

(3.5) Theorem. *Let T be a tree and let $f\colon T \to T$ be any map. Suppose that (T, f) is nondegenerate. Then the shift homeomorphism $\tilde{f}\colon (T, f) \to (T, f)$ of f is not expansive.*

Proof. Suppose, on the contrary, that $\tilde{f}$ is expansive. Let $c > 0$ be an expansive constant for $\tilde{f}$. Fix $0 < \varepsilon < c/2$. According to (3.3), there is a nondegenerate subcontinuum A of (T, f) such that

$$A \subset W^s_\varepsilon(a) \quad \text{or} \quad A \subset W^u_\varepsilon(a) \qquad \text{for some } a \in A.$$

Put $A_i = p_i(A)$ for $i \geqslant 0$ and $A_{-i} = f^i(A_0)$ for $i \geqslant 1$. Then we have

$$p_i\tilde{f}^n(A) = f^n(A_i) = A_{i-n}, \qquad p_i\tilde{f}^{-n}(A) = A_{i+n} \qquad (n \geqslant 0).$$

Let $\delta > 0$ be as in (3.4). Then we can choose a sufficiently large natural number n_0 and a sufficiently small positive number $\eta = \eta(n_0, \delta)$ such that if E is a subset of (T, f) and $\operatorname{diam} p_{n_0}(E) < \eta$, then $\operatorname{diam} E < \delta$.

Now, we consider the following two cases.

Case (I): $A \subset W^s_\varepsilon(a)$. By (3.2) and (3.4), if B is a nondegenerate subcontinuum of A, there is a natural number $n(B)$ such that

$$\operatorname{diam} \tilde{f}^{-n}(B) \geqslant \delta$$

for all $n \geqslant n(B)$. This implies that

$$\operatorname{diam} B_{n_0+n} = \operatorname{diam} p_{n_0}(\tilde{f}^{-n}(B)) \geqslant \eta \qquad \text{for all } n \geqslant n(B).$$

Now, for any natural number N, we can choose subcontinua $B^1, B^2, \ldots, B^N$ of A such that each B^i is nondegenerate and $B^i \cap B^j = \emptyset$ $(i \neq j)$. Next, we choose a natural number $n(N)$ such that $p_{n(N)}(B^i) \cap p_{n(N)}(B^j) = \emptyset$ $(i \neq j)$. Let n be a natural number such that

$$n \geqslant \max\{n(N), n_0 + n(B^1), \ldots, n_0 + n(B^N)\}.$$

Since $f^{n-n(N)}(B^i_n) = B^i_{n(N)}$ and $B^i_{n(N)} \cap B^j_{n(N)} = \emptyset$ $(i \neq j)$, we see that $B^i_n \cap B^j_n = \emptyset$ $(i \neq j)$. Note that $\operatorname{diam} B^i_n \geqslant \eta$ for each $i = 1, \ldots, N$. Then the tree T has a collection of arbitrarily many mutually disjoint subcontinua of T whose diameters are larger than or equal to η. This is a contradiction.

Case (II): $A \subset W^u_\varepsilon(a)$. First, we shall prove that for some m, $n \geqslant 0$, $f^n|A_m\colon A_m \to T$ is not injective. Suppose, on the contrary, that for any $m, n \geqslant 0$, $f^n|A_m$ is injective. Let N be an arbitrary natural number. As above, choose nondegenerate subcontinua $B^1, \ldots, B^N$ of A such that $B^i \cap B^j = \emptyset$ $(i \neq j)$. Choose a natural number $n(N)$ such that $p_{n(N)}(B^i) \cap p_{n(N)}(B^j) = \emptyset$ $(i \neq j)$. By (3.2) and (3.4), we can easily see that for some natural number n, $\operatorname{diam} f^n(B^i_{n(N)}) \geqslant \eta$ for each $i = 1, \ldots, N$. Then T has the mutually disjoint collection $\{f^n(B^i_{n(N)}) \mid 1 \leqslant i \leqslant N\}$. This is a contradiction. Hence, for some m, $n \geqslant 0$, $f^n|A_m\colon A_m \to T$ is not injective. Choose two points s and t $(s \neq t)$ of A_m with $f^n(s) = f^n(t)$ $(= v)$. Let P be the arc from s to t in A_m. Consider two cases.

Case (i): $f^n(P)$ is a nondegenerate tree. Choose an end point e $(e \neq v)$ of $f^n(P)$ and choose a point c of P such that $f^n(c) = e$. If the component of $(f^n|P)^{-1}(e) = f^{-n}(e) \cap P$ containing c is degenerate, we can easily see that there are two points c' and c'' of P such that $s < c' < c$, $c < c'' < t$ and $f^n(c') = f^n(c'')$ $(\neq e)$, because e is an end point of the tree $f^n(P)$ and $e \neq v$. Moreover, we may assume that c' and c'' are sufficiently near. If the component of $(f^n|P)^{-1}(e)$ containing c is nondegenerate, we can choose two points c' and c'' of P such that $c' \neq c''$ and $f^n(c') = f^n(c'') = e$. Also, we may assume that c' and c'' are sufficiently near.

Case (ii): $f^n(P)$ is degenerate. In this case, we also choose c' and c'' in P as above.

Next, we shall define two points $\tilde{a}$ and $\tilde{b}$ of (T, f) as follows. Since $f(A_{i+1}) = A_i$, we can choose two sequences $a_{m+1}, a_{m+2}, \ldots,$ and $b_{m+1}, b_{m+2}, \ldots,$ of points of T such that

$$f(a_{m+1}) = c', \quad f(a_{j+1}) = a_j \quad (j \geqslant m+1),$$
$$f(b_{m+1}) = c'', \quad f(b_{j+1}) = b_j \quad (j \geqslant m+1).$$

Define $\tilde{a}$ and $\tilde{b}$ in (T, f) by

$$(\tilde{a})_i = \begin{cases} f^{m-i}(c') & \text{if } i < m, \\ c' & \text{if } i = m, \\ a_i & \text{if } i \geqslant m+1, \end{cases} \qquad (\tilde{b})_i = \begin{cases} f^{m-i}(c'') & \text{if } i < m, \\ c'' & \text{if } i = m, \\ b_i & \text{if } i \geqslant m+1. \end{cases}$$

Then $\tilde{a}, \tilde{b} \in A$. Since $A \subset W^u_\varepsilon(a)$, $\tilde{d}(\tilde{f}^{-n}(\tilde{a}), \tilde{f}^{-n}(\tilde{b})) < 2\varepsilon$ for $n \geqslant 0$. Since we can choose c' and c'' to be near, we may assume that $\tilde{d}(\tilde{f}^n(\tilde{a}), \tilde{f}^n(\tilde{b})) < 2\varepsilon$ for all $n \geqslant 0$. This is a contradiction.

(3.6) Theorem. *Let X be a 1-dimensional compact connected ANR and let $f\colon X \to X$ be a map. Suppose that (X, f) is nondegenerate. If f is null-homotopic, then the shift homeomorphism $\tilde{f}\colon (X, f) \to (X, f)$ of f is not expansive.*

Outline of proof. Suppose, on the contrary, that $\tilde{f}$ is expansive and let $c > 0$ be an expansive constant for $\tilde{f}$. Fix $0 < \varepsilon < c/2$. Since X is a 1-dimensional compact ANR, for any $\eta > 0$ we can choose a natural number $n = n(\eta)$ such that if $A^1, \ldots, A^n$ are subcontinua of X and $\operatorname{diam} A^i \geqslant \eta$ for each $i = 1, \ldots, n$, then $A^i \cap A^j \neq \emptyset$ for some i and j $(i \neq j)$. In the same way as in the proof of (3.5), we can choose a nondegenerate subcontinuum A of (X, f) such that $A \subset W_\varepsilon^u(a)$ for some $a \in A$. Also, we can choose $m, n \geqslant 0$ such that $f^n | A_m$: $A_m \to X$ is not injective. Take the universal covering p: $\tilde{X} \to X$ of X. Since f is null-homotopic, there is a lifting g: $X \to \tilde{X}$ of f, i.e., $pg = f$. Note that $gf^n | A_m$ is not injective and $g(X)$ is a 1-dimensional compact AR. In the same way as in the proof of (3.5), we can choose two points c' and c'' of A_m such that $c' \neq c''$, c' and c'' are sufficiently near and $gf^n(c') = gf^n(c'')$, which implies $f^{n+1}(c') = f^{n+1}(c'')$. The rest of the proof is as for (3.5).

(3.7) Corollary. *Let f: $X \to X$ be a map of a 1-dimensional compact connected ANR X. Suppose that (X, f) is nondegenerate and the shift homeomorphism $\tilde{f}$ of f is expansive. Then there is a sequence $\{A_i\}_{i=n_0}^{\infty}$ of arcs in X such that*

(1) *f is positively expansive on A_{n_0} and $f^n(A_{n_0})$ contains a simple closed curve for each $n \geqslant 1$,*

(2) *$f(A_{i+1}) = A_i$ $(i \geqslant n_0)$ and $f | A_{i+1}$ is injective, and*

(3) $\lim_{i \to \infty} \operatorname{diam} A_i = 0$.

Outline of proof. In the same way as in the proof of (3.6), there is a nondegenerate subcontinuum A of (X, f) such that $A \subset W_\varepsilon^u(a)$ for some $a \in A$. We may assume that for some n_0, $p_{n_0}(A)$ is a tree. By the proof of (3.6), $f^{n-n_0} | p_n(A)$: $p_n(A) \to p_{n_0}(A)$ is injective $(n \geqslant n_0)$. Also, we may assume that A_{n_0} is an arc and $f^n(A_{n_0})$ contains a simple closed curve. Clearly, $A \subset W_\varepsilon^u(a)$ implies (3).

(3.8) Remark. Let S be the unit circle. If f: $S \to S$ is a map such that the shift homeomorphism $\tilde{f}$ of f is expansive, then there is an integer p with $|p| \geqslant 2$ such that f is isotopic to the $\times p$ map g: $S \to S$, i.e., $g(x) = px$. Hence (S, f) is homeomorphic to the $|p|$-adic solenoid (see (3.7)).

(3.9) Remark. In [6], M. Dateyama proved that the shift homeomorphisms of arc-like continua are not expansive. His proof is different from our proof of (3.5) and deeply depends on the structure of the interval $[0, 1]$.

4. Expansiveness and positively pseudo-expansive maps.

In this section, we investigate the relations between positively pseudo-expansive maps and expansiveness of shift homeomorphisms.

First, we prove the following

(4.1) Theorem. *If f: $X \to X$ is a positively pseudo-expansive map of a compact metric space X, then the shift homeomorphism $\tilde{f}$ of f is expansive.*

Proof. Since f is a positively pseudo-expansive map, there is a closed finite covering $\mathscr{A}$ satisfying the conditions (P_1) and (P_2). Choose $c > 0$ such that

$$0 < c < \min\{d(A_i, A_j) |\ A_i, A_j \in \mathscr{A} \text{ and } A_i \cap A_j = \emptyset\}.$$

Also, we may assume that for each $A \in \mathscr{A}$, c is a positively expansive constant for $f | A$ and if $A_i, A_j \in \mathscr{A}$, $A_i \cap A_j \neq \emptyset$ and f is positively expansive on $A_i \cup A_j$, then c is also a positively expansive constant for $f | A_i \cup A_j$. We shall prove that c is an expansive constant for $\tilde{f}$: $(X, f) \to (X, f)$. Let $\tilde{x} = (x_i)_{i=0}^{\infty}$, $\tilde{y} = (y_i)_{i=0}^{\infty} \in (X, f)$ and $\tilde{x} \neq \tilde{y}$. Choose m such that $x_m \neq y_m$. Consider the following cases.

Case (I): $x_m \in A' \in \mathscr{A}$, $y_m \in A'' \in \mathscr{A}$ and $A' \cap A'' = \emptyset$.

Then we have

$$\tilde{d}(\tilde{f}^{-m}(\tilde{x}), \tilde{f}^{-m}(\tilde{y})) = \tilde{d}((x_m, x_{m+1}, \ldots), (y_m, y_{m+1}, \ldots)) \geqslant d(x_m, y_m) > c.$$

Case (II): $x_m, y_m \in A \in \mathscr{A}$. Since f is positively expansive on A, there is $n \geqslant 0$ such that $d(f^n(x_m), f^n(y_m)) > c$. Then

$$\tilde{d}(\tilde{f}^{n-m}(\tilde{x}), \tilde{f}^{n-m}(\tilde{y})) = \tilde{d}((\tilde{f}^n(x_m, x_{m+1}, \ldots), \tilde{f}^n(y_m, y_{m+1}, \ldots)) \geqslant d(f^n(x_m), f^n(y_m)) > c.$$

Case (III): $x_m \in A_i \in \mathscr{A}$, $y_m \in A_j \in \mathscr{A}$, $x_m \in A_i - A_j$ and $y_m \in A_j - A_i$. If f is positively expansive on $A_i \cup A_j$, the proof is the same as in case (II).

Suppose that f is not positively expansive on $A_i \cup A_j$. Let n be a natural number satisfying (P_2). Choose $A', A'' \in \mathscr{A}$ such that $x_{m+n} \in A'$ and $y_{m+n} \in A''$. By (P_2), $A' \cap A'' = \emptyset$, which implies that $d(x_{m+n}, y_{m+n}) > c$. Hence we have

$$\tilde{d}(\tilde{f}^{-(m+n)}(\tilde{x}), \tilde{f}^{-(m+n)}(\tilde{y})) = \tilde{d}((x_{m+n}, x_{m+n+1}, \ldots), (y_{m+n}, y_{m+n+1}, \ldots)) \geqslant d(x_{m+n}, y_{m+n}) > c.$$

This completes the proof.

(4.2) Corollary. *If an onto map f: $G \to G$ of a graph G is null-homotopic, then it is not positively pseudo-expansive. In particular, no trees admit positively pseudo-expansive maps.*

By (2.2), a graph containing a simple closed curve does not always admit a positively expansive map (see (2.3)). But we have

(4.3) Theorem. *Every graph G containing a simple closed curve admits positively pseudo-expansive maps.*

Outline of proof. Consider the following condition (#): For any point x of a graph G' and for every component of $G' - \{x\}$, its closure contains a simple closed curve.

Let G be a graph containing a simple closed curve. Choose a maximal subgraph G' of G such that G' satisfies (#). By (2.2), there is a positively expansive map f: $G' \to G'$. Note that $G - G' = \bigcup X_k$, where each X_k is a component of $G - G'$. Then $\mathrm{Cl}(X_k)$ may be written as a union of $I_{k,s}$, where each $I_{k,s}$ is a maximal interval in X_k with one end e_s belonging to $Z_{k,s} = G' \cup \bigcup_{t<s} I_{k,t}$ and the rest disjoint from $Z_{k,s}$.

We can suppose that a positively pseudo-expansive map f is defined on $Z_{k,s}$ and

$f(e_s) \neq e_s$. Then define f on $I_{k,s}$ so that the image covers $I_{k,s'}$, the interval joining e_s to $f(e_s)$ is in $Z_{k,s}$ and $f \mid I_{k,s}$ is positively expansive. This is done by induction. Then make the appropriate division. If we do this carefully, we obtain the desired positively pseudo-expansive map $f\colon G \to G$.

(4.4) COROLLARY. *For any graph G containing a simple closed curve, there is a continuum X such that X is G-like and admits expansive homeomorphisms.*

(4.5) PROPOSITION. *If $f\colon G \to G$ is a positively expansive map of a graph G, then the inverse limit (G, f) of f is not movable* (see [2] or [19] for the definition of movable). *In particular, (G, f) cannot be embedded in the plane $\boldsymbol{R}^2$.*

Proof. By (2.2), G contains a simple closed curve. $H = \pi_1(G)$ is a finitely generated free group whose base is $\{a_1, \ldots, a_n\}$.

For each element x of H, write $x = b_1^{\alpha_1} \circ \ldots \circ b_k^{\alpha_k}$, where $b_i \in \{a_1, \ldots, a_n\}$, $b_i \neq b_{i+1}$ and $\alpha_i \in \boldsymbol{Z}$. Set $L(x) = \sum_{j=1}^{k} |\alpha_j|$. Since f is a local expansion with respect to some metric, there is a natural number m such that

$$L(\pi_1(f^m)(x)) \geqslant 2L(x) \quad \text{for each } x \in H$$

(consider a locally injective map $g\colon S \to G$ which represents x, and note that $f^m \circ g$ is locally injective).

Now, suppose, on the contrary, that (G, f) is movable. Then there is $n_0 \geqslant 0$ such that for any n, there is a map $h\colon G \to G$ such that $f^{n_0} \simeq f^{mn} \circ h$ (see [19, p. 159 and 183]).

Then $L(\pi_1(f^{n_0})(x)) \geqslant 2^n L(\pi_1(h)(x))$ for each x. This is a contradiction. By Borsuk's theorem [2], every plane continuum is movable. Hence (G, f) cannot be embedded in $\boldsymbol{R}^2$.

(4.6) PROPOSITION. *Let $f\colon G \to G$ be a map of a graph G such that for any arc A in G, there is a natural number $n > 0$ such that $f^n(A) = G$. Then the shift homeomorphism $\tilde{f}$ of f is expansive if and only if f is positively pseudo-expansive.*

Proof. Suppose that $\tilde{f}$ is expansive. We must show that f is positively pseudo-expansive. By (3.5), G contains a simple closed curve S. If $G = S$, by (3.7) we see that f is positively expansive. Now, assume that $G \neq S$ and $G = |K|$, where K is a 1-dimensional simplicial complex. Put

(1) $Z = \{p \in G \mid \operatorname{ord}_p G \geqslant 3\}$ $(\neq \emptyset)$, and

(2) $k = {}^{\#}\{(p, e, e') \mid p \in \boldsymbol{Z}, e \text{ and } e' \text{ are edges of } K \text{ such that } e \neq e' \text{ and } e \cap e' = \{p\}\}$, where ${}^{\#}A$ denotes the cardinal number of a set A.

By (3.7), there is an arc $A' = [a', b']$ in G as in (3.7). Choose an arc $A = [a, b]$ in $[a', b'] - \{a', b'\}$.

By the hypothesis, $f^m(A) = G$ for some m. Then we can choose a finite sequence $a \leqslant a_1 \leqslant \ldots \leqslant a_N \leqslant b$ of points of A such that $K^* = \{f^m(a_i), f^m(\langle a_i, a_{i+1} \rangle) \mid 0 \leqslant i \leqslant N\}$ is a subdivision of K. We may assume that if $\{e_j\}$ is any set of edges of K^* and $\bigcap e_j \neq \emptyset$, then for $n = 1, \ldots, k$, there is a vertex v of K such that $f^n(\bigcup e_j) \subset \operatorname{St}(v; K)$, where $\operatorname{St}(v; K)$ denotes the open star of v in K. Note that

(3) for any edge e of K^*, there is a sequence $e = e^1, e^2, \ldots$ of arcs in G such that $f \mid e^{i+1}\colon e^{i+1} \to e^i$ is a homeomorphism and $\lim_{i \to \infty} \operatorname{diam} e^i = 0$.

Set $\mathscr{A} = \{e \mid e \text{ is an edge of } K^*\}$. Clearly, $\mathscr{A}$ satisfies (P_1). We shall show that $\mathscr{A}$ satisfies (P_2). Suppose that $e_1, e_2 \in \mathscr{A}$ $(e_1 \neq e_2)$ such that $e_1 \cap e_2 \neq \emptyset$ and f is not positively expansive on $e_1 \cup e_2$. Let $e_1 \cap e_2 = \{p\}$. Clearly, $f^j \mid e_1 \cup e_2$ is not locally injective at p for some $j > 0$.

Suppose, on the contrary, that there are $e_1', e_2' \in \mathscr{A}$ $(e_1' \neq e_2')$ such that $e_1' \cap e_2' \neq \emptyset$ and

$$f^k(e_1' \cup e_2') \cap (e_1 - e_2) \neq \emptyset \neq f^k(e_1' \cup e_2') \cap (e_2 - e_1). \tag{4}$$

Since $\tilde{f}$ is expansive, by (3) and (4), we see that $f^k(p') = p$ (see the proof of (3.5) and (3.6)), where $e_1' \cap e_2' = \{p'\}$. Also, $f^j(p') \in \boldsymbol{Z}$ and $f^j \mid e_1' \cup e_2'$ is locally injective at p' for $0 \leqslant j \leqslant k$. By (2), we conclude that for any $j \geqslant 0$, $f^j \mid e_1' \cup e_2'$ is locally injective at p'. This is a contradiction. Hence (P_2) is true for $n = k$. This completes the proof.

5. Remarks and problems. In Section 4, we showed that the notion of positively pseudo-expansive map is very useful and important for constructing various types of expansive homeomorphisms. Also, it is known that "Plykin's attractors" are 1-dimensional continua in the plane $\boldsymbol{R}^2$ and are examples of Williams' 1-dimensional expanding attractors, whose homeomorphisms are not only expansive homeomorphisms but even hyperbolic diffeomorphisms (see [21], [23], [24] and [28]). Note that Plykin's attractors can be represented as inverse limits of maps $g\colon K \to K$ of graphs K (see [23], p. 243] and [24, p. 121]). In fact, we can easily check that the maps $g\colon K \to K$ are positively pseudo-expansive. Hence we see that for each $n \geqslant 3$, there exists a graph G_n and a positively pseudo-expansive map $f_n\colon G_n \to G_n$ such that the inverse limit (G_n, f_n) of f_n is homeomorphic to a plane continuum X_n and $\boldsymbol{R}^2 - X_n$ has $n+1$ components. We refer the reader to [23] and [24] for properties of Plykin's attractors.

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We close this paper with the following problems.

PROBLEM 1. For each $n = 0, 1, 2$, is it true that there is a plane continuum X_n such that X_n admits an expansive homeomorphism and $\boldsymbol{R}^2 - X_n$ has $n+1$ components?

PROBLEM 2. Let $f\colon G \to G$ be an onto map of a graph G. Is it true that the shift homeomorphism $\tilde{f}$ of f is expansive if and only if f is a positively pseudo-expansive map? (Note that (4.6) gives a partial positive solution to Problem 2.)

Added in proof. Recently, the author proved that Problem 2 has an affirmative answer.

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