all codes k of those pairs (τ, I) for which $|I| = m$. We put $m(k) = |I|$ and $m(k) = 0$ for $k \notin \bigcup K_m$. (Of course $m(k) = 0$ also if $|I| = 0$.) So there is a natural bijection between S_0 and the set B defined prior to Theorem 3 with $\mathfrak{A} = \mathfrak{A}_0$. Let us identify S_0 with B via this bijection. So we can write $\mathfrak{M}_0 = \langle B, R_0, R_1, \ldots \rangle$.

Now we have to show that all the relations R_i have the form prescribed prior to Theorem 3. Let $\langle (k(i, j, 1), \ldots, k(i, j, n(i))) \colon j = 0, 1, \ldots \rangle$ be an enumeration of all $n(i)$-tuples of integers. Let

$$R_{ij} = R_i \cap \left(\{k(i, j, 1)\} \times A_<^{m(k(i,j,1))} \cup \ldots \cup \{k(i, j, n(i))\} \times A_<^{k(i,j,n(i))}\right).$$

Then, if we look again at the meaning of the codes $k \in \bigcup K_m$ and we use the fact that $\mathfrak{M}_0$ satisfies the axioms (*), by Lemma (i) it is clear that R_{ij} is of the form required prior to Theorem 3, with a formula φ_{ij} in the language of $\mathfrak{A}_0$ with $\sum_{r=1}^{n(i)} m(k(i, j, r))$ variables.

This concludes our proof that $\mathfrak{M}_0$ is an $\mathfrak{A}_0$-model of T. The inequality $K_1 \neq \emptyset$ follows from the fact that K_1 must contain a code of the pair $(x_1, \{1\})$.

Now let $\mathfrak{A}$ be an arbitrary dense linear order without endpoints. The same functions $k(i, j, r)$, $m(k)$ and formulas φ_{ij} which we found for $\mathfrak{M}_0$ yield a certain $\mathfrak{A}$-model $\mathfrak{N}$ with $K_1 \neq \emptyset$. It remains to check that $\mathfrak{N}$ satisfies T. But it is clear that every finite part of $\mathfrak{N}$ is isomorphic to some finite part of $\mathfrak{M}_0$. Since $\mathfrak{M}_0 \models T^*$ and the axioms of T^* are universal, $\mathfrak{N} \models T^*$. Since $T^* \vdash T$, the proof is complete.

Note added in July 1989. After this paper was written the authors learned that the problem of existence of Borel models was independently posed and solved by H. Friedman, see

[a] C. I. Steinhorn, *Borel structures for first order and extended logics*, in: *Harvey Friedman Research in the Foundations of Mathematics*, L. A. Harrington et al. (eds.), Elsevier Science Publishers B. V. (North-Holland), 1985, 161–178.

[b] — *Borel structures and measure and category logics*, in: *Model-Theoretic Logics*, J. Barwise and S. Feferman (eds.), Springer, New York 1985, 579–596.

The proofs of the existence of Borel models presented in those papers are closely related to ours, but we decided to keep Theorem 2 and its proof because it gives a sharper estimate of the Borel classes of the relations and because the concrete structure of the model described in Theorem 3 may be of independent interest. The papers [a] and [b] discuss additional aspects and extensions of Theorem 2 but its proofs given there are not as detailed as ours.

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The shrinking property of products of cardinals

by

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Abstract. It is known that for cardinals $\kappa > \omega$ and $\lambda > 1$, κ^λ is normal if and only if κ is regular and $\lambda < \kappa$. We show that normality can be replaced by the shrinking property in this result.

Ordinals and cardinals are considered as sets of smaller ordinals. In particular, $n = \{0, 1, \ldots, n-1\}$ for each $n \in \omega$. Let $\{\chi_\alpha \colon \alpha \in \lambda\}$ be a collection of spaces. $\prod_{\alpha\in\lambda} X_\alpha$ denotes the usual Tikhonov product space of X_α's. Each element f of $\prod_{\alpha\in\lambda} X_\alpha$ is considered as a function whose domain is λ and $f(\alpha)$ is in X_α for each $\alpha \in \lambda$. Whenever X_α is a single space X for each $\alpha \in \lambda$, $\prod_{\alpha\in\lambda} X_\alpha$ is denoted by X^λ.

Let X be a space and let κ be a cardinal. Assume $\mathscr{U}$ is an open cover of X. A cover $\mathscr{V} = \{V(U) \colon U \in \mathscr{U}\}$ is said to be a *shrinking* of $\mathscr{U}$ if $\operatorname{cl} V(U) \subset U$ for each $U \in \mathscr{U}$. In particular, $\mathscr{V}$ is said to be an *open* (*closed*) *shrinking* of $\mathscr{U}$ if each member of $\mathscr{V}$ is open (closed, respectively). X is said to have the *κ-shrinking property* if every open cover of size $\leqslant \kappa$ has an open shrinking. A space has the *shrinking property* if it has the κ-shrinking property for every infinite cardinal κ. Note that 2-shrinking property is normality and that ω-shrinking property is countable paracompactness plus normality. It is easy to show that a normal space which has the property that every open cover of size $\leqslant \kappa$ has a closed shrinking has the κ-shrinking property. Note that paracompact spaces, in particular compact Hausdorff spaces and regular Lindelöf spaces, have the shrinking property. On the other hand, ω_1 with the order topology has the shrinking property but is not paracompact. In general, ordered spaces have the shrinking property, see [Ke]. But the product space $\omega_1 \times (\omega_1 + 1)$ does not have the shrinking property, in fact it is not normal, see [Pr, 2.2]. But note that it is countably paracompact since it is a perfect preimage of the countably paracompact space ω_1. Note that κ-shrinking property implies normality if $\kappa \geqslant 2$. It is strangely difficult to find an example of a normal space without the κ-shrinking property for $\kappa \geqslant \omega$. For each $\kappa \geqslant \omega$, we know of essentially one real such example, namely the κ-Dowker space, see [Ru1], [Ru2].

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It was known that ordered spaces are normal (and countably paracompact), see [En]. But now it is known that they have the shrinking property, see [Ke]. Similarly it was known that Σ-products of metric spaces are normal, see [Pr, 7.4] or [Gu]. But now it is known that they also have the shrinking property, see [Ru3]. It is known that for cardinals $\kappa > \omega$ and $\lambda > 1$, κ^λ is normal if and only if κ is regular and $\lambda < \kappa$, see [Pr, 6.7]. In this paper, we shall show that normality can be replaced by the shrinking property in this result. Our main theorem is the following;

THEOREM. *Let κ be a regular infinite cardinal. Then κ^n has the shrinking property for every $n \in \omega$.*

Proof. If $\kappa = \omega$, then κ^n is a metric (paracompact) space. Thus assume $\kappa \geqslant \omega_1$. Since $|\kappa^n| \leqslant \kappa$, it suffices to show that κ^n has the κ-shrinking property (cf. [Pr, 6.8]). We shall show this by induction on $n \in \omega$. For $n = 0$, κ^n clearly has the κ-shrinking property. Assume κ^{n-1} has the κ-shrinking property. First we shall show;

FACT 1. *For every $\delta \in \kappa$ and $j \in n$, $Y_{j\delta} = \kappa^j \times [0, \delta]^{n-j}$ has the κ-shrinking property.*

Proof of Fact 1. Here $[0, \delta]$ denotes the usul closed interval with end points $0, \delta$. Similarly (α, β), $(\alpha, \beta]$, ... denote the open interval, half open interval, .. with end points α and β. We shall show Fact 1 by induction on $j \in n$. Since $Y_{0\delta}$ is homeomorphic to the compact space $[0, \delta]^n$, $Y_{0\delta}$ has the κ-shrinking property. Next assume that $j \in n$ and that for every $j' \in j$ and $\delta \in \kappa$, $Y_{j'\delta}$ has the κ-shrinking property. To show that $Y_{j\delta}$ has the κ-shrinking property for every $\delta \in \kappa$, assume that $\mathscr{U} = \{U_\gamma \colon \gamma \in \kappa\}$ is an open cover of $Y_{j\delta}$; it suffices to find its closed shrinking since κ^n is normal.

For each $\alpha \in \kappa$, define $\hat{\alpha} \in \kappa^j$ by $\hat{\alpha}(i) = \alpha$ for each $i \in j$. Since $\langle \hat{\alpha}, g \rangle \in Y_{j\delta}$, fix a $\gamma(\alpha, g) \in \kappa$ such that $\langle \hat{\alpha}, g \rangle \in U_{\gamma(\alpha, g)}$, for each $\alpha \in \kappa$ and $g \in [0, \delta]^{n-j}$. Moreover, by the openness of $U_{\gamma(\alpha, g)}$, fix a $\beta(\alpha, g) \in \alpha$ and $h(\alpha, g) \in [0, \delta]^{n-j}$ with $h(\alpha, g) < g$ such that

$$(\beta(\alpha, g), \alpha]^j \times \prod_{i \in n-j} (h(\alpha, g)(i), g(i)] \subset U_{\gamma(\alpha, g)}$$

for each $\alpha \in \kappa$ and $g \in [0, \delta]^{n-j}$. Here $h < g$ means $h(i) < g(i)$ for each $i \in n-j$ whenever $h, g \in [0, \delta]^{n-j}$. For a fixed $g \in [0, \delta]^{n-j}$, since $\beta(\alpha, g) \in \alpha$ for each $\alpha \in \kappa$, by the pressing down lemma, we can find a stationary set $S''(g) \subset \kappa$ and a $\beta(g) \in \kappa$ such that $\beta(\alpha, g) = \beta(g)$ for each $\alpha \in S''(g)$. Furthermore, since $|\{h \in [0, \delta]^{n-j} \colon h < g\}| < \kappa$, there is an $h(g) < g$ and a stationary set $S'(g) \subset S''(g)$ such that $h(\alpha, g) = h(g)$ for each $\alpha \in S'(g)$. This means

$$(*) \qquad (\beta(g), \alpha]^j \times \prod_{i \in n-j} (h(g)(i), g(i)] \subset U_{\gamma(\alpha, g)}$$

for each $\alpha \in S'(g)$. Note that for each $g \in [0, \delta]^{n-j}$, $V_g = \prod_{i \in n-j} (h(g)(i), g(i)]$ is a clopen set of $[0, \delta]^{n-j}$ and $g \in V_g$. Since $\{V_g \colon g \in [0, \delta]^{n-j}\}$ is an open cover of the compact set $[0, \delta]^{n-j}$, there is a finite subset $G \subset [0, \delta]^{n-j}$ such that $\{V_g \colon g \in G\}$ covers $[0, \delta]^{n-j}$. Put $\beta = \max\{\beta(g) \colon g \in G\}$. Then $S(g) = S'(g) - [0, \beta]$ is stationary for each $g \in G$. Then by $(*)$,

$$(\beta, \alpha]^j \times V_g \subset U_{\gamma(\alpha, g)} \quad \text{for each} \quad \alpha \in S(g) \text{ with } g \in G.$$

For α and $\alpha' \in S(g)$ with $g \in G$, define $\alpha \simeq \alpha'$ by $\gamma(\alpha, g) = \gamma(\alpha', g)$. Then clearly $\simeq$ is an equivalence relation on $S(g)$. Let $S(g)/\simeq$ be the quotient of $S(g)$ by $\simeq$. And for each $E \in S(g)/\simeq$, define $\gamma(E) = \gamma(\alpha, g)$ for some (in fact every) $\alpha \in E$. Then clearly $\gamma(E) \neq \gamma(E')$ whenever E and $E' \in S(g)/\simeq$ with $E \neq E'$. Then $\big(\bigcup_{\alpha \in E} (\beta, \alpha]^j\big) \times V_g \subset U_{\gamma(E)}$ holds and $\{\bigcup_{\alpha \in E} (\beta, \alpha]^j \colon E \in S(g)/\simeq\}$ is an open cover of $(\beta, \kappa)^j$ for each $g \in G$. Note that $(\beta, \kappa)^j$ has the κ-shrinking property, since $(\beta, \kappa)^j$ is a clopen subspace of κ^j and κ^j has the κ-shrinking property by the inductive assumption and $j \leqslant n-1$. Thus there is a closed cover $\{F'_{\gamma(E)} \colon E \in S(g)/\simeq\}$ of $(\beta, \kappa)^j$ such that $F'_{\gamma(E)} \subset \bigcup_{\alpha \in E} (\beta, \alpha]^j$ for each $E \in S(g)/\simeq$. Define $F_{\gamma(E)} = F'_{\gamma(E)} \times V_g$ for each $E \in S(g)/\simeq$. Then for each $g \in G$, $\mathscr{F}_g = \{F_{\gamma(E)} \colon E \in S(g)/\simeq\}$ is a collection of closed sets in $Y_{j\delta}$ such that $F_{\gamma(E)} \subset U_{\gamma(E)}$ for each $E \in S(g)/\simeq$. Note that $\mathscr{F}_g$ covers $(\beta, \kappa)^j \times V_g$ for each $g \in G$. For each $\gamma \in \kappa$, define

$$F_\gamma = \bigcup_{g \in G} \{F_{\gamma(E)} \colon E \in S(g)/\simeq \text{ and } \gamma(E) = \gamma\}.$$

Then each F_γ is closed in $Y_{j\delta}$ and contained in U_γ. Furthermore, since each $\mathscr{F}_g$ covers $(\beta, \kappa)^j \times V_g$, $\{F_\gamma \colon \gamma \in \kappa\}$ covers $\bigcup_{g \in G} (\beta, \kappa)^j \times V_g = (\beta, \kappa)^j \times [0, \delta]^{n-j}$. Next put

$$Z_i = \kappa^i \times [0, \beta] \times \kappa^{j-(i+1)} \times [0, \delta]^{n-j} \quad \text{for each } i \in j.$$

Then $Y_{j\delta} - (\beta, \kappa)^j \times [0, \delta]^{n-j} = \bigcup_{i \in j} Z_i$. Since Z_i is homeomorphic to $\kappa^{j-1} \times [0, \beta] \times [0, \delta]^{n-j}$, by putting $\delta' = \max\{\beta, \delta\}$, Z_i is homeomorphic to a closed subspace of

$$Y_{j-1, \delta'} = \kappa^{j-1} \times [0, \delta']^{n-(j-1)}.$$

Since $Y_{j-1, \delta'}$ has the κ-shrinking property by the inductive assumption, Z_i also has the κ-shrinking property for each $i \in j$.

Thus there is a closed cover $\{H_{\gamma i} \colon \gamma \in \kappa\}$ of Z_i such that $H_{\gamma i} \subset U_\gamma$ for each $\gamma \in \kappa$. Then $\{(\bigcup_{i \in j} H_{\gamma i}) \cup F_\gamma \colon \gamma \in \kappa\}$ is a closed shrinking of $\mathscr{U}$. Thus the proof of Fact 1 is complete.

To show κ^n has the κ-shrinking property, assume $\mathscr{U} = \{U_\gamma \colon \gamma \in \kappa\}$ be an open cover of κ^n. As above, for each $\alpha \in \kappa$, define $\hat{\alpha} \in \kappa^n$ by $\hat{\alpha}(i) = \alpha$ for each $i \in n$. For each $\alpha \in \kappa$, fix a $\gamma(\alpha) \in \kappa$ such that $\hat{\alpha} \in U_{\gamma(\alpha)}$ and fix a $\beta(\alpha) \in \alpha$ such that $(\beta(\alpha), \alpha]^n \subset U_{\gamma(\alpha)}$. Then by the pressing down lemma, we can find a stationary set $S \subset \kappa$ and a $\beta \in \kappa$ such that $\beta(\alpha) = \beta$ for each $\alpha \in S$. Define $S_\gamma = \{\alpha \in S \colon \gamma(\alpha) = \gamma\}$ for each $\gamma \in \kappa$. Then $S = \bigcup_{\gamma \in \kappa} S_\gamma$. We shall show the next fact.

FACT 2. *There is a collection $\{H_\gamma \colon \gamma \in \kappa\}$ of closed sets such that $H_\gamma \subset U_\gamma$ for each $\gamma \in \kappa$ and $\bigcup_{\gamma \in \kappa} H_\gamma = (\beta, \kappa)^n$.*

Proof of Fact 2. There are two cases.

Case 1. *There is a $\gamma \in \kappa$ such that S_γ is unbounded in κ.*

In this case, $(\beta, \kappa)^n \subset U_\gamma$. To show this, let g be in $(\beta, \kappa)^n$. Since S_γ is unbounded, take an $\alpha \in S_\gamma$ with $\max\{g(i) \colon i \in n\} \in \alpha$. Then $g \in (\beta, \alpha]^n \subset U_\gamma$. Thus $(\beta, \kappa)^n \subset U_\gamma$. Put $H_\gamma = (\beta, \kappa)^n$ and $H_{\gamma'} = 0$ if $\gamma' \in \kappa$ with $\gamma' \neq \gamma$. Then clearly $\{H_\gamma \colon \gamma \in \kappa\}$ is the desired collection of closed sets.

Case 2. *S_γ is bounded in κ for each $\gamma \in \kappa$.*

In this case first we shall construct two strictly increasing sequences $\{\gamma(\theta) \colon \theta \in \kappa\}$

and $\{\delta(\theta): \theta \in \kappa\}$ in κ such that $\delta(\theta) \in S_{\gamma(\theta)}$ for each $\theta \in \kappa$, by induction on $\theta \in \kappa$. First define $\gamma(0) = \min\{\gamma \in \kappa: S_\gamma \neq 0\}$, and fix $\delta(0) \in S_{\gamma(0)}$. Next assume that $\gamma(\theta')$ and $\delta(\theta')$ have already been defined for every $\theta' \in \theta$. Put $\gamma_0 = \sup\{\gamma(\theta'): \theta' \in \theta\} + 1$. Since each S_γ is bounded in κ, $\sup(\bigcup\{S_\gamma: \gamma \in \gamma_0\}) \in \kappa$. Thus there is a $\gamma(\theta) \in \kappa$ such that $S_{\gamma(\theta)} - \sup(\bigcup\{S_\gamma: \gamma \in \gamma_0\}) \neq 0$. Note that $\gamma(\theta') \in \gamma(\theta)$ for every $\theta' \in \theta$, since $\gamma_0 \leqslant \gamma(\theta)$. Fix a $\delta(\theta) \in S_{\gamma(\theta)} - \sup(\bigcup\{S_\gamma: \gamma \in \gamma_0\})$. Then clearly also $\delta(\theta') \in \delta(\theta)$ for each $\theta' \in \theta$. Thus we have constructed the desired sequences. For each $\theta \in \kappa$, put $H_{\gamma(\theta)} = (\beta, \delta(\theta)]^n$. Then since $\delta(\theta) \in S_{\gamma(\theta)}$ (thus $\gamma(\delta(\theta)) = \gamma(\theta)$) we have $H_{\gamma(\theta)} \subset U_{\gamma(\delta(\theta))} = U_{\gamma(\theta)}$. For each $\gamma \in \kappa - \{\gamma(\theta): \theta \in \kappa\}$, put $H_\gamma = 0$. Since $\{\delta(\theta): \theta \in \kappa\}$ is unbounded in κ,

$$\bigcup_{\gamma \in \kappa} H_\gamma = \bigcup_{\theta \in \kappa} H_{\gamma(\theta)} = \bigcup_{\theta \in \kappa} (\beta, \delta(\theta)]^n = (\beta, \kappa)^n.$$

Thus $\{H_\gamma: \gamma \in \kappa\}$ is the desired collection of closed sets. This completes the proof of Fact 2.

Finally, for each $j \in n$, define $Z_j = \kappa^j \times [0, \beta] \times \kappa^{n-(j+1)}$. Then each Z_j is clopen in κ^n and $\kappa^n - (\beta, \kappa)^n = \bigcup_{j \in n} Z_j$. Moreover, since each Z_j is homeomorphic to $Y_{n-1,\beta} = \kappa^{n-1} \times [0, \beta]$, each Z_j has the κ-shrinking property by Fact 1. Therefore for each $j \in n$, there is a collection $\{F_{\gamma j}: \gamma \in \kappa\}$ of closed sets in κ^n such that $F_{\gamma j} \subset U_\gamma$ for each $\gamma \in \kappa$ and $\bigcup_{\gamma \in \kappa} F_{\gamma j} = Z_j$. Then it is easy to show that $\{H_\gamma \cup (\bigcup_{j \in n} F_{\gamma j}): \gamma \in \kappa\}$ is a closed shrinking of $\mathcal{U}$. This completes the proof of the theorem.

Remark. By putting $U_0 \cup U_1 = \kappa^n$ and $U_\gamma = 0$ for $2 \leqslant \gamma \in \kappa$, the above proof shows the normality of κ^n. But in this case, Case 2 of Fact 2 cannot happen.

LEMMA ([Be, 3.4]). *A normal product $\prod_{\alpha \in \lambda} X_\alpha$ has the (κ-) shrinking property if and only if $\prod_{\alpha \in S} X_\alpha$ has the (κ-) shrinking property for every finite $S \subset \lambda$.*

Using this lemma, we can show:

COROLLARY. *For cardinals $\kappa > \omega$ and $\lambda > 1$, the following are equivalent:*

(i) *κ^λ has the (κ-) shrinking property,*

(ii) *κ is regular and $\lambda < \kappa$.*

Proof. (i) ⇒ (ii). If κ^λ has the κ-shrinking property, then κ^λ is normal. Thus this follows from [Pr, 6.7].

(ii) ⇒ (i). Assume that κ is regular and $\lambda < \kappa$. Then by [Pr, 6.7], κ^λ is normal. Furthermore, applying the above theorem and lemma, we can show that κ^λ has the shrinking property. The proof is complete.

The shrinking version of [Pr, 6.9] is also valid.

COROLLARY. *Let κ be an arbitrary infinite cardinal. The space κ^2 has the (κ-) shrinking property if and only if κ is regular.*

To end this paper, we note that we can remove the condition "$\lambda < \kappa$" from 6.8 of [Pr].

PROPOSITION. *Let κ, λ and τ be cardinals with $\omega \leqslant \kappa$ and $\tau < \operatorname{cf}\kappa$. Every cover of κ^λ by τ open sets has a finite subcover. In particular, κ^λ is τ-paracompact for every $\tau < \operatorname{cf}\kappa$.*

Proof. Here $\operatorname{cf}\kappa$ denotes the cofinality of κ. Let τ be the first cardinal for which this proposition fails and let $\mathcal{U} = \{U_\gamma: \gamma \in \tau\}$ be an open cover of κ^λ which does not have a finite subcover. Put $V_\gamma = \bigcup_{\gamma' \in \gamma} U_{\gamma'}$ for each $\gamma \in \tau$. Then $\{V_\gamma: \gamma \in \tau\}$ is an increasing open cover, and by the definition of τ, $\kappa^\lambda - V_\gamma$ is non-empty. So fix $f_\gamma \in \kappa^\lambda - V_\gamma$ for each $\gamma \in \tau$. Define for each $\alpha \in \lambda$, $\delta_\alpha = \sup\{f_\gamma(\alpha): \gamma \in \tau\} \in \kappa$. Then $Z = \prod_{\alpha \in \lambda} [0, \delta_\alpha]$ is compact, thus there is a $\gamma \in \tau$ such that $Z \subset V_\gamma$. But this yields a contradiction, since $f_\gamma \in Z - V_\gamma$.

COROLLARY. *Let κ, λ and τ be cardinals with $\omega \leqslant \kappa$ and $\tau < \operatorname{cf}\kappa$. Then every increasing open cover $\{U_\gamma: \gamma \in \tau\}$ of κ^λ has an increasing open shrinking.*

Proof. For such a cover $\{U_\gamma: \gamma \in \tau\}$, there is a $\gamma' \in \tau$ such that $\kappa^\lambda = U_{\gamma'}$ by the above proposition. Put $V_\gamma = 0$ for $\gamma \in \gamma'$ and $V_\gamma = \kappa^\lambda$ for $\gamma \geqslant \gamma'$. Then $\{V_\gamma: \gamma \in \tau\}$ is the desired shrinking.

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