F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$

by

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Abstract. We show an example of an F_σ-ideal not contained in any summable ideal (see Definition 1.6). Next we give an apparently weak condition on an ideal I for the algebra $P(\omega)/I$ to have $\omega_1\omega_1^*$-gaps. Finally, the problem of $\omega_1\omega_1^*$-gaps is solved for meager ideals under the assumption $2^\omega < 2^{\omega_1}$.

§ 0. Summary of results and notation. In 1936 Hausdorff showed that there exists an $\omega_1\omega_1^*$-gap in the Boolean algebra $P(\omega)/\mathscr{F}in$. One can ask if for an arbitrary meager ideal I there exists an $\omega_1\omega_1^*$-gap in the Boolean algebra $P(\omega)/I$ (shortly an *I-gap*). As far as I know this question remains open.

In this paper I give a partial solution of this problem, based on a generalization of the Hausdorff symbol γ (see [Ha]). This has led me to the notion of pseudosolid ideal (see Definition 3.1). All coanalytic ideals known to me are pseudosolid. It is an interesting question if every coanalytic ideal is pseudosolid.

There is, however, a special situation when we can prove the existence of I-gaps for every meager ideal I. Namely, if we assume $2^\omega < 2^{\omega_1}$ then we can use a cardinality argument in order to show that I-gaps (in fact they are also $\mathscr{F}in$-gaps) do exist (Theorem 3.9). It is not known, however, if this assumption may be dropped.

Theorem 1.9 is a solution of a special case of the following general problem: Assume that we have two classes of ideals K and L and we know that $K \subset L$ and $K \neq L$. Can we find an ideal $I \in L - K$ such that I is not contained in any ideal of the class K? For $K = F_\sigma$ and $L = F_{\sigma\delta}$ Samy Zafrany has got the positive solution of this problem (see [Z] or [M]). For higher Borel classes and for $K = \Delta_1^1$ (Borel sets) and $L = \Pi_1^1$ (coanalytic sets) it is still unsolved.

The main results of this paper were obtained in my master thesis written at the Warsaw University under the supervision of Professor Winfried Just.

Set-theoretical notions used here are standard and can be found in [Je] or [Ku]. By ω we denote the set of natural numbers. By $2^{<\omega}$ (resp. $2^{<\omega_1}$) we denote the set of all finite (resp. countable) sequences of 0's and 1's.

Fix an infinite set X. Recall that $[X]^{<\omega}$ is the set of all finite subsets of X. We say that $I \subset P(X)$ is an *ideal* if I is closed under taking subsets and finite unions, $X \notin I$ and $[X]^{<\omega} \subset I$. By ${}^X2$ we denote the set of all functions from X to $2 = \{0, 1\}$. On the set ${}^\omega 2$

we have the product topology. Thus identifying the sets $P(\omega)$ and ${}^{\omega}2$ we can transfer this topology to the set $P(\omega)$. A base for this topology is the family $\{B_s\colon s\in 2^{<\omega}\}$ where $B_s=\{x\subset\omega\colon x\cap\operatorname{dom}(s)=s^{-1}(\{1\})\}$. Hence we can talk about F_σ, $F_{\sigma\delta}$, Borel, meager ideals on the set ω.

§ 1. F_σ-ideals. We say that a set $Z\subseteq P(\omega)$ is *hereditary* if it is closed under taking subsets. By $\mathscr{F}in$ we denote the ideal of finite subsets of ω. Notice that the functions $\underline{\cup}$, $\underline{\cap}$: $P(\omega)\times P(\omega)\to P(\omega)$ defined by $\underline{\cup}(x,y)=x\cup y$ and $\underline{\cap}(x,y)=x\cap y$ are continuous.

1.1. PROPOSITION. *For any hereditary F_σ-set H there exists a family $\{F_n\colon n\in\omega\}$ of hereditary closed sets such that $H=\bigcup_{n\in\omega}F_n$ and $F_n\subset F_{n+1}$ for $n\in\omega$.*

Proof. Let $H=\bigcup_{n\in\omega}D_n$ where D_n is closed for $n\in\omega$. It is easy to see that $\{F_n\colon n\in\omega\}$ with $F_n=\underline{\cap}(\bigcup_{k\leqslant n}D_k\times P(\omega))$ satisfies the conclusion. ■

1.2. LEMMA. *The following conditions are equivalent*:

(i) *I is an F_σ-ideal.*

(ii) *$I=\bigcup_{n\in\omega}F_n$ where the family $\{F_n\colon n\in\omega\}$ is as in Proposition 1 and*

$$(*)\qquad \forall n\in\omega\,\forall x,y\in F_n\ (x\cup y\in F_{n+1}).$$

(iii) *There exists a function $f\colon \mathscr{F}in\to \boldsymbol{R}_+\cup\{0\}$ such that*:

(a) $a\subset b\Rightarrow f(a)\leqslant f(b)$,

(b) $f(a\cup b)\leqslant f(a)+f(b)$,

(c) $\lim_{n\to\infty}f(n)=+\infty$,

and $I=\{x\subseteq\omega\colon \lim_{n\to\infty}f(x\cap n)<+\infty\}$.

Proof. (i)⇒(ii). By Proposition 1 we can assume that $I=\bigcup_{n\in\omega}F'_n$ where each F'_n is hereditary closed and $F'_n\subset F'_{n+1}$ for each n. Now define inductively $F_0=F'_0,\ldots,F_{n+1}=\underline{\cup}(F_n\times F_n)\cup F'_{n+1},\ldots$ It is easy to see that $(*)$ is satisfied too.

(ii)⇒(iii). Let $\{F_n\colon n\in\omega\}$ be as in (ii). For every $a\in\mathscr{F}in$ put $f(a)=\min\{n+1\colon a\in F_n\}$.

(iii)⇒(i). If $f\colon\mathscr{F}in\to\boldsymbol{R}_+\cup\{0\}$ is as in (iii) then for every n define $F_n=\{x\subseteq\omega\colon \forall k\ [f(x\cap k)\leqslant n]\}$. For fixed k the set $\{x\subset\omega\colon f(x\cap k)\leqslant n\}$ is a finite sum of basic clopen sets, hence F_n is closed and $I=\bigcup_{n\in\omega}F_n$. Condition (a) of (iii) guarantees that I is hereditary, condition (b) gives that I is closed under finite unions and (c) implies that $\omega\notin I$. ■

Notation. By $I_{(f)}$ we will denote the ideal satisfying (iii) for $f\colon\mathscr{F}in\to\boldsymbol{R}_+\cup\{0\}$.

Examples of F_σ-ideals:

1.3. $\mathscr{F}in$.

1.4. This ideal will be defined on $2^{<\omega}$ instead of ω. On $2^{<\omega}$ we have a partial order defined as $s\lessdot t\Leftrightarrow s\supset t$. By a *branch* we mean any maximal linearly ordered subset of $2^{<\omega}$. By an *antichain* we mean any subset of $2^{<\omega}$ consisting of pairwise incomparable elements.

Define an ideal on $2^{<\omega}$ generated by the branches: $x\in\mathrm{Ib}\Leftrightarrow x$ is contained in a finite number of branches in the tree $2^{<\omega}\Leftrightarrow\exists n$ (each antichain in x has cardinality less than n).

Defining $f\colon [2^{<\omega}]^{<\omega}\to\boldsymbol{R}_+\cup\{0\}$ by $f(a)=$ maximal cardinality of an antichain contained in a, we see that $\mathrm{Ib}=I_{(f)}$.

1.5. *Summable ideals.* Let $g\colon\omega\to\boldsymbol{R}_+$ satisfy $\sum_{n\in\omega}g(n)=+\infty$. Define an ideal I_g by

$$x\in I_g\ \Leftrightarrow\ \sum_{n\in x}g(n)<+\infty.$$

1.6. DEFINITION. An ideal I is *summable* if it is of the form I_g for a function $g\colon\omega\to\boldsymbol{R}_+$ satisfying $\sum_{n\in\omega}g(n)=+\infty$.

Obviously $I_g=I_{(f)}$, where $f(a)=\sum_{i\in a}g(i)$ for $a\in\mathscr{F}in$.

1.7. PROPOSITION. Ib *is not summable.*

Proof. Suppose $\mathrm{Ib}=I_g$ for some $g\colon\omega\to\boldsymbol{R}_+$. Since every branch belongs to Ib we have $\forall\varepsilon>0\ [\{s\in 2^{<\omega}\colon g(s)\leqslant\varepsilon\}$ is dense in $(2^{<\omega},\lessdot)]$. By induction it is easy to construct an antichain $A=\{s_n\colon n\in\omega\}$ such that $g(s_n)\leqslant 1/2^n$. Of course $A\in I_g-\mathrm{Ib}$. ■

Nevertheless, it is true that the ideal Ib is contained in a summable ideal I_g. To see this define $g\colon 2^{<\omega}\to\boldsymbol{R}_+$ by $g(s)=1/2^{|s|}$. Then every branch (and hence the ideal Ib) is contained in I_g.

Our aim is to construct an example of an F_σ-ideal which is not contained in any summable ideal. First we prove:

1.8. LEMMA. *For every $n\in\omega-\{0\}$ and $\varepsilon\in\boldsymbol{R}_+$ there exist a finite set K_n and a family $\mathscr{R}_n\subset P(K_n)$ such that*:

(a) $\forall v_1,\ldots,v_n\in\mathscr{R}_n\ (v_1\cap\ldots\cap v_n\neq\emptyset)$.

(b) *If P is a probability distribution on K_n then there is a $v\in\mathscr{R}_n$ such that $P(v)<\varepsilon$.*

Proof [1]. Let m be a natural number. For $i\in m$ define $\hat{i}=\{f\in{}^n m\colon i\in\operatorname{Rg}(f)\}$. The family $\mathscr{R}=\{\hat{i}\colon i\in m\}$ satisfies condition (a). Let P be a probability distribution on ${}^n m$. We have

$$\sum_{i=0}^{m-1}P(\hat{i})=\sum_{f\in{}^n m}P(\{f\})\times|\operatorname{Rg}(f)|\leqslant n.$$

Hence there exists an $i_0\in m$ such that $P(\hat{i}_0)\leqslant n/m$. If $m\geqslant n/\varepsilon$ then the pair $K_n={}^n m$ and $\mathscr{R}_n=\{\hat{i}\colon i\in m\}$ satisfies our requirement. ■

1.9. THEOREM. *There exists an F_σ-ideal which is not contained in any summable ideal.*

Proof. For every $n\in\omega$ and for $\varepsilon=1/2$ let K_n and $\mathscr{R}_n$ satisfy the conditions of Lemma 1.8. Let $\mathscr{R}_n^*=\{K_n-v\colon v\in\mathscr{R}_n\}$. The family $\mathscr{R}_n^*$ satisfies the following:

(a′) $\forall w_1,\ldots,w_n\in\mathscr{R}_n^*\ (K_n\neq w_1\cup\ldots\cup w_n)$.

(b′) If P is a probability distribution on K_n then there exists a $w\in\mathscr{R}_n^*$ such that $P(w)\geqslant 0.5$.

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[1] This proof, simplifying the author's original one, is due to J. Cichoń.

Assume now that $\{K_n\colon n\in\omega\}$ is a partition of ω into intervals. Define

$$F=\{x\subset\omega\colon \forall n\ (x\cap K_n\in\mathscr{R}_n^*)\},\quad F_0=\cap[F\times P(\omega)],\ \ldots,\ F_{m+1}=\cup[F_m\times F_m],\ \ldots,$$

$$I_{ns}=\bigcup_{m\in\omega}F_m.$$

From the construction it follows that I_{ns} is a hereditary F_σ-set closed under taking finite unions.

We now check that:

(i) All singletons belong to F_0.
(ii) $\omega\notin I_{ns}$.
(iii) $\forall f\colon \omega\to\boldsymbol{R}_+[(\sum_{n\in\omega}f(n)=+\infty)\Rightarrow\neg(I_{ns}\subset I_f)]$.

(i) Let $j\in\omega$. Take n such that $j\in K_n$. Consider the probability distribution δ_j concentrated on $\{j\}$. By (b′) there is a $w\in\mathscr{R}_n^*$ such that $\delta_j(w)\geqslant 0.5$. Hence $j\in w\in F_0$.

(ii) Assume that $\omega=x_1\cup\ldots\cup x_n$ and $x_i\in F$ for $i=1,\ldots,n$. Then $w_i=x_i\cap K_n\in\mathscr{R}_n^*$ and $K_n=\bigcup_{i=1}^n w_i$, which contradicts (a′).

(iii) Let $f\colon\omega\to\boldsymbol{R}_+$ satisfy $\sum_{n\in\omega}f(i)=+\infty$. For every $n\in\omega$ define a probability distribution P_f^n on K_n by $P_f^n(\{i\})=f(i)/\sum_{j\in K_n}f(j)$. Applying (b′) to P_f^n we see that for each n there exists a $w_n\in\mathscr{R}_n^*$ such that $\sum_{i\in w_n}f(i)\geqslant\frac{1}{2}\sum_{j\in K_n}f(j)$. If $w=\bigcup_{n\in\omega}w_n$ then $w\in I_{ns}-I_f$. ■

The following is a refinement of Theorem 1.9:

1.10. Theorem. *Assume that $\lambda\geqslant\omega$ is a cardinal and any union of λ meager sets does not cover the real line. Then the ideal I_{ns} from Theorem 1.9 is not contained in any union of λ summable ideals.*

Proof. We keep the notation of the proof of Theorem 1.9. We use an equivalent form of our assumption, namely: for every countable poset P and every family $\{D_\alpha\colon \alpha<\lambda\}$ of dense subsets of P there exists $G\subset P$ which is $\{D_\alpha\colon\alpha<\lambda\}$ generic (see [Je], [Ku]).

Let $\{I_{g_\alpha}\colon \alpha<\lambda\}$ be a family of summable ideals. Define a countable poset $P=\bigcup_{m\in\omega}P_m$ where $P_m=\prod_{i=1}^m\mathscr{R}_i^*$. P is ordered by reverse inclusion. Let

$$D_{\alpha,k}=\{p\in P\colon \sum_{i\in\operatorname{dom}(p)}\sum_{j\in p(i)}g_\alpha(j)\geqslant k\}\qquad\text{for } \alpha<\lambda,\ k\in\omega.$$

It is easy to see that each $D_{\alpha,k}$ is dense. Let $G\subset P$ be $\{D_{\alpha,k}\colon \alpha<\lambda,\ k<\omega\}$ generic and let $x_G=\bigcup G$. Then

$$x_G\in F-\bigcup_{\alpha<\lambda}I_{g_\alpha}\subset I_{ns}-\bigcup_{\alpha<\lambda}I_{g_\alpha}.\ ■$$

§ 2. Solid ideals.

2.1. Definition. An ideal I is *solid* if there is a hereditary F_σ-set H such that $I\subset H$ and $\cup(H\times H)$ is meager.

The following problem is open: Are there $F_{\sigma\delta}$-ideals which are not solid?

The following ideals are solid:

2.2. Intersection of any number of F_σ-ideals.

2.3. *The ideals of f-density*:

2.4. Definition. We say that $f\colon\omega\to\boldsymbol{R}_+$ is an EU-*function* (EU = Erdös–Ulam) iff $\sum_{n\in\omega}f(n)=+\infty$ and

$$\lim_{n\to\infty}f(n)/\sum_{k<n}f(k)=0.\ ■$$

For every EU-function f we define an ideal I^f as follows:

$$x\in I^f\Leftrightarrow\lim_{n\to\infty}\frac{\sum_{k\in x\cap n}f(k)}{\sum_{k<n}f(k)}=0.$$

Putting

$$H^f=\left\{x\subset\omega\colon \limsup\frac{\sum_{k\in x\cap n}f(k)}{\sum_{k<n}f(k)}\leqslant\frac{1}{3}\right\}$$

one can easily check that H^f satisfies Definition 2.1.

2.5. *The ideal dual to the filter of open dense subsets of* $(2^{<\omega},\ll)$. For $t\in 2^{<\omega}$ define $t^{\leqslant}=\{u\in 2^{<\omega}\colon u\ll t\}$. Recall that $x\subset 2^{<\omega}$ is open dense $\Leftrightarrow\forall s\in 2^{<\omega}\exists t\ll s\ (t^{\leqslant}\subset x)$. Hence the dual ideal I_{nd} is defined by: $x\in I_{nd}\Leftrightarrow\forall s\exists t\ll s\ (t^{\leqslant}\cap x=\emptyset)$.

Define the set $H_{nd}\supset I$ by

$$(*)\qquad x\in H_{nd}\Leftrightarrow\exists n[\exists t\in 2^n\ (t^{\leqslant}\cap x=\emptyset)\wedge\forall s\in 2^n\exists r\ll s\ (r^{\leqslant}\cap x=\emptyset)].$$

We will show that $\cup(H_{nd}\times H_{nd})\subset H'_{nd}=\{x\subset\omega\colon \exists t\ (t^{\leqslant}\cap x=\emptyset)\}$.

Take $x,y\in H_{nd}$ and let x, y satisfy $(*)$ with $n=n_x$, $n=n_y$ respectively. W.l.o.g. we can assume that $n_x\leqslant n_y$. Take $t\in 2^{n_x}$ such that $t^{\leqslant}\cap x=\emptyset$ and extend it to $s\in 2^{n_y}$. Finally, find $r\ll s$ such that $r^{\leqslant}\cap y=\emptyset$. It is obvious that $r^{\leqslant}\cap(x\cup y)=\emptyset$ and that H'_{nd} is meager.

2.6. Proposition. *Let $H\supset\mathscr{F}in$ be a hereditary F_σ-set and let $H'=\cup(H\times H)$ be meager. Then there exist two families $\{H_n\colon n\in\omega\}$ and $\{H'_n\colon n\in\omega\}$ of hereditary closed sets and an increasing sequence $(l_n)_{n\in\omega}$ of natural numbers such that $H=\bigcup_{n\in\omega}H_n$, $H'=\bigcup_{n\in\omega}H'_n$ and*

$$(*)\qquad \forall n\ [H_n\subset H_{n+1},\ \cup(H_n\times H_n)=H'_{n+1},\ [l_n,l_{n+1})\notin H'_n].$$

Proof. By Proposition 1.1 we must only construct the sequence $(l_n)_{n\in\omega}$. First put $l_0=0$. If for any n there is no $l_{n+1}>l_n$ satisfying $(*)$ we obtain $\forall l>l_n\ [[l_n,l)\in H'_n]$. Now using the fact that H'_n is closed and hereditary we obtain $P(\omega-l_n)\subset H'_n\subset H'$. But this implies that H' is not meager, a contradiction. ■

§ 3. $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$. For every ideal I we denote by $<_I$ the transitive antisymmetric relation on $P(\omega)$ defined by: $x <_I y \Leftrightarrow [x-y \in I \wedge y-x \notin I]$. We say that the pair $\langle (a_\xi)_{\xi<\omega_1}, (b_\eta)_{\eta<\omega_1}\rangle$ of sequences is an *I-gap* if

$$\forall \xi, \eta\ (\xi < \eta < \omega_1 \Rightarrow a_\xi \dot{<}_I a_\eta <_I b_\eta <_I b_\xi \quad \text{and} \quad \neg\exists c \forall \xi < \omega_1\ (a_\xi <_I c <_I b_\xi).$$

Thus usual $\omega_1\omega_1^*$-gaps are $\mathcal{F}in$-gaps in this terminology.

3.1. Definition. An ideal $I \subset P(\omega)$ is *pseudosolid* iff there exist a partition $\{L_n\colon n \in \omega\}$ of ω into nonempty sets and a family $\{\mathcal{F}_n\colon n \in \omega\}$ of hereditary sets such that

(1) $\forall n\ [L_n \in I \wedge \mathcal{F}_n \subset P(L_n) \wedge L_n \notin \bigcup(\mathcal{F}_n \times \mathcal{F}_n)]$.
(2) $x \in I \Rightarrow \forall_n^\infty\ (x \cap L_n \in \mathcal{F}_n)$.

3.2. Proposition. *Every solid ideal is pseudosolid.*

Proof. Take $H \supset I$ as in Definition 2.1 and then find $(l_n)_{n\in\omega}$ and $\{H_n\colon n \in \omega\}$ as in Proposition 2.6. For every n put $L_n = [l_n, l_{n+1})$ and $\mathcal{F}_n = H_n \cap P([l_n, l_{n+1}))$. ■

The following problem is open: Are there coanalytic (or even Borel) ideals which are not pseudosolid? The examples of meager not pseudosolid ideals can be constructed e.g. under MA$+\neg$CH.

3.3. Theorem. *If I is a pseudosolid ideal then there exists an I-gap.*

Proof. This proof is similar to Hausdorff's original one. Fix a pseudosolid ideal I. Let $\{L_n\colon n \in \omega\}$, $\{\mathcal{F}_n\colon n \in \omega\}$ be as in Definition 3.1 for I. For every n let $\mathcal{F}_n' = \bigcup(\mathcal{F}_n \times \mathcal{F}_n)$. Let α be an ordinal, let $(a_\xi)_{\xi<\alpha}$ be an $<_I$ increasing sequence and let $b \subset \omega$. We define two notions of nearness for the ideal I:

$$(a_\xi)_{\xi<\alpha}\gamma_0^I b \Leftrightarrow \forall n[\{\xi < \alpha\colon \forall i \geqslant n\ [(a_\xi - b) \cap L_i \in \mathcal{F}_i]\} \text{ is finite}],$$
$$(a_\xi)_{\xi<\alpha}\gamma_1^I b \Leftrightarrow \forall n[\{\xi < \alpha\colon \forall i \geqslant n\ [(a_\xi - b) \cap L_i \in \mathcal{F}_i']\} \text{ is finite}].$$

Of course γ_1^I is stronger than γ_0^I.

3.4. Claim. *If $(a_\xi)_{\xi<\alpha}\gamma_1^I b$ and $\forall \xi < \alpha\ (a_\xi <_I c <_I b)$ then $(a_\xi)_{\xi<\alpha}\gamma_0^I c$.*

Proof. There exists n_0 such that $\forall i \geqslant n_0\ [(c-b) \cap L_i \in \mathcal{F}_i]$. It is enough to prove the result for $n \geqslant n_0$. Using the fact that $a_\xi - b \subset (a_\xi - c) \cup (c-b)$ we have for $n \geqslant n_0$

$$\{\xi < \alpha\colon \forall i \geqslant n\ (a_\xi - c) \cap L_i \in \mathcal{F}_i\} \subset \{\xi < \alpha\colon \forall i \geqslant n\ (a_\xi - b) \cap L_i \in \mathcal{F}_i'\}$$

and the second set is finite by assumption. ■

3.5. Claim. *If the pair $\langle (a_\xi)_{\xi<\omega_1}, (b_\eta)_{\eta<\omega_1}\rangle$ of sequences satisfies*

$$\forall \xi, \eta\ (\xi < \eta < \omega_1 \Rightarrow a_\xi <_I a_\eta <_I b_\eta <_I b_\xi), \qquad \forall \eta < \omega_1\ [(a_\xi)_{\xi<\eta}\gamma_1^I b_\eta]$$

then it is an I-gap.

Proof. Assume on the contrary that c is such that $\forall \xi \langle \omega_1\ (a_\xi <_I c <_I b_\xi)$. From Claim 3.4 we have $\forall \alpha < \omega_1\ [(a_\xi)_{\xi<\alpha}\gamma_0^I c]$. Let $F\colon \omega_1 \to \omega$ be defined as follows:

$$F(\alpha) = \min\{n\colon \forall i \geqslant n\ [(a_\alpha - c) \cap L_i \in \mathcal{F}_i\}.$$

Let n_0 and $\beta < \omega_1$ be such that $\beta \cap F^{-1}(\{n_0\})$ is infinite. But this contradicts $(a_\xi)_{\xi<\beta}\gamma_0^I c$. ■

Hausdorff defined his nearness symbol γ for the ideal $\mathcal{F}in$ just as we did. He took $\forall n\ (L_n^{\mathcal{F}in} = \{n\}, \mathcal{F}_n^{\mathcal{F}in} = \mathcal{F}_n'^{\mathcal{F}in} = \{\emptyset\})$. In this case we have $\gamma = \gamma_0^{\mathcal{F}in} = \gamma_1^{\mathcal{F}in}$. He built an $\omega_1\omega_1^*$-gap with the properties as in the assumption of Claim 3.5 for $I = \mathcal{F}in$ (see [Ha]). We will call such $\mathcal{F}in$-gaps *Hausdorff gaps.*

Now we define a function $\phi_I\colon P(\omega) \to P(\omega)$ by $\phi_I(x) = \bigcup_{i\in x} L_i$. Since $\{L_i\colon i \in \omega\} \subset I$ and $x \in I \Rightarrow \forall_n^\infty\ (x \cap L_n \in \mathcal{F}_n)$ we can infer that

$$x \subset^* y \Leftrightarrow \phi_I(x) <_I \phi_I(y),$$

i.e. ϕ determines an embedding of quotient Boolean algebras.

3.6. Claim. *If $\langle (a_\xi)_{\xi<\omega_1}, (b_\eta)_{\eta<\omega_1}\rangle$ is an $\omega_1\omega_1^*$ Hausdorff gap (i.e. for every $n \in \omega$ and $\alpha < \omega_1$, $\{\xi < \alpha\colon \forall i \geqslant n\ (a_\xi - b_\alpha) \cap \{i\} = \emptyset\}$ is finite) then $\langle \phi_I(a_\xi)_{\xi<\omega_1}, \phi_I(b_\eta)_{\eta<\omega_1}\rangle$ is an I-gap.*

Proof. We show that the premises of Claim 3.5 are satisfied. Fix $n \in \omega$ and $\eta < \omega_1$. We have

$$\begin{aligned}\{\xi < \eta\colon \forall i \geqslant n\ &[(\phi_I(a_\xi) - \phi_I(b_\eta)) \cap L_i \in \mathcal{F}_i]\}\\ &= \{\xi < \eta\colon \forall i \geqslant n\ [\phi_I(a_\xi - b_\eta) \cap L_i = \emptyset]\}\\ &= \{\xi < \eta\colon \forall i \geqslant n\ [(a_\xi - b_\eta) \cap \{i\} = \emptyset]\}\end{aligned}$$

and the last set is finite. ■

This finishes the proof of Theorem 3.3. ■

To convince the reader that the class of pseudosolid ideals is sufficiently wide let us introduce the following operation on ideals: Let $\{A_n\colon n \in \omega\}$ be a partition of ω into infinite blocks, let I be an ideal on ω and for every n let I_n be an ideal on A_n. We define $\forall^I\{(I_n)_{n\in\omega}\}$, a new ideal on ω, by

$$x \in \forall^I\{(I_n)_{n\in\omega}\} \Leftrightarrow \{n \in \omega\colon x \cap A_n \notin I_n\} \in I.$$

3.7. Proposition. (a) *If $I = \bigcup_{n\in\omega} J_n$ where for every n, J_n is an ideal, $J_n \subset J_{n+1}$ and $J_{n+1} - J_n \neq \emptyset$, then I is pseudosolid.*
(b) *If I is pseudosolid then so is $\forall^I\{(I_n)_{n\in\omega}\}$.*

Proof. In case (a) pick inductively a disjoint family $\{L_n\colon n \in \omega\}$ such that $L_{n+1} \in I_{n+1} - I_n$ and put $\mathcal{F}_n = \mathcal{F}_n' = L_{n+1} \cap I_n$. In case (b) let $L_n^1 = \bigcup_{i\in L_n} A_i$ and $\mathcal{F}_n^1 = \{x \subset L_n^1\colon \{i\colon x \cap A_i \notin I_i\} \in \mathcal{F}_n\}$. ■

Notice that a necessary condition for the existence of I-gaps is the infiniteness of the quotient Boolean algebra $P(\omega)/I$. But this condition is not sufficient. Let e.g. $\{A_n\colon n \in \omega\}$ be a partition of ω into infinite blocks and for every n let M_n be a maximal ideal on A_n. Consider the ideal $J = \forall^{\{\emptyset\}}\{(M_n)_{n\in\omega}\}$.

We have $x \in J \Leftrightarrow \forall n\ (x \cap A_n \in M_n)$. We can extend each M_n to a maximal ideal M_n' on ω if we put, for $B \subseteq \omega$, $B \in M_n' \Leftrightarrow (B \cap A_n \in M_n)$. Then of course we have $J = \bigcap_{n\in\omega} M_n'$.

Hence $P(\omega)/J \approx P(\omega)$ so there are no J-gaps. From the theorem of Talagrand [Tal] stating that no intersection of countably many maximal ideals has the Baire property we see that J is not meager. Now we restrict our attention to meager ideals.

The proof of the following proposition may be found in [Tal] or [JK].

3.8. Proposition. *Let I be a meager ideal. Then there exists an injection $\phi^I\colon P(\omega)\to P(\omega)$ which induces an embedding of Boolean algebras $P(\omega)/\mathscr{F}in$ into $P(\omega)/I$.* ■

Now we are sure that if I is a meager ideal then there are sequences which are ordered by the relation $<_I$ to the order type $\omega_1\omega_1^*$. We now show that under an additional set-theoretical assumption one of such sequences is an I-gap.

3.9. Theorem. *If $2^\omega < 2^{\omega_1}$ then for every meager ideal I on ω there exists an I-gap.*

Proof. Define a partial order $\ll^+$ on $2^{<\omega_1}\times\{0,1\}$ by

$$[s,\varepsilon]\ll^+[s',\varepsilon'] \Leftrightarrow (\varepsilon=\varepsilon'=0\wedge s\subset s')\vee(\varepsilon=\varepsilon'=1\wedge s'\subset s)$$
$$\vee\,[\varepsilon=0\wedge\varepsilon'=1\wedge(s\cup s' \text{ is a function})].$$

We construct now a set $T=\{\tau([s,\varepsilon])\colon [s,\varepsilon]\in 2^{<\omega_1}\times\{0,1\}\}$ such that $T\subset P(\omega)$ and the following equivalence holds:

$$(*)\qquad [s,\varepsilon]\ll^+[s',\varepsilon'] \Leftrightarrow \tau([s,\varepsilon])\subset^*\tau([s',\varepsilon']).$$

The set T will be built by induction on $\alpha=\mathrm{lh}(s)$. For $\alpha=0$ put $\tau([\emptyset,0])=\emptyset$, $\tau([\emptyset,1])=\omega$. Assume that we have constructed $\tau([t,\varepsilon])$ for $\beta=\mathrm{lh}(t)<\alpha$ and $\varepsilon\in\{0,1\}$ such that for all such elements $\tau([s,\varepsilon])$, $(*)$ holds. We have two cases:

(i) α limit. If $s\in{}^\alpha 2$ then take for $\tau([s,0])$ any $x\subset\omega$ such that $\forall\beta<\alpha$ $\{\tau([s\restriction\beta,0])\subset^* x\subset^*\tau([s\restriction\beta,1])\}$, and next take for $\tau([s,1])$ any $y\subset\omega$ such that $\forall\beta<\alpha$ $\{\tau([s,0])\subset^* y\subset^*\tau([s\restriction\beta,1])\}$.

(ii) $\alpha=\beta+1$. For $s\in{}^\alpha 2$ put $t=s\restriction\beta$. Let $a=\tau([t,1])-\tau([t,0])$. We can divide the set a into three infinite subsets $a=x_0\cup x_1\cup x_2$ and then put

$$\tau([t^\wedge\langle\varepsilon\rangle,0])=\tau([t,0])\cup x_\varepsilon,$$
$$\tau([t^\wedge\langle\varepsilon\rangle,1])=\tau([t,1])-x_{1-\varepsilon}\quad \text{for } \varepsilon\in\{0,1\}.$$

Thus the construction of T is finished.

Fix a meager ideal I. Let $\phi^I\colon P(\omega)\to P(\omega)$ be as in Proposition 3.8. The proof of the following fact will give the final result:

3.10. Claim. *If $2^\omega<2^{\omega_1}$ then there exists a $b\in 2^{\omega_1}$ such that*

$$\langle\phi^I(\tau([b\restriction\xi,0]))_{\xi<\omega_1},\ \phi^I(\tau([b\restriction\eta,1]))_{\eta<\omega_1}\rangle\quad \textit{is an } I\textit{-gap.}$$

Proof. Suppose that we can pick, for every $b\in 2^{\omega_1}$, a $y_b\subset\omega$ such that $\forall\xi<\omega_1$ $[\phi^I(\tau([b\restriction\xi,0]))_{\xi<\omega_1}<_I y_b<_I\phi^I(\tau([b\restriction\xi,1]))]$. We show that this choice must be one-to-one. Let b_0, b_1 be distinct branches and let $\alpha<\omega_1$ be least such that $b_0(\alpha)\neq b_1(\alpha)$. Let $t=b_0\restriction\alpha=b_1\restriction\alpha$ and let, e.g., $b_0(\alpha)=0$, $b_1(\alpha)=1$. According to the construction of $\tau([t^\wedge\langle\varepsilon'\rangle,\varepsilon])$ for $\varepsilon,\varepsilon'\in\{0,1\}$ we have $\tau([t,1])-\tau([t,0])=x_0\cup x_1\cup x_2$ for some infinite disjoint x_0, x_1, x_2. Hence

$$\emptyset<_I\phi^I(x_0)<_I\phi^I(\tau([b_0\restriction\alpha+1,0]))<_I y_{b_0},$$
$$y_{b_1}<_I\phi^I(\tau([b_1\restriction\alpha+1,1]))=\phi^I(\tau([t,1]))-\phi^I(x_0).$$

And this implies that $y_{b_0}\neq_I y_{b_1}$. But this contradicts the fact that $2^{\omega_1}>2^\omega\geqslant|\{y_b\colon b\in 2^{\omega_1}\}|$. ■

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