$\mu\langle xA_{\sigma i}\rangle < \frac{1}{2}$, $\mu\langle xa_i^{-1}A_i\rangle > \frac{1}{2}$). So $2[X] \leqslant [X]$ in the semigroup of equidecomposability types (see [W, § 8]), i.e. X is F-paradoxical.

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Ideals of the second category

by

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Abstract. We show that the intersection of less than h ideals of the second category is an ideal of the second category and there exists a family of d ideals of the second category which has an empty intersection.

Introduction. A family of infinite subsets of the set $N = \{0, 1, \ldots\}$ of all natural numbers is an *ideal* if it is closed under forming finite unions, taking infinite subsets and adding finite sets of natural numbers; we assume that the set of all natural numbers does not belong to an ideal.

If A and B are sets, then let $\langle A, B\rangle$ be the family of all infinite subsets of B containing A.

An ideal is of the *second category* if it is of the second category with respect to the topology on the set of all infinite sets of natural numbers generated by the sets $\langle x, N\backslash y\rangle$, where x and y are finite sets of natural numbers. This topology is called the *natural topology.*

The *Ellentuck topology* on the set of all infinite sets of natural numbers is generated by the sets $\langle x, V\rangle$, where x is a finite set of natural numbers and V is an infinite set of natural numbers. Let h denote the least cardinality among the cardinalities of families consisting of open and dense sets in the Ellentuck topology which have empty intersections. This definition of h is equivalent to that of Balcar and Simon [1], as shown in [4].

We prove that the intersection of less than h ideals of the second category is an ideal of the second category. This strengthens a result of Talagrand [6]. In Fremlin [3], p. 55, is was noticed that, in fact, Talagrand proved that the intersection of less than p ideals of the second category is an ideal of the second category, where p is a cardinal about which it is known that it is not greater than h (cf. Balcar and Simon [1]). There exists a model of ZFC in which p is less than h (see Dordal [2]).

It is known (see [5]) that the intersection of less than continuum many maximal ideals (they are always of the second category) is an ideal of the second category. We prove that without the assumption of maximality such a result cannot be proved in

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ZFC. Namely, we prove that there exists a family of d ideals of the second category which has an empty intersection, where d is a cardinal about which it is known that there exists a model of ZFC in which d is less than continuum (cf. Balcar and Simon [1]).

1. A theorem concerning h. We begin with the following characterization of ideals of the second category, which is a reformulation of Theorem 21 from Talagrand [6].

LEMMA 1. *An ideal I is of the second category iff for any sequence of non-empty, pairwise disjoint and finite sets of natural numbers some infinite union of them belongs to I.* ■

The next two lemmas have been known to many mathematicians. Lemma 2 is an easy corollary of Lemma 1, so we omit its proof.

LEMMA 2. *A maximal ideal is of the second category.* ■

LEMMA 3. *An ideal of the second category is open and dense in the Ellentuck topology.*

Proof. If a set W belongs to an ideal I, then the family $\langle\emptyset, W\rangle$ is contained in I. Therefore any ideal is open in the Ellentuck topology.

Suppose that I is not dense in the Ellentuck topology. Take a non-empty base set $\langle x, V\rangle$ disjoint from I. Infinite subsets of V do not belong to I. Therefore the ideal I is contained in the union of the sets $\langle\emptyset, (N\setminus V)\cup y\rangle$, where y runs through finite sets of natural numbers. Sets of the form $\langle\emptyset, N\setminus W\rangle$, where W is infinite, are nowhere dense in the natural topology. Thus I is contained in a countable union of nowhere dense sets. So it cannot be of the second category. ■

Let $a_0, a_1, \ldots$ be a sequence of non-empty, pairwise disjoint and finite sets of natural numbers. For an ideal I such that $a_0\cup a_1\cup\ldots$ does not belong to I we define I^a as the family of all sets A of natural numbers such that $\bigcup\{a_n\colon n\in A\}$ belongs to I. Clearly, if I is an ideal, then I^a is either empty or is an ideal.

LEMMA 4. *If I is an ideal of the second category, then so is I^a.*

Proof. Suppose $b_0, b_1, \ldots$ is a sequence of non-empty, pairwise disjoint and finite sets of natural numbers. For each natural number n we set $c_n = \bigcup\{a_k\colon k\in b_n\}$. The sequence $c_0, c_1, \ldots$ consists of non-empty, pairwise disjoint and finite sets of natural numbers. By Lemma 1, there exists an infinite set A such that $\bigcup\{c_n\colon n\in A\}$ belongs to I. This means that $\bigcup\{b_n\colon n\in A\}$ belongs to I^a. So, by Lemma 1, I^a is of the second category. ■

THEOREM 1. *The intersection of less than h ideals of the second category is an ideal of the second category.*

Proof. Suppose that a family U has cardinality less than h and consists of ideals of the second category. Let $a_0, a_1, \ldots$ be a sequence of non-empty, pairwise disjoint and finite sets of natural numbers. For any ideal I from U the ideal I^a is of the second category, by Lemma 4, and is open and dense in the Ellentuck topology, by Lemma 3. Thus $\bigcap\{I^a\colon I\in U\}$ is non-empty as the intersection of less than h open and dense sets in the Ellentuck topology. This means that for any set A which belongs to this intersection $\bigcup\{a_n\colon n\in A\}$ belongs to each ideal I from U. Therefore $\bigcap U$ is non-empty and, by

Lemma 1, is an ideal of the second category since the sequence $a_0, a_1, \ldots$ was taken arbitrarily. ■

THEOREM 2. *If U is a family of ideals of the second category and its intersection is not of the second category, then there exists a family of ideals of the second category which has cardinality not greater than the cardinality of U and which has an empty intersection.*

Proof. The desired family is $\{I^a\colon I\in U\}$, where $a_0, a_1, \ldots$ is a sequence of non-empty, pairwise disjoint and finite sets of natural numbers such that no infinite union of them belongs to $\bigcap U$. ■

2. A theorem concerning d. Let $p_0, p_1, \ldots$ and $r_0, r_1, \ldots$ be increasing sequences of natural numbers. The set $R = \{r_0, r_1, \ldots\}$ is *sparser* than $P = \{p_0, p_1, \ldots\}$ if p_n is less than r_n for all but finitely many natural numbers n. A family F of infinite sets of natural numbers is *dominating* if for any infinite set A of natural numbers there exists a set B from F sparser than A. Let d be the least cardinality among the cardinalities of dominating families.

Let $f_0, f_1, \ldots$ be an increasing sequence of natural numbers. For an ideal I we define I_f as the family of all infinite sets of natural numbers which are contained in some union $\bigcup\{\{f_n, f_n+1, \ldots, f_{n+1}-1\}\colon n\in A\}$ which belongs to I. Clearly, I_f is either empty or is an ideal.

LEMMA 5. *If I is an ideal of the second category, then so is I_f.*

Proof. Suppose $a_0, a_1, \ldots$ is a sequence of non-empty, pairwise disjoint and finite sets of natural numbers. Let g_0 be a natural number from the sequence $f_0, f_1, \ldots$ which is greater than $\sup a_0$. If natural numbers $g_0, g_1, \ldots, g_{n-1}$ have been defined, then let k be a natural number such that $\inf a_k$ is greater than g_{n-1}. Let g_n be a natural number from the sequence $f_0, f_1, \ldots$ which is greater than $\sup a_k$. We set $p_n = \{g_{n-1}, g_{n-1}+1, \ldots, g_n-1\}$. Then $a_k\subset p_n$ and p_n is the union of sets of the form $\{f_t, f_t+1, \ldots, f_{t+1}-1\}$. By Lemma 1, some infinite union of the p_n belongs to I. Therefore some infinite union of the a_n belongs to I_f. Thus the family I_f is non-empty and Lemma 1 implies that it is of the second category. ■

The next lemma was used by many authors. Its proof can be found in Balcar and Simon [1], p. 355.

LEMMA 6. *If $q_0, q_1, \ldots$ is an increasing sequence of natural numbers, then there exists a sequence $Q_0, Q_1, \ldots$ of natural numbers such that for any set P sparser than $\{q_0, q_1, \ldots\}$ the intersection $P\cap\{Q_n, Q_n+1, \ldots, Q_{n+1}-1\}$ is non-empty for all but finitely many natural numbers n.* ■

THEOREM 3. *There exists a family of d ideals of the second category which has an empty intersection.*

Proof. Let I be a maximal ideal and let F be a dominating family of cardinality d. We consider the family U consisting of the ideals I_Q, where the sequences $Q_0, Q_1, \ldots$ are as in Lemma 6 for $\{q_0, q_1, \ldots\}$ from F. The family U has cardinality not greater than d and, by Lemmas 2 and 5, it consists of ideals of the second category. Suppose, to the contrary, that there exists a set P which belongs to $\bigcap U$. Let $\{q_0, q_1, \ldots\}$ from F be sparser than P. By Lemma 6, for all but finitely many natural numbers n the intersection $P\cap\{Q_n, Q_n+1, \ldots, Q_{n+1}-1\}$ is non-empty. Thus P does not belong to I_Q, since in this case I has to contain all but finitely many natural numbers, which is impossible as I is an ideal. ∎

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The space of Lipschitz maps from a compactum to an absolute neighborhood LIP extensor

by

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Abstract. Let X be a non-discrete metric compactum and Y a separable locally compact absolute neighborhood LIP extensor. The spaces of continuous maps and Lipschitz maps from X to Y are denoted by $C(X, Y)$ and $\mathrm{LIP}(X, Y)$, respectively. Let l_2 be the Hilbert space and

$$l_2^Q = \{(x_i)\in l_2 \mid \sup|i\cdot x_i| < \infty\}.$$

It is proved that $(C(X, Y), \mathrm{LIP}(X, Y))$ is an (l_2, l_2^Q)-manifold pair if each point of Y has a neighborhood V admitting a map $\gamma\colon V\to \mathrm{LIP}(I, Y)$ such that each $\gamma(y)$ is an embedding with $\gamma(y)(0) = y$ and the Lipschitz constant of each $\gamma(y)$ does not exceed some $k > 0$, e.g., Euclidean polyhedra without isolated points and Lipschitz n-manifolds ($n > 0$) have such a property.

Introduction. Let l_2 be Hilbert space and l_2^Q the subspace of l_2 which is the linear span of the Hilbert cube $\prod_{i\in N}[-i^{-1}, i^{-1}]\subset l_2$, that is,

$$l_2^Q = \{(x_i)\in l_2 \mid \sup|i\cdot x_i| < \infty\}.$$

An l_2-*manifold* or an l_2^Q-*manifold* is a separable metrizable space locally homeomorphic to l_2 or l_2^Q, respectively. An (l_2, l_2^Q)-*manifold pair* is a pair (M, N) of an l_2-manifold M and an l_2^Q-manifold N which admits an open cover $\mathcal{U}$ of M and open embeddings $\varphi_U\colon U\to l_2$, $U\in\mathcal{U}$, such that $\varphi_U(N\cap U) = l_2^Q\cap\varphi_U(U)$.

Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be separable metric spaces. In case there is no confusion, d stands for both metrics d_X and d_Y. We assume that

X is non-discrete compact and Y has no isolated point.

The spaces of (continuous) maps and Lipschitz maps from X to Y are denoted by $C(X, Y)$ and $\mathrm{LIP}(X, Y)$, respectively. The topology of these spaces is induced by

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