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A new proof of Kelley's Theorem

by

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Abstract. Kelley's Theorem is a purely combinatorial characterization of measure algebras. We first apply linear programming to exhibit the duality between measures and this characterization for finite algebras. Then we give a new proof of the Theorem using methods from nonstandard analysis.

First some notation and definitions. $\mathcal{B}$ always denotes a Boolean algebra of the form $(\mathcal{B}, 0, 1, +, \cdot, -)$ with the induced ordering $\leq$. We sometimes write $\sum$ and $\prod$ for $+$ and $\cdot$, especially when the operations are infinitary. By a *measure* μ on a subalgebra $\mathcal{A} \subseteq \mathcal{B}$ we mean a finitely additive monotone function $\mu : \mathcal{A} \to [0,1]$ so that $\mu(0) = 0$ and $\mu(1) = 1$. A Boolean algebra $\mathcal{B}$ is called a *measure algebra* if there is a measure $\mu : \mathcal{B} \to [0,1]$ which is *strictly positive*, i.e. $\mu(b) = 0$ iff $b = 0$. A σ-algebra $\mathcal{B}$ is called a σ*-measure algebra* if there is such a σ-additive μ. Following [K], we define for $\mathcal{A} \subseteq \mathcal{B}$ the *intersection number* of $\mathcal{A}$ as

$$\alpha(\mathcal{A}) = \inf\{\widehat{\alpha}(a_1, \ldots, a_n) : a_1, \ldots, a_n \in \mathcal{A},\ n < \omega\},$$

where $\widehat{\alpha}(a_1, \ldots, a_n) = n^{-1} \max\{|I| : I \subseteq \{1, \ldots, n\}, \prod_{i \in I} a_i \neq 0\}$. (The a_i's are not necessarily distinct.) We also define the *measure number* of $\mathcal{A}$ as

$$\beta(\mathcal{A}) = \sup\{r \in [0,1] : \text{ there is a measure } \mu \text{ on the subalgebra generated by } \mathcal{A} \text{ such that } \mu(a) \geq r \text{ for all } a \in \mathcal{A}\}.$$

A σ-algebra $\mathcal{B}$ is said to be *weakly* ω*-distributive* if given $\{b_{ij} : i, j < \omega\}$ such that $b_{i,j} \geq b_{ij+1}$, then

$$\sum_{i=0}^{\omega} \prod_{j=0}^{\omega} b_{ij} = \prod_{n=0}^{\omega} \sum_{i=0}^{\omega} b_{i f_n(i)}$$

for some $f_n : \omega \to \omega$ such that $f_n \leq f_{n+1}$ (i.e. $\forall i,\ f_n(i) \leq f_{n+1}(i)$).

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We assume some basic knowledge of nonstandard analysis (cf. e.g. [HL] and [L]) and work in a $\aleph_1$-saturated nonstandard universe that has the enlargement property (i.e. every standard set has a hyperfinite extension). We use nonstandard analysis and the duality theorem for linear programming to prove the following theorem of J. L. Kelley. See [K] or [F] for a standard proof.

KELLEY'S THEOREM. (1) *A Boolean algebra $\mathcal{B}$ is a measure algebra iff*

$$(*) \qquad \textit{there are } \mathcal{A}_n \subset \mathcal{B} \textit{ such that } \alpha(\mathcal{A}_n) > 0 \textit{ and } \mathcal{B} = \bigcup_{n<\omega} \mathcal{A}_n \cup \{0\}\,.$$

(2) *A σ-algebra $\mathcal{B}$ is a σ-measure algebra iff $\mathcal{B}$ is weakly ω-distributive and also satisfies* $(*)$. ■

The proof is based on the following lemma, which exhibits the duality between measures and Kelley's characterization.

LEMMA. *Let $\mathcal{A} \subseteq \mathcal{B}$ such that $\mathcal{A}$ is* finite. *Then $\alpha(\mathcal{A}) = \beta(\mathcal{A})$ and both the infimum and supremum in the definitions of α and β are attained.*

Proof. Write $\mathcal{A} = \{a_0, \ldots, a_n\}$, and identify the finite Boolean algebra generated by $\mathcal{A}$ as a power set algebra $\mathcal{P}(X)$ for some $X = \{p_0, \ldots, p_r\}$.

For $i \le r$, $j \le n$, define

$$m_{ij} = \begin{cases} 1 & \text{if } p_i \in a_j, \\ 0 & \text{otherwise,} \end{cases}$$

and write $M = [m_{ij}]$, an $(r+1) \times (n+1)$ binary matrix.

CLAIM 1. (a) *$\alpha(\mathcal{A})$ is the minimal $\alpha \in \mathbb{R}$ such that*

$$M \cdot \begin{bmatrix} x_0 \\ \vdots \\ x_n \end{bmatrix} \le \begin{bmatrix} \alpha \\ \vdots \\ \alpha \end{bmatrix}$$

for some $x_0 + \ldots + x_n \ge 1$, $x_0, \ldots, x_n \ge 0$.

(b) *$\alpha(\mathcal{A}) = \widehat{\alpha}(a_{i_0}, \ldots, a_{i_l})$ for some $a_{i_0}, \ldots, a_{i_l} \in \mathcal{A}$.*

Proof. Let $(\alpha_0, \ldots, \alpha_n, \alpha) \in [0, \infty)^{n+2}$ be an extreme point in a region determined by some of the $r+2$ hyperplanes

$$m_{i0}x_0 + \ldots + m_{in}x_n = x_{n+1}, \qquad i = 0, \ldots, r\,,$$
$$x_0 + \ldots + x_n = 1\,,$$

with the property that α is minimal.

Then this is the minimal α such that $M \cdot [x_0, \ldots, x_n]^T \le \alpha$ among all $x_0, \ldots, x_n \ge 0$ satisfying $x_0 + \ldots + x_n \ge 1$. Since the coefficient of each one of the x_i is either 0 or 1, the extreme point is a rational point of the form

$$(\alpha_0, \ldots, \alpha_n, \alpha) = (s_0/s, \ldots, s_n/s, \alpha)\,,$$

where $s_0, \ldots, s_n \in \mathbb{N}$ and $s_0 + \ldots + s_n = s$.

Any such rational point $(s_0/s, \ldots, s_n/s)$ is in one-to-one correspondence with the following sequence from $\mathcal{A}$:

$$\sigma = (a_0 \ldots a_0 \; a_1 \ldots a_1 \ldots a_n \ldots a_n)\,,$$

where a_0 repeats s_0 times, a_1 repeats s_1 times, etc.

Moreover, for the above extreme point, we have $\widehat{\alpha}(\sigma) = \alpha$, and for any other sequence τ from $\mathcal{A}$, we have $\widehat{\alpha}(\tau) \geq \alpha$. Therefore $\alpha(\mathcal{A}) = \alpha$ and (a) is proved. Since $\alpha(\mathcal{A}) = \widehat{\alpha}(\sigma)$, (b) holds as well.

CLAIM 2. (a) $\beta(\mathcal{A})$ *is the maximal* $\beta \in \mathbb{R}$ *such that*

$$M^T \cdot \begin{bmatrix} y_0 \\ \vdots \\ y_r \end{bmatrix} \geq \begin{bmatrix} \beta \\ \vdots \\ \beta \end{bmatrix}$$

for some $y_0 + \ldots + y_r \leq 1$, $y_0, \ldots, y_r \geq 0$.

(b) $\beta(\mathcal{A})$ *is attained by some measure on the subalgebra generated by* $\mathcal{A}$.

Proof. Similar to Claim 1, the maximal β satisfying the matrix inequality is given by some rational point $(\beta_0, \ldots, \beta_r, \beta)$. This corresponds to the measure that assigns weights $\beta_0, \ldots, \beta_r$ to $p_0, \ldots, p_r$ respectively. Conversely, any such weights $\beta_0, \ldots, \beta_r$ satisfy the matrix inequality for some β, thus the claim is proved.

The linear programming problems in the above two claims are dual to each other, so $\alpha(\mathcal{A}) = \beta(\mathcal{A})$ and the lemma is proved. ∎

COROLLARY. $\alpha(\mathcal{A}) \geq \beta(\mathcal{A})$ *for arbitrary* $\mathcal{A} \subseteq \mathcal{B}$.

Proof. $\alpha(\mathcal{A}) = \inf \alpha(\mathcal{A}_0) = \inf \beta(\mathcal{A}_0) \geq \beta(\mathcal{A})$, where the infimum is taken over all finite $\mathcal{A}_0 \subseteq \mathcal{A}$. ∎

Proof of Kelley's Theorem (1). ($\Rightarrow$). Let $\mu : \mathcal{B} \to [0,1]$ be a strictly positive measure and let $\mathcal{A}_n = \{b \in \mathcal{B} : \mu(b) \geq 2^{-n}\}$. Then $\mathcal{B} = \bigcup_{n<\omega} \mathcal{A}_n \cup \{0\}$. Since $\beta(\mathcal{A}_n) \geq 2^{-n}$, so by the Corollary, $\alpha(\mathcal{A}_n) > 0$.

($\Leftarrow$). Suppose $\mathcal{B} = \bigcup_{n<\omega} \mathcal{A}_n \cup \{0\}$ and $\alpha(\mathcal{A}_n) > 0$. Use the enlargement property and let $\mathcal{B}'$ be a hyperfinite algebra such that $\mathcal{B} \subseteq \mathcal{B}' \subseteq {}^*\mathcal{B}$. Let $\mathcal{B}_n = \mathcal{B}' \cap {}^*\mathcal{A}_n$. It follows from $\{{}^*a : a \in \mathcal{A}_n\} \subseteq \mathcal{B}_n \subseteq {}^*\mathcal{A}_n$ and the transfer principle that ${}^*\alpha(\mathcal{B}_n) \approx \alpha(\mathcal{A}_n) > 0$ and both are noninfinitesimal. By transferring the Lemma, ${}^*\alpha(\mathcal{B}_n) = {}^*\beta(\mathcal{B}_n)$, and hence there is an internal measure ν_n on the subalgebra generated by $\mathcal{B}_n$ such that for each $b \in \mathcal{B}_n$, $\nu_n(b) \geq {}^*\alpha(\mathcal{B}_n) > 0$, noninfinitesimal. For each $m < \omega$, there is an internal measure ν on $\mathcal{B}'$ such that $\nu \geq 2^{-n}\nu_n$ on $\mathcal{B}_n$ for all $n < m$. So by $\aleph_1$-saturation, there is an internal measure ν on $\mathcal{B}'$ such that for all $n < \omega$, $\nu \geq 2^{-n}\nu_n$ on $\mathcal{B}_n$. Let $\mu = {}^\circ\nu \restriction \mathcal{B}$, i.e. $\forall b \in \mathcal{B}$, $\mu(b) = {}^\circ\nu({}^*b)$. Then μ is a strictly positive measure on $\mathcal{B}$. ∎

P r o o f o f K e l l e y ' s T h e o r e m (2). ($\Rightarrow$). Let $\mu : \mathcal{B} \to [0,1]$ be a strictly positive σ-measure. By (1), $(*)$ holds. To show the weak ω-distributivity, let $\{b_{ij}\}$ be such that $b_{ij} \geq b_{i,j+1}$. For each n and $i < \omega$, let $f_n(i) =$ least j such that

$$\mu\Big(b_{ij} - \prod_{k=0}^{\omega} b_{ik}\Big) \leq 1/2^{n+i+1}\,.$$

Then $\mu(\sum_{i=0}^{\omega} b_{if_n(i)} - \sum_{i=0}^{\omega}\prod_{j=0}^{\omega} b_{ij}) \leq 1/2^n$. Hence

$$\prod_{n=0}^{\omega}\sum_{i=0}^{\omega} b_{if_n(i)} = \sum_{i=0}^{\omega}\prod_{j=0}^{\omega} b_{ij}\,.$$

($\Leftarrow$). Let $\mathcal{B}'$ be a hyperfinite algebra so that $\mathcal{B} \subseteq \mathcal{B}' \subseteq {}^*\mathcal{B}$. Let ν be the internal measure given by the proof in (1). Then ${}^\circ\nu(b) > 0$ for each $b \in \mathcal{B}' \cap {}^*\mathcal{A}_n$. Let

$$\mathcal{D} = \{b \in \mathcal{B}' : \text{ there is a decreasing sequence } \{c_n\}_{n<\omega} \text{ from } \mathcal{B} \text{ so that } \textstyle\prod_{n=0}^{\omega} c_n = 0 \text{ and each } {}^*c_n \geq b\}\,.$$

Note that $\prod_{n=0}^{\omega} c_n$ is defined for the σ-algebra $\mathcal{B}$, while in general, $\mathcal{B}'$ is not a σ-algebra, and $\{{}^*c_n\}_{n<\omega}$ may have nonzero lower bounds in $\mathcal{B}'$.

Define $r = \sup\{{}^\circ\nu(b) : b \in \mathcal{D}\}$.

CLAIM. *r is attained by some $a' \in \mathcal{D}$. In other words, there are $a' \in \mathcal{B}'$ and decreasing $d_n \in \mathcal{B}$ so that $\prod_{n=0}^{\omega} d_n = 0$, each ${}^*d_n \geq a'$ and $r \approx \nu(a')$.*

P r o o f. Choose $b_0, b_1, \ldots$ from $\mathcal{D}$ so that ${}^\circ\nu(b_i) \to r$. For each i, choose $c_{in} \in \mathcal{B}$ $(n < \omega)$ so that c_{in} decreases to 0 as $n \to \omega$ and each ${}^*c_{in} \geq b_i$. By the weak ω-distributivity, $\sum_{i=0}^{\omega}\prod_{j=0}^{\omega} c_{ij} = \prod_{n=0}^{\omega}\sum_{i=0}^{\omega} c_{if_n(i)}$ for some $f_n : \omega \to \omega$ such that $f_n \leq f_{n+1}$. The left hand side equals 0. Write $d_n = \sum_{i=0}^{\omega} c_{if_n(i)}$. Then $\prod_{n=0}^{\omega} d_n = 0$. Note also that $d_n \in \mathcal{B}$ and decreases. For any n, m, there is $a \in \mathcal{B}'$ such that $b_0 + \ldots + b_m = a \leq {}^*d_n$, so by $\aleph_1$-saturation, there is an internal $a' \in \mathcal{B}'$ such that $b_i \leq a' \leq {}^*d_n$ for all $i, n < \omega$ (so in particular $a' \in \mathcal{D}$), and ${}^\circ\nu(a') \geq \sup_{i<\omega} {}^\circ\nu(b_i)$. Hence ${}^\circ\nu(a') = r$. The claim is proved.

Notice that $\nu(a') \not\approx 1$. Suppose otherwise; then for each $n < \omega$, ${}^\circ\nu({}^*d_n) = 1$, so ${}^\circ\nu(1 - {}^*d_n) = 0$, and by the strict positivity of ${}^\circ\nu \restriction \mathcal{B}$, it follows that $d_n = 1$, contradicting $\prod_{n=0}^{\omega} d_n = 0$.

Now define the internal measure $\mu(b) = \nu(b - a')/(1 - \nu(a'))$, $b \in \mathcal{B}'$. For $b \in \mathcal{B}$, write ${}^\circ\mu(b) = {}^\circ(\mu({}^*b))$. Then ${}^\circ\mu$ is a strictly positive σ-measure on $\mathcal{B}$. To show this, it suffices to verify the following.

(i) *If $0 \neq b \in \mathcal{B}$ then ${}^\circ\mu(b) > 0$.*

Proof. Let d_n decrease to 0 in $\mathcal{B}$ and ${}^*d_n \geq a'$ for any $n < \omega$, as in the claim. Since $b \neq 0$, b is not a lower bound of $\{d_n\}$, therefore $b - d_n \neq 0$ for some n. Then ${}^\circ\nu({}^*b - {}^*d_n) > 0$, so ${}^\circ\nu({}^*b - a') > 0$, so ${}^\circ\mu(b) > 0$.

(ii) ${}^\circ\mu$ *is* σ*-additive on* $\mathcal{B}$.

Proof. Let $\{c_n\}_{n<\omega}$ be a sequence decreasing to 0 in $\mathcal{B}$. By enlargement, this extends to a sequence $\{c_n\}_{n<\mathrm{H}}$ in $\mathcal{B}'$, where H is an infinite hyperfinite integer, and for convenience, we omit the "*" from ${}^*c_n$ for finite n. From the definition of $\mathcal{D}$, $c_\mathrm{N} \in \mathcal{D}$ for each infinite N. By a' being maximal, for any infinite N, $\nu(c_\mathrm{N} + a') \approx \nu(a')$, so $\nu(c_\mathrm{N} - a') \approx 0$. Thus $\mu(c_\mathrm{N}) \approx 0$ for any infinite N. Therefore ${}^\circ\mu(c_n) \to 0$. ∎

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