p -convex functions in linear spaces

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Abstract. The object of this paper is to investigate the continuity of p -convex functions under very general conditions. Let X and Y be partially ordered real linear spaces (with partial orders compatible with the linear structure of X and Y , respectively) endowed with semilinear topologies. By a p -convex function we mean any function f defined on an open and convex subset D of X with values in Y fulfilling the condition $\Delta_h^{p+1} f(x) \geq 0$ for all $h \in X$ and $x \in D$ such that $h \geq 0$ and $x + (p+1)h \in D$, where Δ_h^i denotes the i -th iterate of the difference operator Δ_h with increment h . The results contain theorems on the continuity of p -convex functions obtained in earlier papers ([7], [6], [11], [13]) as particular cases. In particular it is shown that every such function f bounded bilaterally (in the sense of the partial order in Y) on a non-empty open subset of D is continuous in D . In the case $p = 1$ (which is discussed in more detail) the assumption of bilateral boundedness of f may be weakened to the condition that f is bounded above on an open subset of D . Our results are compared with those of Ger [6], where similar problems are discussed.

1. Introduction and preliminaries. This paper is a variant of, and an annex to, Ger's paper [6] bearing almost the same title. The results of [6] are here reproved under slightly different assumptions about the underlying spaces X and Y . Moreover, they are supplemented by theorems on the continuity of restrictions of the functions considered (in the spirit of [10], [11], [13]; cf. also [15]), which have no counterparts in [6]. In the last section the particular case $p = 1$ is discussed in more detail.

Let X be a real linear space endowed with a topology $\mathcal{T}$. The topology $\mathcal{T}$ is called *linear* (and X itself is called a *linear topological space*) iff the function $\psi: \mathbb{R} \times X \times X \rightarrow X$ defined by

$$(1) \quad \psi(\lambda, x, y) = \lambda x + y$$

is continuous. $\mathcal{T}$ is said to be *semilinear* (cf. [14] and also [9], [1]) iff ψ is separately continuous with respect to each variable. The most important feature of such a topology is that for every fixed $y \in X$ and $\lambda \in \mathbb{R} \setminus \{0\}$, $\psi(\lambda, \cdot, y)$ is a homeomorphism (cf. [14]). Consequently, for every y and $\lambda \neq 0$ the set $\lambda A + y$ ($A \subset X$) has the same topological properties (e.g. being open, closed, of the first or second category, etc.) as A .

The notion of a linear space with a semilinear topology is a simple and very natural generalization of the notion of a linear topological space and most properties of convex functions in topological vector spaces extend to the case where the topology is semilinear (cf. [14], [12]). The importance of the latter topologies lies, among other things, in the fact that the so-called core topology

(determined by the linear structure of the underlying space; cf. [9], [18], [1], [12]–[14]) is semilinear.

Let $D \subset X$ be a convex open set, and let Y be another real linear space. If $f: D \rightarrow Y$ is a function, the difference (with increment h) $\Delta_h f(x)$ for x such that $x, x+h \in D$ is defined by

$$\Delta_h f(x) = f(x+h) - f(x),$$

and the iterates $\Delta_h^i f$ by

$$\Delta_h^0 f = f, \quad \Delta_h^{i+1} f = \Delta_h(\Delta_h^i f), \quad i = 0, 1, 2, \dots$$

Thus $\Delta_h^i f$ is defined for x such that $x, x+ih \in D$ (cf., e.g., [15]).

Suppose that in X and in Y we have a relation $\leq$. Despite of being denoted by the same symbol, the relations in X and Y need not be related to each other in any way. Also, for the moment we do not assume that $\leq$ is an order in X or in Y ; the properties of $\leq$ will be specified later. But here and in the sequel we will sometimes write $a \geq b$ instead of $b \leq a$, and $a < b$ ($a > b$) is equivalent to $a \leq b$ and $a \neq b$ ($a \geq b$ and $a \neq b$).

Let p be a positive integer. A function $f: D \rightarrow Y$ is called *p-convex* (convex of order p) iff

$$(2) \quad \Delta_h^{p+1} f(x) \geq 0$$

for all $h \geq 0$ and $x \in D$ such that $x + (p+1)h \in D$. A function $f: X \rightarrow Y$ is called a *polynomial function of order p* iff

$$(3) \quad \Delta_h^{p+1} f(x) = 0$$

for all $x, h \in X$.

For real-valued functions of a real variable the above notions were introduced by Popoviciu [17] (cf. [5], [6], [15] and also the references in [15] concerning generalizations). The present definition follows the pattern of [6].

In other words, it can also be said that a function $f: D \rightarrow Y$ is *p-convex* iff (2) holds with

$$(4) \quad h = (y-x)/(p+1)$$

for all $x, y \in D$ such that $x \leq y$. When $p = 1$, (2) with (4) becomes

$$(5) \quad f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}$$

for $x, y \in D$, $x \leq y$. By symmetry, (5) also holds for $y \leq x$. Thus $f: D \rightarrow Y$ is 1-convex iff (5) holds for all comparable $x, y \in D$ (i.e. $x, y \in D$ such that either $x \leq y$ or $y \leq x$). In the literature, functions $f: D \rightarrow \mathbb{R}$ fulfilling (5) for all $x, y \in D$ are usually referred to as *J-convex* (convex in the sense of Jensen). The notion of a 1-convex function generalizes that of a *J-convex* function and reduces to the latter when $C = X$ (cf. (7) below) and $Y = \mathbb{R}$.

For every positive integer p and every $T \subset X$ the set $H_p(T)$ (cf. also [5], [6], [15]) is defined as

$$(6) \quad H_p(T) = \{x \in X; \text{ there exists an } h \geq 0 \text{ such that } x + ih \in T \text{ for } i \in I_p\},$$

where $I_p = \{-p-1, -p, \dots, -1, 1, \dots, p, p+1\}$. Observe that if $D \subset X$ is convex and $T \subset D$, then also $H_p(T) \subset D$. In fact, if $x \in H_p(T)$, then by (6) $x \pm h \in T \subset D$ for an $h \in X$. By the convexity of D we have $x = \frac{1}{2}[(x+h) + (x-h)] \in D$, whence $H_p(T) \subset D$.

The iterates H_p^j of the operation H_p are defined by $H_p^0(T) = T$, $H_p^{j+1}(T) = H_p(H_p^j(T))$, $j = 0, 1, 2, \dots$. Clearly, if $T \subset D \subset X$, where D is convex, then $H_p^j(T) \subset D$ for $j = 0, 1, 2, \dots$.

2. Assumptions. The following assumptions will be used throughout the paper:

- (i) X is a real linear space endowed with a semilinear topology.
- (ii) $C \subset X$ is a non-empty open set such that $C + C \subset C$ and $\alpha C \subset C$ for real $\alpha > 0$. (In other words, C is a non-empty open cone in X .) The relation $\leq$ is defined in X by

$$(7) \quad x \leq y \Leftrightarrow y - x \in C \quad \text{or} \quad x = y.$$

- (iii) $D \subset X$ is a non-empty open convex set.
- (iv) Y is a real linear space endowed with a semilinear topology.
- (v) $K \subset Y$, $\emptyset \neq K \neq Y$, is an open cone in Y such that for every $c \in K$ and for every neighbourhood V of the origin in Y there is a positive integer n such that the set

$$(8) \quad \Gamma_n(c) = n^{-1}[(c - K) \cap (K - c)]$$

is contained in V . (In other words, for every $c \in K$ the family $\{\Gamma_n(c)\}_{n \in \mathbb{N}}$ is a neighbourhood base at the origin in Y .) The relation $\leq$ is defined in Y by

$$(9) \quad a \leq b \Leftrightarrow b - a \in K \quad \text{or} \quad a = b.$$

Remark 1. Unless $C = X$, the relation $\leq$ defined by (7) is a partial order in X , compatible with the linear structure

$$(10) \quad x \leq y \quad \text{implies} \quad x + z \leq y + z \quad \text{and} \quad \alpha x \leq \alpha y$$

for every $x, y, z \in X$ and real $\alpha > 0$. (Thus C is the positive cone in X : the condition $h > 0$ is equivalent to $h \in C$.) It follows from (10) that if $x \leq y$ and α is a negative real number, then $\alpha y \leq \alpha x$.

If $C = X$, then $x \leq y$ for all $x, y \in X$. Relation (10) is then trivially fulfilled. The condition $h > 0$ is equivalent to $h \neq 0$.

If $C \neq X$, then the origin does not belong to C , but is a boundary point of C . If $C = X$, then the origin is an interior point of C . In either case we have $C \cap U \neq \emptyset$ for every neighbourhood U of the origin in X .

Instead of assuming that C is open and non-empty, it is enough to assume that $C_0 = \text{int } C \neq \emptyset$. In fact, then C_0 fulfils (ii), and if $f: D \rightarrow Y$ is p -convex with respect to C , then it is also p -convex with respect to C_0 .

Remark 2. It follows from (v) that every $e \in K$ fulfils the condition:

(vi) For every $a \in Y$ there is a positive integer n such that $-ne \leq a \leq ne$.

In fact, since K is open, to a given a we may find an n such that $e - n^{-1}a \in K$ and $e + n^{-1}a \in K$, which implies that $-ne \leq a \leq ne$.

In the literature an $e \in Y$ fulfilling condition (vi) is called a *strong unity* (cf., e.g., [1], [2]). Thus (v) implies that every $e \in K$ is a strong unity in Y .

Remark 3. Since $n^{-1}K = K$ (use $\alpha K \subset K$ with $\alpha = n$ and $\alpha = n^{-1}$), (8) may be written as

$$\Gamma_n(c) = \{y \in Y; -n^{-1}c < y < n^{-1}c\}.$$

We now compare our assumptions (i)–(v) with those in [6].

In [6], X is assumed to be a real Hausdorff linear topological space (so (i) clearly is weaker), and $C = X$ is not admitted. The condition $C \neq \emptyset$ is not stated explicitly, but it is tacitly assumed and used throughout the paper. Anyhow, if $C = \emptyset$, then every function $f: D \rightarrow Y$ trivially satisfies

$$\Delta_h^{p+1} f(x) \geq 0 \quad \text{for } x \in D, h \in C \cup \{0\},$$

and consequently is p -convex (for any $p \in \mathbb{N}$).

Regarding Y Ger assumes that

(vii) Y is a real normed linear space, partially ordered by a relation $\leq$ which is compatible with the linear structure of Y (cf. Remark 1). Moreover,

$$(11) \quad 0 \leq a \leq b \quad \text{implies} \quad \|a\| \leq \|b\|.$$

In general it would be difficult to compare (vii) with (iv) and (v). But if (vii) is fulfilled and the set

$$(12) \quad K = \{y \in Y; y > 0\}$$

is open and non-void ($K \neq Y$ since $\leq$ is a partial order), then (iv) and (v) are fulfilled too. For take a $c \in K$ and a neighbourhood V of the origin in Y . Choose a positive real r such that $B = \{y \in Y; \|y\| < r\} \subset V$. Next choose a positive integer $n > 3\|c\|/r$. For every $y \in \Gamma_n(c)$ we have (cf. Remark 3)

$$-c/n < y < c/n,$$

whence

$$0 < y + c/n < 2c/n$$

and by (11)

$$\|y\| - n^{-1}\|c\| \leq \|y + c/n\| \leq 2\|c\|/n.$$

Hence

$$\|y\| < 3\|c\|/n < r$$

so that $y \in B$. Consequently $\Gamma_n(c) \subset B \subset V$. (iv) and (9) are trivial.

In [6], in the definition of H_p (cf. (6)) the condition $h \geq 0$ is missing, but this is an evident oversight (cf. Remark 6 below).

In the sequel whenever we take $X = \mathbf{R}^N$ (or $Y = \mathbf{R}^N$) it is always understood that X (resp. Y) is endowed with the natural (linear) topology. If $x \in \mathbf{R}^N$ has coordinates $\xi_1, \dots, \xi_N$, we write $x = \langle \xi_1, \dots, \xi_N \rangle$ and $\|x\| = (\xi_1^2 + \dots + \xi_N^2)^{1/2}$. The Lebesgue measure in $\mathbf{R}^N$ always means the N -dimensional Lebesgue measure. Moreover, when $Y = \mathbf{R}$ we always take $K = (0, \infty)$ so that (9) gives the usual inequality between real numbers.

3. Main result. The main result of this paper is (similarly to [6], where, however, the assumptions were somewhat different) the following:

THEOREM 1. *Let hypotheses (i)–(v) be fulfilled, and suppose that a p -convex function $f: D \rightarrow Y$ is bounded on a set $T \subset D$. If there exists a non-negative integer k such that $\text{int } H_p^k(T) \neq \emptyset$, then f is continuous in D .*

As in [6], Theorem 1 is a direct consequence of three facts, here presented as Propositions 1–3. We assume that (i)–(v) are fulfilled, and $f: D \rightarrow Y$ is a p -convex function.

PROPOSITION 1. *If f is bounded in a neighbourhood of some point of D , then it is bounded in a neighbourhood of every point of D .*

PROPOSITION 2. *If f is bounded in a neighbourhood of $x_0 \in D$, then it is continuous at x_0 .*

PROPOSITION 3. *If f is bounded on $T \subset D$, then it is also bounded on $H_p(T)$.*

The proofs of Propositions 1 and 2 will be given in the next section. The proof of Proposition 3 does not differ from that given in [5] (cf. also [15]) in the particular case $X = \mathbf{R}^N$, $Y = \mathbf{R}$, and is therefore omitted.

Remark 4. Apart from any other existing differences, there is a difference between [6] and the present paper which is not so apparent at first glance. The expression: $f: D \rightarrow Y$ is bounded on a set $T \subset D$ in [6] meant: there exists a positive real number M such that

$$(13) \quad \|f(t)\| \leq M \quad \text{for } t \in T,$$

whereas here it means: there exists a $c \in K$ such that

$$(14) \quad -c \leq f(t) \leq c \quad \text{for } t \in T.$$

However, if (vii) is fulfilled and the set (12) is open and non-void, then (13) and (14) are equivalent. To see this, assume that (13) holds, and let $B = \{y \in Y; \|y\| \leq M\}$. Take a $d \in K$. Since K is open there exists a positive integer n such

that

$$(15) \quad d + n^{-1}B = d - n^{-1}B \subset K.$$

With $c = nd$ we obtain from (15) for $y \in B$

$$-c < y < c,$$

and so (14) is fulfilled. Conversely, assuming (14) we have $0 \leq f(t) + c \leq 2c$ for $t \in T$, whence by (11)

$$\|f(t)\| - \|c\| \leq \|f(t) + c\| \leq 2\|c\| \quad \text{for } t \in T,$$

and (13) results with $M = 3\|c\|$.

Remark 5. Let (iv) and (v) be fulfilled. It would seem that the condition: there exist $a, b \in Y$ such that

$$(16) \quad a \leq f(t) \leq b \quad \text{for } t \in T$$

is more general than (14). But actually the two conditions are equivalent. In fact, take any $e \in K$ and choose a positive integer n such that $-a \leq ne$ and $b \leq ne$ (cf. Remark 2). Then $-ne \leq a \leq b \leq ne$, and (16) implies (14) with $c = ne$.

Remark 6. Put (cf. (6))

$$(17) \quad \hat{H}_p(T) = \{x \in X; \text{ there exists an } h \in X \text{ such that } x + ih \in T \text{ for } i \in I_p\}.$$

The following example shows that Proposition 3 with $\hat{H}_p$ in place of H_p is not true.

EXAMPLE 1. Let $X = \mathbb{R}^2$, $Y = \mathbb{R}$. Take $D = X = \mathbb{R}^2$ and $C = \{x = \langle \xi, \eta \rangle \in \mathbb{R}^2; \xi > 0 \text{ and } \eta > 0\}$. Define $f_1: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f_1(x) = f_1(\langle \xi, \eta \rangle) = \xi\eta$ and $f_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ by $f_2(x) = 0$. Both functions are 1-convex ((5) holds for all $x = \langle \xi, \eta \rangle$ and $y = \langle \zeta, \omega \rangle$ such that $(\xi - \zeta)(\eta - \omega) \geq 0$), consequently so is $f = \max(f_1, f_2)$:

$$f(x) = f(\langle \xi, \eta \rangle) = \begin{cases} \xi\eta & \text{if } \xi\eta > 0, \\ 0 & \text{if } \xi\eta \leq 0. \end{cases}$$

Put $T = \{x = \langle \xi, \eta \rangle \in \mathbb{R}^2; \xi\eta \leq 0\}$. Then clearly f is bounded on T , but not on $\hat{H}_1(T) = X = \mathbb{R}^2$.

The iterates $\hat{H}_p^j$ of the operation $\hat{H}_p$ are defined like those of H_p .

Taking $D = C = X$ we obtain from Theorem 1:

COROLLARY 1. Let (i), (iv), (v) be fulfilled, and suppose that a polynomial function (of order p) $f: X \rightarrow Y$ is bounded on a set $T \subset X$. If there exists a non-negative integer k such that $\text{int } \hat{H}_p^k(T) \neq \emptyset$, then f is continuous in X .

4. Proofs. Throughout this section we assume that (i)–(v) are fulfilled and $f: D \rightarrow Y$ is a p -convex function.

For any real numbers $\xi_1, \dots, \xi_{p+1}$ let $V(\xi_1, \dots, \xi_{p+1})$ be the Vandermonde determinant

$$(18) \quad V(\xi_1, \dots, \xi_{p+1}) = \begin{vmatrix} 1 & \xi_1 & \xi_1^2 & \dots & \xi_1^p \\ 1 & \xi_2 & \xi_2^2 & \dots & \xi_2^p \\ \dots & \dots & \dots & \dots & \dots \\ 1 & \xi_{p+1} & \xi_{p+1}^2 & \dots & \xi_{p+1}^p \end{vmatrix} = \prod_{1 \leq i < j \leq p+1} (\xi_j - \xi_i).$$

Let $z_0 \in D$ and $h \geq 0$ be arbitrary, take arbitrary real numbers $\xi_1, \dots, \xi_{p+1}$, and write

$$(19) \quad z_i = z_0 + \xi_i \cdot h, \quad i = 1, \dots, p+1.$$

LEMMA 1. If ξ_i , $i = 0, 1, \dots, p+1$, are rational and such that $0 = \xi_0 < \xi_1 < \dots < \xi_{p+1}$, and if the points (19) belong to D , then

$$(20) \quad \sum_{i=0}^{p+1} (-1)^{p+1+i} V(\xi_0, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{p+1}) f(z_i) \geq 0.$$

Proof. The function φ of the real variable τ :

$$\varphi(\tau) = f(x_0 + \tau h), \quad \tau \in \{\varrho \in \mathbf{R}; \varrho h \in D - x_0\},$$

is *p*-convex, and a formula equivalent to (20) may be obtained as in the case $Y = \mathbf{R}$ (cf. [17], [15]).

In the rest of this section we assume that $x_0 \in D$ and f fulfils (14) with a $c \in K$, where $T \subset D$ is a neighbourhood of x_0 .

LEMMA 2. If $x \in D$ is comparable with x_0 , then f is bounded in a neighbourhood of x .

Proof. Assume that $x < x_0$. (If $x > x_0$ the proof is similar, and if $x = x_0$ there is nothing to prove.) There is a rational number $\alpha > 1$ such that $z_0 = x_0 + \alpha(x - x_0)$ belongs to D . Replacing, if necessary, c by nc (with a suitable $n \in \mathbf{N}$; cf. Remark 2) we may assume that

$$(21) \quad -c \leq f(z_0) \leq c.$$

Put $h = x - z_0 = (1 - \alpha)(x - x_0) > 0$ and observe that $x = z_0 + \xi_1 h$, $x_0 = z_0 + \xi_2 h$, where $\xi_1 = 1$, $\xi_2 = \alpha/(\alpha - 1) > \xi_1$. We may find rational numbers $\xi_3, \dots, \xi_{p+1}$ (if $p = 1$ this step is superfluous) such that

$$0 = \xi_0 < \xi_1 < \xi_2 < \xi_3 < \dots < \xi_{p+1}$$

and the points z_i given by (19) belong to T for $i = 2, \dots, p+1$. For every $i = 0, 1, \dots, p+1$ the mapping $\varphi_i: X \rightarrow X$ given by

$$\varphi_i(u) = z_0 + \xi_i(u - z_0), \quad u \in X,$$

is continuous and $\varphi_i(x) \in T$ for $i = 2, \dots, p+1$. Consequently for $i = 2, \dots, p+1$ the sets $\varphi_i^{-1}(T)$ are neighbourhoods of x and so is also the set

$$U = (z_0 + C) \cap \bigcap_{i=2}^{p+1} \varphi_i^{-1}(T).$$

For $u \in U$ we have $\varphi_0(u) = z_0$, $\varphi_1(u) = u$ and $\varphi_i(u) \in T$ for $i = 2, \dots, p+1$. By Lemma 1

$$\sum_{i=0}^{p+1} (-1)^{p+1+i} V_i f[\varphi_i(u)] \geq 0,$$

where $V_i = V(\xi_0, \dots, \xi_{i-1}, \xi_{i+1}, \dots, \xi_{p+1})$ are positive numbers (cf. (18)) independent of u . Hence by (14) and (21)

$$(22) \quad (-1)^p f(u) \geq -c V_1^{-1} (V_0 + V_2 + \dots + V_{p+1}).$$

Now find a positive rational $\zeta_1 < 1$ such that $w_1 = x + \zeta_1(x_0 - x)$ belongs to T , and put $\zeta_0 = 0$, $\zeta_i = (\alpha - 1)(\xi_i - 1)$ for $i = 2, \dots, p+1$, and

$$w_i = x + \zeta_i(x_0 - x), \quad i = 0, 1, \dots, p+1.$$

Then $w_i = z_i \in T$ for $i = 2, \dots, p+1$. Also $w_1 \in T$ by definition. Since $\zeta_2 = 1$ we have $0 = \zeta_0 < \dots < \zeta_{p+1}$ and obviously all ζ_i are rational.

For every $i = 0, 1, \dots, p+1$ the mapping $\psi_i: X \rightarrow X$ given by

$$\psi_i(u) = u + \zeta_i(x_0 - u) = \zeta_i x_0 + (1 - \zeta_i)u, \quad u \in X,$$

is continuous and $\psi_i(x) = w_i \in T$ for $i = 1, \dots, p+1$. Consequently for $i = 1, \dots, p+1$ the sets $\psi_i^{-1}(T)$ are neighbourhoods of x and so is also the set

$$W = (x_0 - C) \cap \bigcap_{i=1}^{p+1} \psi_i^{-1}(T).$$

For $u \in W$ we have $\psi_0(u) = u$ and $\psi_i(u) \in T$ for $i = 1, \dots, p+1$. By Lemma 1

$$\sum_{i=0}^{p+1} (-1)^{p+1+i} \bar{V}_i f[\psi_i(u)] \geq 0,$$

where $\bar{V}_i = V(\zeta_0, \dots, \zeta_{i-1}, \zeta_{i+1}, \dots, \zeta_{p+1})$ are positive numbers independent of u . By (14)

$$(23) \quad (-1)^p f(u) \leq c \bar{V}_0^{-1} (\bar{V}_1 + \dots + \bar{V}_{p+1}).$$

In view of Remark 5 relations (22) and (23) show that f is bounded in $U \cap W$, i.e., in a neighbourhood of x .

Now we are in a position to give:

Proof of Proposition 1. We argue as in [6]. Let D^* be the set of $x \in D$ such that f is bounded in a neighbourhood of x . Evidently D^* is open and

$x_0 \in D^*$. Suppose that there exists an $x \in D \setminus D^*$ and define $z: [0, 1] \rightarrow D$ by

$$z(\lambda) = \lambda x + (1 - \lambda)x_0 = x_0 + \lambda(x - x_0), \quad \lambda \in [0, 1].$$

Since z is continuous and $z(1) = x$ belongs to the closed set $X \setminus D^*$, there is a smallest $\mu \in [0, 1]$ such that $z_0 = z(\mu) \notin D^*$. Clearly, $\mu > 0$ and $z_0 \in D$.

The set $(z_0 + C) \cap D$ is open and non-empty (cf. Remark 1). Fix a $u \in (z_0 + C) \cap D$. Then $z_0 \in (u - C) \cap D$ and the latter set is open and non-empty. By the continuity of z we have

$$(24) \quad \{\lambda \in (0, \mu); z(\lambda) \in (u - C) \cap D\} \neq \emptyset.$$

Choose a λ_0 belonging to the set (24) and put $v = z(\lambda_0)$. Then $u, v, z_0 \in D$ and

$$(25) \quad z_0 \leq u, \quad v \leq u.$$

According to the extremal property of μ we have $v \in D^*$, whence by (25) and Lemma 2 also $z_0 \in D^*$, a contradiction. Consequently $D^* = D$ and the proof is complete.

Now fix rational numbers $\delta_1, \dots, \delta_p$ such that

$$(26) \quad 0 = \delta_0 < \delta_1 < \dots < \delta_{p-1} < 1 < \delta_p.$$

For every positive integer n we may find a rational $\delta = \delta(n)$ such that $\delta_{p-1} < \delta < 1$ and

$$(27) \quad \left[1 - \frac{V(\delta_0, \dots, \delta_{p-1}, \delta)}{V(\delta_0, \dots, \delta_{p-1}, 1)} \right] + \sum_{i=0}^{p-1} \frac{V(\delta_0, \dots, \delta_{i-1}, \delta_{i+1}, \dots, \delta_{p-1}, \delta, 1)}{V(\delta_0, \dots, \delta_{p-1}, 1)} < \frac{1}{2n},$$

$$(28) \quad \left| 1 - \frac{V(\delta_0, \dots, \delta_{p-2}, \delta, \delta_p)}{V(\delta_0, \dots, \delta_{p-2}, 1, \delta_p)} \right| + \sum_{i=0}^{p-2} \frac{V(\delta_0, \dots, \delta_{i-1}, \delta_{i+1}, \dots, \delta_{p-2}, \delta, 1, \delta_p)}{V(\delta_0, \dots, \delta_{p-2}, 1, \delta_p)} + \frac{V(\delta_0, \dots, \delta_{p-2}, \delta, 1)}{V(\delta_0, \dots, \delta_{p-2}, 1, \delta_p)} < \frac{1}{2n}$$

(we just choose $\delta = \delta(n)$ sufficiently close to 1, cf. (18)).

We put

$$(29) \quad \lambda_i = \lambda_i(n) = \frac{1 - \delta}{1 - \delta_i}, \quad i = 0, \dots, p,$$

and for every $\bar{x} \in T$ we define $S_n[\bar{x}]$ by

$$(30) \quad S_n[\bar{x}] = T \cap \bigcap_{i=0}^p [\bar{x} + \lambda_i(T - \bar{x})].$$

It is easily seen that $S_n[\bar{x}]$ is a neighbourhood of $\bar{x}$.

LEMMA 3. If $x \in S_n[\bar{x}]$ and $x \leq \bar{x}$, then

$$(31) \quad f(x) - f(\bar{x}) \in \Gamma_n(\tfrac{1}{2}c).$$

Proof. Let $x \in S_n[\bar{x}]$, $x \leq \bar{x}$. By virtue of (30) there exist $t_i \in T$, $i = 0, \dots, p$, such that

$$(32) \quad x = \bar{x} + \lambda_i(t_i - \bar{x}), \quad i = 0, \dots, p.$$

Hence in particular

$$\lambda_i(t_i - \bar{x}) = \lambda_0(t_0 - \bar{x}), \quad i = 0, \dots, p,$$

whence, by (29), $t_i - \bar{x} = (1 - \delta_i)(t_0 - \bar{x})$, and

$$(33) \quad t_i = t_0 + \delta_i(\bar{x} - t_0), \quad i = 0, \dots, p.$$

Similarly, we obtain from (32) for $i = 0$

$$x = t_0 - (1 - \lambda_0)t_0 + (1 - \lambda_0)\bar{x},$$

whence by (29)

$$(34) \quad x = t_0 + \delta(\bar{x} - t_0).$$

Evidently

$$(35) \quad \bar{x} = t_0 + (\bar{x} - t_0).$$

Observe also that by (32)

$$\bar{x} - x = \lambda_0(\bar{x} - t_0).$$

Consequently, $x \leq \bar{x}$ implies $\bar{x} - t_0 \geq 0$.

Lemma 1 now yields by virtue of (26) and (33)-(35)

$$\begin{aligned} & \sum_{i=0}^{p-1} (-1)^{p+1+i} V(\delta_0, \dots, \delta_{i-1}, \delta_{i+1}, \dots, \delta_{p-1}, \delta, 1) f(t_i) \\ & - V(\delta_0, \dots, \delta_{p-1}, 1) f(x) + V(\delta_0, \dots, \delta_{p-1}, \delta) f(\bar{x}) \geq 0 \end{aligned}$$

and

$$\begin{aligned} & \sum_{i=0}^{p-2} (-1)^{p+1+i} V(\delta_0, \dots, \delta_{i-1}, \delta_{i+1}, \dots, \delta_{p-2}, \delta, 1, \delta_p) f(t_i) \\ & + V(\delta_0, \dots, \delta_{p-2}, 1, \delta_p) f(x) - V(\delta_0, \dots, \delta_{p-2}, \delta, \delta_p) f(\bar{x}) \\ & + V(\delta_0, \dots, \delta_{p-2}, \delta, 1) f(t_p) \geq 0. \end{aligned}$$

Hence according to (14), (27) and (28), since $t_0, \dots, t_p, \bar{x} \in T$, we obtain

$$-c/(2n) < f(x) - f(\bar{x}) < c/(2n),$$

which in view of Remark 3 is equivalent to (31).

LEMMA 4. For every positive integer n there exists a neighbourhood $U_n \subset D$ of x_0 such that

$$(36) \quad f(x) - f(x_0) \in \Gamma_n(c) \quad \text{for } x \in U_n.$$

Proof. Let $\delta_0, \dots, \delta_p$, $\delta = \delta(n)$, and $\lambda_0, \dots, \lambda_p$ have the same meaning as previously. Put

$$G = T \cap \bigcap_{i=0}^p \left[x_0 + \frac{\lambda_i}{1-\lambda_i} (x_0 - T) \right].$$

G is a neighbourhood of x_0 . In view of Remark 1, $(x_0 + C) \cap G$ is non-empty. Take an $\hat{x} \in (x_0 + C) \cap G$. Then

$$(37) \quad x_0 \leq \hat{x},$$

$$(38) \quad \hat{x} = x_0 + \frac{\lambda_i}{1-\lambda_i} (x_0 - s_i), \quad i = 0, \dots, p,$$

with some $s_0, \dots, s_p \in T$. Relation (38) may be written as

$$x_0 = \hat{x} + \lambda_i (s_i - \hat{x}), \quad i = 0, \dots, p.$$

In other words (cf. (30)),

$$(39) \quad x_0 \in T \cap \bigcap_{i=0}^p [\hat{x} + \lambda_i (T - \hat{x})] = S_n[\hat{x}].$$

Now we put

$$(40) \quad U_n = (\hat{x} - C) \cap S_n[\hat{x}].$$

Since $\hat{x} \in x_0 + C$, we have $x_0 \in \hat{x} - C$, whence according to (39) and (40), $x_0 \in U_n$. It follows from (ii) and (30) that U_n is open, so U_n is a neighbourhood of x_0 . Moreover, $U_n \subset T \subset D$ in view of (39) and (40). According to (40), $x \leq \hat{x}$ and $x \in S_n[\hat{x}]$ for every $x \in U_n$. In particular (cf. (37) and (39)), $x_0 \leq \hat{x}$ and $x_0 \in S_n[\hat{x}]$. By Lemma 3

$$f(x) - f(\hat{x}) \in \Gamma_n(\tfrac{1}{2}c) \quad \text{and} \quad f(x_0) - f(\hat{x}) \in \Gamma_n(\tfrac{1}{2}c),$$

whence $f(x) - f(x_0) \in \Gamma_n(\tfrac{1}{2}c) - \Gamma_n(\tfrac{1}{2}c) = \Gamma_n(c)$, since by (8) the sets Γ_n are convex and symmetric ($\Gamma_n = -\Gamma_n$), and (36) follows.

Proposition 2 is an immediate consequence of (v) and Lemma 4.

5. Measure and category. The significance of the operation H_p may well be seen from Propositions 4 and 5 below. Note that formula (6) may also be written as

$$(41) \quad H_p(T) = T \cup \{x \in X; C \cap \bigcap_{i \in I_p} i^{-1}(T - x) \neq \emptyset\}.$$

PROPOSITION 4. *Let (i) and (ii) be fulfilled, and let $T \subset X$ be a second category Baire set. Then $\text{int } H_p(T) \neq \emptyset$.*

Proof. We have $T = (G \setminus P) \cup R$, where G is open and non-empty (otherwise T would be of the first category), and P, R are first category sets.

Thus $G \setminus P \subset T$, whence for arbitrary $x \in X$

$$(42) \quad C \cap \bigcap_{i \in I_p} i^{-1}[(G \setminus P) - x] \subset C \cap \bigcap_{i \in I_p} i^{-1}(T - x).$$

If $x \in G$, then

$$U = \bigcap_{i \in I_p} i^{-1}(G - x)$$

is a neighbourhood of the origin. Moreover,

$$(43) \quad \bigcap_{i \in I_p} i^{-1}[(G \setminus P) - x] = U \setminus \bigcup_{i \in I_p} i^{-1}(P - x).$$

This means that the set (43) is residual in U , since P is of the first category. Consequently we have for $x \in G$

$$C \cap \bigcap_{i \in I_p} i^{-1}[(G \setminus P) - x] \neq \emptyset$$

(note that X is a Baire space since it contains a second category set T ; cf. [16]), whence in view of (42) and (41) we obtain $G \subset H_p(T)$ so that $\text{int } H_p(T) \neq \emptyset$.

Remark 7. The above proof remains valid also in the case where X (endowed with a semilinear topology) is a linear space over the rationals.

PROPOSITION 5. Let $X = \mathbb{R}^N$, let C satisfy (ii), and let $T \subset \mathbb{R}^N$ be a set of positive inner Lebesgue measure. Then $\text{int } H_p(T) \neq \emptyset$.

Proof. Without loss of generality we may assume that T is measurable. Let x_0 be a density point of T . Put $T_0 = T - x_0$ and

$$(44) \quad A = \bigcap_{i \in I_p} i^{-1} T_0.$$

Thus the origin is a density point of A .

Let B be an open ball centered at the origin, and choose a positive number α such that (m denotes the Lebesgue measure)

$$(45) \quad \alpha < \frac{1}{2} \cdot \frac{m(B \cap C)}{m(B)}.$$

The expression on the right-hand side of (45) is independent of the choice of B . In fact, for every positive number γ we have $m(\gamma B) = \gamma^N m(B)$, and $C = \gamma C$ (cf. (ii)), whence $m[(\gamma B) \cap C] = m[\gamma(B \cap C)] = \gamma^N m(B \cap C)$. Therefore we may assume that B has been chosen small enough to satisfy

$$(46) \quad m(B \cap A) > (1 - \alpha)m(B).$$

Finally, we may find an open ball G_0 (centered at the origin) such that

$$(47) \quad m[(B \cap A) \setminus ((B \cap A) - x)] < \frac{\alpha}{2(p+1)} m(B) \quad \text{for } x \in G_0.$$

We have by (46) and (47) for arbitrary $x_i \in G_0$, $i \in I_p$,

$$\begin{aligned}
 (1-\alpha)m(B) &< m(B \cap A) \\
 &= m\left[(B \cap A) \setminus \bigcap_{i \in I_p} ((B \cap A) - x_i)\right] + m\left[B \cap A \cap \bigcap_{i \in I_p} ((B \cap A) - x_i)\right] \\
 &\leq m\left[\bigcup_{i \in I_p} ((B \cap A) \setminus ((B \cap A) - x_i))\right] + m\left[B \cap \bigcap_{i \in I_p} ((B \cap A) - x_i)\right] \\
 &\leq \sum_{i \in I_p} m\left[(B \cap A) \setminus ((B \cap A) - x_i)\right] + m\left[B \cap \bigcap_{i \in I_p} ((B \cap A) - x_i)\right], \\
 &\leq \alpha m(B) + m\left[B \cap \bigcap_{i \in I_p} ((B \cap A) - x_i)\right],
 \end{aligned}$$

i.e.,

$$(48) \quad m\left[B \cap \bigcap_{i \in I_p} ((B \cap A) - x_i)\right] > (1-2\alpha)m(B).$$

Relations (48) and (45) imply that

$$C \cap \bigcap_{i \in I_p} ((B \cap A) - x_i) \neq \emptyset$$

so that in view of (44)

$$C \cap \bigcap_{i \in I_p} \left(\bigcap_{j \in I_p} j^{-1} T_0 - x_i\right) \neq \emptyset$$

and

$$(49) \quad C \cap \bigcap_{i \in I_p} (i^{-1} T_0 - x_i) \neq \emptyset$$

for every $x_i \in G_0$, $i \in I_p$.

Now we write $G = G_0 + x_0$ (so that G is a non-empty open set), take an arbitrary $x \in G$, and put $x_i = i^{-1}(x - x_0)$, $i \in I_p$. Clearly $x_i \in G_0$ for $i \in I_p$, whence by (49)

$$(50) \quad C \cap \bigcap_{i \in I_p} i^{-1}(T - x) \neq \emptyset.$$

Relation (50) holds for every $x \in G$. According to (41) this means that $G \subset H_p(T)$. Consequently $\text{int } H_p(T) \neq \emptyset$.

If $C = X$, then the difference between H_p and $\hat{H}_p$ (cf. (17)) disappears. Thus we obtain from Propositions 4 and 5

COROLLARY 2. *Propositions 4 and 5 without hypothesis (ii) are valid also when H_p is replaced by $\hat{H}_p$.*

The measure-theoretical part of Corollary 2 is known; it is a direct consequence of a result of Kemperman [8]. In the case where X is a linear topological space over the rationals, the topological part of Corollary 2 is proved in [6]. However, our proof is somewhat simpler.

Theorem 1 together with Propositions 4 and 5 yield the following results.

THEOREM 2. *Let (i)–(v) be fulfilled, and suppose that a p -convex function $f: D \rightarrow Y$ is bounded on $T \subset D$. If there exists a non-negative integer k such that $H_p^k(T)$ is a second category Baire set, then f is continuous in D .*

THEOREM 3. *Let $X = \mathbb{R}^N$, let (ii)–(v) be fulfilled, and suppose that a p -convex function $f: D \rightarrow Y$ is bounded on $T \subset D$. If there exists a non-negative integer k such that the inner Lebesgue measure of $H_p^k(T)$ is positive, then f is continuous in D .*

Hence we obtain (cf. Corollaries 1 and 2)

COROLLARY 3. *Theorems 2 and 3 without hypotheses (ii) and (iii) are valid if $f: X \rightarrow Y$ is a polynomial function of order p , and D and $H_p^k(T)$ are replaced by X and $\hat{H}_p^k(T)$, respectively.*

6. Restrictions. In the literature there are several results to the effect that—under suitable assumptions—if $f: D \rightarrow Y$ is a function whose restriction $f|_T$ to a set $T \subset D$ is continuous, then f is continuous in D (cf., e.g., [15]). Here we shall prove theorems of this kind for p -convex functions.

We start with a result of a more general character. We first introduce further hypotheses.

(viii) T is a class of sets $T \subset X$ with the property that for every $T \in T$ there exists an $x_0 \in T$ such that $T \cap U \in T$ for every neighbourhood U of x_0 .

(ix) F is a class of functions $f: D \rightarrow Y$ such that if $f \in F$ is bounded on a set $T \in T$, $T \subset D$, then it is continuous in D .

(x) Y is partially ordered by a relation $\leq$ which is compatible with the linear structure of Y , and $\text{int } K \neq \emptyset$, where K is defined by (12).

Remark 8. The family of all second category Baire subsets of X , or (when $X = \mathbb{R}^N$) the family of all sets $T \subset \mathbb{R}^N$ of positive Lebesgue measure are examples of set classes T fulfilling (viii). Every point at which T is locally of the second category (cf., e.g., [16] or [15]) resp. every density point of T may serve as x_0 occurring in (viii). It follows from Theorems 2 and 3 that for such T the family F of all p -convex functions $f: D \rightarrow Y$ fulfils (ix).

THEOREM 4. *Let (i), (iv), (viii), (ix) and (x) be fulfilled with a non-empty open set $D \subset X$, and let $f: D \rightarrow Y$ and $T \subset D$ belong to F and T , respectively. If $f|_T$ is continuous, then f is continuous in D .*

Proof. Choose a $c \in \text{int } K$, and let $x_0 \in T$ be such that $T \cap U \in T$ for every neighbourhood U of x_0 . Then (cf. Remark 3) $V = \Gamma_1(c)$ is a neighbourhood of the origin in Y . It follows from the continuity of $f|_T$ at x_0 that there exists a neighbourhood $U \subset D$ of x_0 such that $f(t) \in f(x_0) + V = f(x_0) + \Gamma_1(c)$ for $t \in T \cap U$. This means that f is bounded on $T \cap U \in T$, and hence it is continuous in D .

Hence we obtain according to Remark 8:

THEOREM 5. *Let (i)–(v) be fulfilled, let $T \subset D$ be a second category Baire set, and let $f: D \rightarrow Y$ be p -convex. If $f|_T$ is continuous, then f is continuous in D .*

THEOREM 6. *Let $X = \mathbb{R}^N$, let (ii)–(v) be fulfilled, let $T \subset D$ be a set of positive Lebesgue measure, and let $f: D \rightarrow Y$ be p -convex. If $f|_T$ is continuous, then f is continuous in D .*

COROLLARY 4. *Theorems 5 and 6 without hypotheses (ii) and (iii) are valid if $f: X \rightarrow Y$ is a polynomial function of order p , and D is replaced by X .*

Taking, in Theorem 5, X to be a linear topological space, $C = X$, and $p = 1$, we obtain a result proved in [13]. Taking further $X = \mathbb{R}^N$ we obtain an earlier version found in [11].

Taking, in Theorem 6, $C = \mathbb{R}^N$ and $p = 1$ we get a result from [10].

A function $f: X \rightarrow \mathbb{R}$ satisfying for all $u, v \in X$ the equation

$$(51) \quad f(u+v) + f(u-v) = 2f(u) + 2f(v)$$

is called a *quadratic functional*. Relation (51) can be written as

$$(52) \quad \Delta_h^2 f(x) = 2f(h),$$

where $x = u+v$, $h = v$, valid for all $x, h \in X$. Relation (52) implies that $\Delta_h^3 f(x) = 0$ for $x, h \in X$, and thus every quadratic functional is a polynomial function of order 2. (More exactly, (52) shows that a quadratic functional is a monomial of degree 2, and conversely; cf. [3].)

Corollary 4 with $X = Y = \mathbb{R}$ applied to quadratic functionals $f: \mathbb{R} \rightarrow \mathbb{R}$ yields a result proved in [4] under the additional assumption of the continuum hypothesis.

We terminate this section with a

PROBLEM. Let $T \subset \mathbb{R}^N$ be an analytic set containing a Hamel basis of $\mathbb{R}^N$. Does there exist a point $x_0 \in T$ such that for every neighbourhood U of x_0 the set $T \cap U$ contains a Hamel basis of $\mathbb{R}^N$?

7. 1-convex functions. In the present section we show that in the case $p = 1$ some of our results remain valid when the boundedness condition (14) is replaced by boundedness from above:

$$(53) \quad f(t) \leq c \quad \text{for } t \in T.$$

As pointed out in the Introduction, a function $f: D \rightarrow Y$ is 1-convex if and only if (5) holds for all comparable $x, y \in D$.

The following lemma is of a key importance.

LEMMA 5. *Let (i)–(v) be fulfilled, and let $f: D \rightarrow Y$ be 1-convex. If f is bounded above on a non-empty open set $T \subset D$, then there exists a non-empty open set $U \subset D$ such that f is bounded (bilaterally) on U .*

Proof. Assume that (53) holds with a $c \in Y$, and take an $x_0 \in T$. The set

$$(54) \quad W = T \cap (2x_0 - T) \cap (x_0 - C)$$

is open and non-empty, $W \subset T$. For $y \in W$ we have

$$(55) \quad y \leq x_0,$$

and by (53)

$$(56) \quad f(y) \leq c.$$

Put $U = 2x_0 - W$. Then U is also open and non-empty. Moreover, by (54), $U \subset 2x_0 - (2x_0 - T) = T \subset D$, and by (53) we have for $x \in U$

$$(57) \quad f(x) \leq c.$$

Every $x \in U$ may be written as $x = 2x_0 - y$ with a $y \in W$. Hence $x - x_0 = x_0 - y$, which implies according to (55) that $x_0 \leq x$, whence, again by (55), $y \leq x$. Thus x and y are comparable and we may use (5). We get by (57), (5), and (56), for every $x \in U$ and $y = 2x_0 - x \in W$,

$$c \geq f(x) \geq 2f(x_0) - f(y) \geq 2f(x_0) - c.$$

Consequently f is bounded on U (cf. Remark 5).

As an immediate consequence of Lemma 5 and Theorem 1 (with $p = 1$ and $k = 0$) we obtain:

THEOREM 7. *Let (i)–(v) be fulfilled, and let $f: D \rightarrow Y$ be 1-convex. If f is bounded above on a non-empty open set $T \subset D$, then it is continuous in D .*

Remark 9. Theorem 7 may be regarded as a very general version of the famous theorem of Bernstein and Doetsch (cf. e.g., [15]). In the particular case where $C = X$ and $Y = \mathbb{R}$, Theorem 7 has been proved in [14].

For every set $T \subset X$ we put

$$(58) \quad H(T) = T \cup \{x \in X; C \cap (T - x) \cap (x - T) \neq \emptyset\}.$$

By (58) and (41) we have for every $T \subset X$

$$(59) \quad H_p(T) \subset H(T), \quad p = 1, 2, 3, \dots$$

LEMMA 6. *Let (i)–(v) be fulfilled, and let $f: D \rightarrow Y$ be 1-convex. If f is bounded above on $T \subset D$, then it is also bounded above on $H(T)$.*

Proof. Assume that (53) holds with a $c \in Y$, and take an arbitrary $u \in H(T)$. According to (58) there exists an $h \geq 0$ such that $x = u - h$ and $y = u + h$ belong to T , whence by (53)

$$(60) \quad f(x) \leq c \quad \text{and} \quad f(y) \leq c.$$

We have $y - x = 2h \geq 0$, so x and y are comparable and we get by (5) and (60)

$$f(u) \leq \frac{1}{2} [f(x) + f(y)] \leq c.$$

Consequently f is bounded above on $H(T)$.

In view of (59) we obtain from Propositions 4 and 5, Lemma 6, and Theorem 7

THEOREM 8. *Let (i)–(v) be fulfilled, let $f: D \rightarrow Y$ be 1-convex, and let $T \subset D$ be a second category Baire set. If f is bounded above on T , then it is continuous in D .*

THEOREM 9. *Let $X = \mathbf{R}^N$, let (ii)–(v) be fulfilled, let $f: D \rightarrow Y$ be 1-convex, and let $T \subset D$ be a set of positive inner Lebesgue measure. If f is bounded above on T , then it is continuous in D .*

Remark 10. In Theorems 7–9, the condition on T to be non-empty open, or of the second category with the Baire property, or of positive inner Lebesgue measure, may be replaced by the condition that there exists a non-negative integer k such that $\text{int } H^k(T) \neq \emptyset$, or $H^k(T)$ contains a second category Baire set, or $H^k(T)$ has positive inner Lebesgue measure, respectively.

Remark 11. When $C = X$ (i.e. (5) holds for all $x, y \in D$), Theorem 8 has been proved in [7] under the assumption that X is a real linear topological Baire space and Y fulfils the condition:

(xi) Y is a Banach lattice with a strong unit.

This means (cf. [2]) that Y is a Banach space endowed with a lattice structure such that the partial order $\leq$ is compatible with the linear structure of Y and

$$(61) \quad |a| \leq |b| \quad \text{implies} \quad \|a\| \leq \|b\|$$

for all $a, b \in Y$, where $|a| = \max(a, -a)$, $a \in Y$; moreover, there exists an $e \in Y$ fulfilling (vi).

Although it is not explicitly stated, the assumptions of Theorem 8 force X to be a Baire space (still the topology in X may be semilinear), but in Theorem 7 this is not so. Thus Theorem 7 is not a particular case of Theorem 8: unless X is a Baire space, a non-empty open set $T \subset X$ need not be of the second category.

As to Y , condition (xi) implies (vii). The only thing to check is that (11) is fulfilled. For $a \geq 0$, $a \in Y$, we have $2a \geq 0$, i.e. $a \geq -a$, whence $|a| = \max(a, -a) = a$. Thus if $0 \leq a \leq b$, then $|a| \leq |b|$ and by (61), $\|a\| \leq \|b\|$.

We end this paper with two examples bearing on the mutual relationship of (v), (vii), and (xi). In these examples $Y = \mathbf{R}^2$ and (9) holds with a K which will be specified in each instance.

EXAMPLE 2. Let

$$K = \{y = \langle \xi, \eta \rangle \in \mathbf{R}^2; \xi \in \mathbf{R}, \eta > \max(0, -\xi)\}.$$

Then (iv) and (v) are fulfilled, but (vii) is not. For take $a = \langle 5, 1 \rangle \in K$, $b = \langle 4, 3 \rangle \in K$. Then $b - a = \langle -1, 2 \rangle \in K$ so that $0 < a < b$. But $\|a\| = \sqrt{26} > 5 = \|b\|$, contrary to (11).

EXAMPLE 3. Let

$$K = \{y = \langle \xi, \eta \rangle \in \mathbb{R}^2; \xi > 0, 0 < \eta < \xi\}.$$

Then (vii) is fulfilled, but (61) is not true (so that, in particular, (xi) cannot be fulfilled). For take $a = \langle 7, 6 \rangle$, $b = \langle 0, 9 \rangle$. Then $|a| = a = \langle 7, 6 \rangle$ and $|b| = \langle 18, 9 \rangle$. Hence $|b| - |a| = \langle 11, 3 \rangle \in K$ so that $|a| < |b|$. But $\|a\| = \sqrt{85} > 9 = \|b\|$, contrary to (61).

References

- [1] J. Bair et R. Fourneau, *Etude géométrique des espaces vectoriels*, Lecture Notes in Math. 489, Springer, Berlin 1975.
- [2] G. Birkhoff, *Lattice Theory*, 3rd ed., Amer. Math. Soc. Colloq. Publ. 25, Amer. Math. Soc., Providence, R. I., 1967.
- [3] D. Ž. Djoković, *A representation theorem for $(X_1-1)(X_2-1) \dots (X_n-1)$ and its applications*, Ann. Polon. Math. 22 (1969), 189–198.
- [4] Z. Gajda and H. Miller, *On quadratic functionals and some properties of Hamel bases*, J. Math. Anal. Appl. 145 (1) (1990), 45–51.
- [5] R. Ger, *Convex functions of higher orders in Euclidean spaces*, Ann. Polon. Math. 25 (1972), 293–302.
- [6] — *n-Convex functions in linear spaces*, Aequationes Math. 10 (2/3) (1974), 172–176.
- [7] — *Convex transformations with Banach lattice range*, Stochastica 11 (1987), 13–23.
- [8] J. H. B. Kemperman, *A general functional equation*, Trans. Amer. Math. Soc. 86 (1957), 28–56.
- [9] V. Klee, *Convex sets in linear spaces, I, II, III*, Duke Math. J. 18 (1951), 443–466 and 877–883; 20 (1953), 105–112.
- [10] B. Kominek and Z. Kominek, *On some set classes connected with the continuity of additive and Q-convex functions*, Prace Naukowe Uniwersytetu Śląskiego Nr 218, Prace Mat. 8 (1978), 60–63.
- [11] Z. Kominek, *On the continuity of Q-convex and additive functions*, Aequationes Math. 23 (1981), 146–150.
- [12] —, *Convex functions in linear spaces*, Prace Naukowe Uniwersytetu Śląskiego Nr 1087, 1989.
- [13] Z. Kominek and M. Kuczma, *On the lower hull of convex functions*, Aequationes Math. 38 (1989), 192–210.
- [14] —, —, *Theorems of Bernstein–Doetsch, Piccard and Mehdi and similinear topology*, Arch. Math. (Basel) 52 (1989), 595–602.
- [15] M. Kuczma, *An introduction to the theory of functional equations and inequalities. Cauchy's equation and Jensen's inequality*, PWN (Polish Scientific Publishers) and Uniwersytet Śląski, Warszawa–Kraków–Katowice 1985.
- [16] K. Kuratowski, *Topologie*, Vol. I, Monografie Mat. 35, PWN (Polish Scientific Publishers), Warszawa 1958.
- [17] T. Popoviciu, *Sur quelques propriétés des fonctions d'une ou de deux variables réelles*, Mathematica (Cluj) 8 (1934), 1–85.
- [18] F. A. Valentine, *Convex Sets*, McGraw-Hill, New York 1973.

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Reçu par la Rédaction le 25.07.1988

Révisé le 27.11.1989