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Equivariant maps of joins of finite G-sets and an application to critical point theory

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Abstract. A lower estimate is proved for the number of critical orbits and critical values of a G-invariant C^1 function $f : S^n \to \mathbb{R}$, where G is a finite nontrivial group acting freely and orthogonally on $\mathbb{R}^{n+1} \setminus \{0\}$. Neither Morse theory nor the minimax method is applied. The proofs are based on a general version of Borsuk's Antipodal Theorem for equivariant maps of joins of G-sets.

Introduction. Let S^n denote the unit sphere in $\mathbb{R}^{n+1}$, G a finite group acting orthogonally on S^n and let $f : S^n \to \mathbb{R}$ be a G-invariant C^1 function. Let δ denote the greatest common divisor of $|Gx|$ for all $x \in S^n$, where $Gx = \{gx \mid g \in G\}$ and $|Gx|$ is the cardinality of Gx.

Benci and Pacella [2] showed that if $\delta > 1$ then f has at least $n + 1$ orbits of critical points provided that each critical point is counted with its multiplicity. Furthermore, if the action of G on S^n is free then f has at least $(n + 1)|G|$ critical points. Those results were established with the aid of Morse theory.

The problem of finding lower bounds for the number of orbits of an invariant functional occurs in the papers of various authors.

Let M be a paracompact and complete Banach manifold endowed with an action by diffeomorphisms of a finite group G and let $M \setminus M^G$ contain a G-invariant subset S which is G-homeomorphic to a sphere S^n. Assume that $f : M \to \mathbb{R}$ is a G-invariant C^1 function bounded from below satisfying the Palais–Smale condition and such that

$$\forall s \in S \ \forall x \in M^G \qquad f(s) < r < f(x)$$

for some $r \in \mathbb{R}$.

Under the above assumptions Krawcewicz and Marzantowicz [9] gave

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a lower estimate for the number of critical points of f. The proof of this result involved the Lusternik–Schnirelmann method. In [10] they extended this method to the class of locally Lipschitzian functionals invariant with respect to a finite group action.

Fadell and Rabinowitz [6] introduced the cohomological index to study the case of general compact Lie group actions.

The case of an action of an arbitrary compact Lie group G on the sphere SV, where V denotes a finite-dimensional real vector space on which G acts orthogonally, was studied by Bartsch [1]. Using the notion of the equivariant Lusternik–Schnirelmann category he provided lower bounds for the number of critical orbits with a given orbit type of G-invariant C^1 functionals $f : SV \to \mathbb{R}$.

The purpose of this paper is to prove a related result. We strengthen the assumptions and obtain an estimate not only for the number of critical orbits but also for the number of critical values of the functional f:

Theorem 2. *If a finite nontrivial group G acts freely and orthogonally on $\mathbb{R}^{n+1} \setminus \{0\}$ and $f \in C^1(S^n, \mathbb{R})$ is G-invariant then f has at least $n+1$ critical orbits. Moreover, if the number of critical orbits is finite then there exist at least $n+1$ critical values.*

In fact Theorem 2 will be obtained as a consequence of the basic theorem of this paper:

Theorem 1. *Let S be the unit sphere in a Hilbert space H. Suppose that G is a nontrivial finite group which acts freely and orthogonally on $H \setminus \{0\}$, $f \in C^1(S, \mathbb{R})$ is bounded, G-invariant and the Palais–Smale condition is satisfied. Then f has an infinite number of critical orbits.*

Let us remark that the results concerning the case of infinite dimensions for $G = \mathbb{Z}_2$ are given for example in [5] and [4].

We would like to emphasize that using either the formula given in [9] ([10]) or in [1] we obtain the same lower estimate on the number of critical points of the functional considered in Theorem 2. Yet, contrary to [2], [1], [9], [10] we apply neither Morse theory nor the minimax method. We use a simple geometrical method. Our proofs are based on the general version of Borsuk's Antipodal Theorem:

Theorem (Extension of Borsuk's Antipodal Theorem). *Let G be a nontrivial finite group. If there exists a G-map $\phi : G_{(n)} \to G_{(k)}$ then $k \geq n$.*

Let C denote the set of critical points of f. To prove Theorem 1 we assume that the number of critical orbits is finite (thus C and $f(C)$ are finite) and construct for an arbitrarily great $n \in \mathbb{N}$, G-maps φ, ψ, η:

$$G_{(n)} \xrightarrow{\phi} S \xrightarrow{\psi} R_0 * \ldots * R_k \xrightarrow{\eta} G_{(k)}$$

where $R_i := f^{-1}(\lambda_i) \cap C$ for all $\lambda_i \in f(C)$ $(i = 0, \ldots, k)$.

In future we hope to generalize the geometrical method used in this paper to the class of nondifferentiable functionals invariant with respect to an orthogonal and free action of a finite group (comp. [10]).

Since we consider free and orthogonal actions of a finite group on a sphere let us remark that there exist finite groups that cannot act freely on any sphere S^n (see [3], III.8.1). The complete classification of the possible groups can be found in [16]. A necessary condition on such a group is given there ([16], (5.3.1)): each of its subgroups of order pq (where p and q are not necessarily different prime numbers) must be cyclic. For solvable finite groups the necessary condition becomes sufficient ([16], (6.1.11)).

In Sections 1 and 2 we recall the definitions and theorems that we use in the next sections. In Section 3 we formulate and prove a generalization of Borsuk's Antipodal Theorem. Section 4 is devoted to construction of G-maps φ and η. We construct a G-map ψ in Section 5. Finally, in Section 6 we give the proofs of Theorems 1 and 2.

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1. Preliminaries. Since our approach involves the notion of join and some elements of the theory of groups including Sylow's Theorem we recall the relevant definitions and theorems.

(1.1) *Join.* This subsection is devoted to an exposition of the notion of join (introduced by J. Milnor [11]) which is needed to formulate and prove Borsuk's Antipodal Theorem.

The *join* $X_0 * \ldots * X_n$ of $n+1$ topological groups (topological spaces) can be defined as follows. A point of the join is specified by:

(1) $n+1$ real numbers $t_0, \ldots, t_n$ satisfying $t_i \geq 0$, $t_0 + \ldots + t_n = 1$;
(2) a point $x_i \in X_i$ for each $t_i > 0$ $(0 \leq i \leq n)$.

Such a point will be denoted by $(t_0, x_0, \ldots, t_n, x_n)$, where x_i may be chosen arbitrarily or omitted whenever the corresponding t_i vanishes.

Now we define a topology in $X_0 * \ldots * X_n$. For $i = 0, \ldots, n$ we consider the following coordinate functions:

(1) $t_{(i)} : X_0 * \ldots * X_n \to [0,1]$, $t_{(i)}(t_0, x_0, \ldots, t_i, x_i, \ldots, t_n, x_n) = t_i$;
(2) $x_{(i)} : t_{(i)}^{-1}]0,1] \to X_i$, $x_{(i)}(t_0, x_0, \ldots, t_i, x_i, \ldots, t_n, x_n) = x_i$.

We endow $X_0 * \ldots * X_n$ with the strongest topology such that the above functions are continuous.

As an immediate consequence we deduce that a sub-basis for the open sets is given by the sets of the following two types:

(1) $\{(t_0, x_0, \ldots, t_i, x_i, \ldots, t_n, x_n) \mid \alpha < t_i < \beta\}$, where $\alpha, \beta \in \mathbb{R}$;

(2) $\{(t_0, x_0, \ldots, t_i, x_i, \ldots, t_n, x_n) \mid t_i \neq 0 \wedge x_i \in U\}$, where U is an arbitrary open subset of X_i.

The next example (cf. [15]) will be particularly useful for our later work:

(1.1.1) Example. Let X be a topological space and let D^k be the k-dimensional unit ball. The cone over X is denoted by $\operatorname{con} X$ and the suspension of X by $\mathbf{S}X$. Note that:

(i) $X * D^0 \approx \operatorname{con} X$;
(ii) $X * S^0 \approx \mathbf{S}X$;
(iii) $\forall k \in \mathbb{N}$, $\mathbf{S}^{k+1} X \approx X * S^k$;
(iv) $\forall n \in \mathbb{N}\ \forall m_1, \ldots, m_n \in \mathbb{N}$, $S^{m_1} * \ldots * S^{m_n} \approx S^{m_1 + \ldots + m_n + n - 1}$;
(v) $\forall n \in \mathbb{N}\ \forall m_1, \ldots, m_n \in \mathbb{N}$, $D^{m_1} * \ldots * D^{m_n} \approx D^{m_1 + \ldots + m_n + n - 1}$.

(1.1.2) Remark. Given $f_i : X_i \to Y_i$ for $i = 0, \ldots, n$, where $X_0, \ldots, X_n, Y_0, \ldots, Y_n$ are topological spaces, we define a mapping $f_0 * \ldots * f_n : X_0 * \ldots * X_n \to Y_0 * \ldots * Y_n$ by $(f_0 * \ldots * f_n)(t_0, x_0, \ldots, t_n, x_n) = (t_0, f_0(x_0), \ldots, t_n, f_n(x_n))$. If $f : X \to Y$ then by $f_{(n)}$ we denote the $(n+1)$-fold join $f * \ldots * f : X * \ldots * X \to Y * \ldots * Y$.

(1.2) *G-spaces.* We recall some basic facts on G-spaces (we refer to [3] for details).

Let G be a finite group, X a topological space, H a Hilbert space with scalar product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{R}$, S the unit sphere in H. Moreover, $Gx := \{gx \in X \mid g \in G\}$ for $x \in X$ is the *orbit* of x; X/G is the orbit space with the quotient topology; $g \cdot A := \{g \cdot x \mid x \in A\}$, where $A \subset X$; $G_{(n)}$ denotes the $(n+1)$-join $G * \ldots * G$ (with discrete topology); and $|\cdot| : H \to \mathbb{R}_+$ is the norm defined by $|x| := (\langle x, x \rangle)^{1/2}$. This notation is valid throughout the paper.

(1.2.1) Definition. (1) An action of G on X is called *free* if

$$\forall x \in X\ \forall g \in G \quad gx = x \to g = e\,.$$

(2) An action of G on H is said to be *orthogonal* if

$$\forall x, y \in H\ \forall g \in G \quad \langle gx, gy \rangle = \langle x, y \rangle\,.$$

(1.2.2) Definition. (i) X is said to be a *G-space* if X is a paracompact Hausdorff space with a fixed action of G.

(ii) X is called a *G-set* if it is a finite G-space.

(iii) A subset $A \subset X$ is called *G-invariant* if:

$$\forall x \in A\ \forall g \in G \quad gx \in A\,.$$

(1.2.3) Definition. For G-sets $X_0, \ldots, X_n$ and $k < n$, the mapping $\varphi : X_0 * \ldots * X_k \to X_0 * \ldots * X_n$ defined by $\varphi(t_0, x_0, \ldots, t_k, x_k) = (t_0, x_0, \ldots, t_k, x_k, 0, x_{k+1}, \ldots, 0, x_n)$ is a *standard embedding*.

(1.2.4) Definition. Let X, Y be G-spaces.

(i) A continuous map $f : X \to \mathbb{R}$ is called *G-invariant* if

$$\forall g \in G \ \forall x \in X \qquad f(gx) = f(x)\,.$$

(ii) A continuous map $f : X \to Y$ is called a *G-map* if

$$\forall g \in G \ \forall x \in X \qquad f(gx) = g \cdot f(x)\,.$$

The following simple facts will be useful in the next section.

(1.2.5) Proposition. (1) *If $X_0, \ldots, X_n$ are G-sets then the mapping $\nu : G \times (X_0 * \ldots * X_n) \to X_0 * \ldots * X_n$ defined by*

$$\nu(g, (t_0, x_0, \ldots, t_n, x_n)) := (t_0, gx_0, \ldots, t_n, gx_n)$$

is an action of G on $X_0 * \ldots * X_n$. Moreover, the join $X_0 * \ldots * X_n$ is a G-space.

(2) *If $f_i : X_i \to Y_i$ for $i = 0, \ldots, n$ are G-maps of G-spaces then $f_0 * \ldots * f_n : X_0 * \ldots * X_n \to Y_0 * \ldots * Y_n$ is a G-map.*

(1.3) *Sylow's Theorem.* We do not recall Sylow's Theorem in whole generality (we refer to [8] for details). The version given here is particularly suitable for an application to the general version of Borsuk's Antipodal Theorem (see (3.3)).

(1.3.1) Theorem (Sylow's Theorem). *If $|G| = p^m \cdot s$, where p is a prime number, then G contains a subgroup of order p.*

2. G-maps on S.

This section can be considered as a preparation to Section 5. We recall the definitions and theorems we need to prove the Critical Point Theorems.

Notice that the unit sphere in a Hilbert space is a C^∞ Finsler manifold (see Ex. (27.2), Prop. (27.7), Cor. (27.2) in [5]).

Denote the *tangent bundle* of S by TS and the *tangent space* at $p \in S$ by T_pS.

Recall that $p \in S$ is a *critical point* of $f \in C^1(S, \mathbb{R})$ if $df(p)(x) = 0$ for all $x \in T_pS$, where $df(p)$ is the *derivative* of f at p which is the element of the *cotangent space* T_p^*S. The function $|df| : S \to \mathbb{R}$, $p \mapsto |df(p)|$, is well-defined, nonnegative and continuous on S.

(2.1) Definition. Let M be a Finsler manifold and let $f : M \to \mathbb{R}$ be differentiable at $p \in M$. Then $v \in T_pM$ is called a *pseudo-gradient vector* for f at p if

$$|v| \le 2|df(p)|\,, \tag{2.1.i}$$

$$df(p)(v) \ge |df(p)|^2\,. \tag{2.1.ii}$$

If f is differentiable at each point of $U \subseteq M$ and v is a C^k vector field on U (M being of class C^l, $l > k$) then v is called a *pseudo-gradient vector field* for f on U if for each $p \in U$, $v(p)$ is a pseudo-gradient vector for f at p.

Similarly to (8.9) in [4] we obtain the following:

(2.2) Lemma. *If $f \in C^1(S,\mathbb{R})$, then there exists a locally Lipschitz pseudo-gradient vector field for f on $S \setminus C$, where C denotes the set of critical points of f.*

(2.3) Remark. Note that if a group G acts orthogonally on S and $f \in C^1(S,\mathbb{R})$ is G-invariant then for any $g \in G$ and $x \in S$

$$T_xS = g^{-1}T_{gx}S\,, \tag{2.3.i}$$

$$|df(x)| = |df(gx)|\,. \tag{2.3.ii}$$

We prove the following theorem:

(2.4) Theorem. *If a group G acts orthogonally on $H \setminus \{0\}$, $f \in C^1(S,\mathbb{R})$ is G-invariant and the set C of critical points of f is finite then for every open neighbourhood N of C there exists a vector field $\widetilde{W} : S \to TS$ such that*:

$$|\widetilde{W}(v)| \le 2|df(v)| \qquad \textit{for } v \in S\,, \tag{2.4.i}$$

$$df(v)(\widetilde{W}(v)) \ge |df(v)|^2 \qquad \textit{for } v \in S \setminus N\,, \tag{2.4.ii}$$

$$df(v)(\widetilde{W}(v)) \ge 0 \qquad \textit{for } v \in S\,, \tag{2.4.iii}$$

$$\widetilde{W} \textit{ is Lipschitz continuous}\,, \tag{2.4.iv}$$

$$\forall h \in G \quad \widetilde{W}(hv) = h\widetilde{W}(v)\,. \tag{2.4.v}$$

Proof. Since by our assumption C is finite, we put $C = \{x_0,\ldots,x_l\}$. Moreover, we can find $\varepsilon > 0$ such that

$$\forall 0 \le i \le l \quad B(x_i,3\varepsilon) \cap (C \setminus \{x_i\}) = \emptyset$$

and $B(C,\varepsilon) \subset N$.

For $i = 0,\ldots,l$ let $g_i : S \to [0,1]$ denote an Urysohn function such that $g_i|_{\{x_i\}} \equiv 0$ and $g_i|_{S\setminus B(x_i,\varepsilon)} \equiv 1$. Obviously, g_i is locally Lipschitz. Note that if $v \notin C$ then $g_i(v) > 0$. Now we use Lemma (2.2) to construct a vector field $\widehat{W} : S \to TS$ (comp. (4.2) in [12])

$$\widehat{W}(v) := \begin{cases} g_0(v)\cdot\ldots\cdot g_1(v)W(v)\,, & v \notin C\,,\\ 0\,, & v \in C\,,\end{cases}$$

where $W : S \setminus C \to TS$ denotes a pseudo-gradient vector field for f. It is easily seen that:

(2.4.1) $$|\widehat{W}(v)| \le |W(v)| \qquad \text{for every } v \in S \setminus C\,,$$

(2.4.2) $$df(v)(W(v)) \ge |df(v)|^2 \qquad \text{for } v \in S \setminus B(C, \varepsilon)\,,$$

(2.4.3) $$\begin{aligned} &df(v)(\widehat{W}(v)) \ge 0 && \text{for every } v \in S\,, \\ &df(v)(\widehat{W}(v)) > 0 && \text{for every } v \in S \setminus C\,, \end{aligned}$$

(2.4.4) $$\widehat{W} : S \to TS \text{ is locally Lipschitz continuous}\,.$$

Now we define a vector field $\widetilde{W} : S \to TS$ by

$$\widetilde{W}(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1} \widehat{W}(gv)\,.$$

It is easily checked that $\widetilde{W}$ has the required properties. ∎

(2.5) Lemma. *Let the assumptions of* (2.4) *hold and let f be bounded. Then there exists a flow $\Phi : \mathbb{R} \times S \to S$ such that*

(2.5.i) $$\begin{cases} \partial\Phi(t,x)/\partial t = \widetilde{W}(\Phi(t,x))\,, \\ \Phi(0,x) = x\,, \end{cases}$$

where $\widetilde{W}$ denotes the vector field from (2.4). *Moreover*,

(2.5.ii) $$\forall t \in \mathbb{R}\ \forall g \in G\ \forall y \in S \qquad \Phi(t, gy) = g\Phi(t, y)\,.$$

Proof. Since the proof of the first part of the conclusion is analogous to the proof of Lemma 5.5 in Section 4.5 of [4] we omit it. (2.5.ii) is easy to check since $\widetilde{W}$ is a G-map. ∎

3. Extension of Borsuk's Antipodal Theorem. In this section we establish a theorem which represents a general version of Borsuk's Antipodal Theorem:

(3.1) Theorem. *Let G be a nontrivial finite group. If there exists a G-map $\phi : G_{(n)} \to G_{(k)}$ then $k \ge n$.*

(3.2) Remark. Note that this is in fact a generalization of Borsuk's Antipodal Theorem (see [7]). Consider the group $G = \mathbb{Z}_2$. Recall that $\mathbb{Z}_2 \approx S^0$. Furthermore, by (1.1.2), $(S^0)_{(n)} \approx S^n$. Since a $\mathbb{Z}_2$-map is antipodal-preserving we conclude that the theorem below results from (3.1):

Theorem (Borsuk's Antipodal Theorem). *There is no antipodal-preserving map $f : S^n \to S^{n-1}$.*

Our aim is to prove a more general result:

(3.3) THEOREM. *Assume that G is a nontrivial finite group which acts freely on G-sets $X_0, \ldots, X_n$ $(n \in \mathbb{N})$. If there exists a G-map $\phi : X_0 * \ldots \ldots * X_n \to G_{(k)}$ then $k \geq n$.*

Now we prove some technical lemmas.

(3.4) LEMMA. *Assume that G is a nontrivial finite group acting freely on G-sets $X_0, \ldots, X_n$, $Y_0, \ldots, Y_n$. Moreover, assume that for each $i = 0, \ldots, n$ there exists a surjective G-map $f_i : X_i \to Y_i$. Then there is a G-map $\mu : Y_0 * \ldots * Y_n \to X_0 * \ldots * X_n$ such that*

$$f \circ \mu = \mathrm{id}_{Y_0 * \ldots * Y_n} ,$$

where $f = f_0 * \ldots * f_n : X_0 * \ldots * X_n \to Y_0 * \ldots * Y_n$.

Proof. Since for each i, G acts freely on X_i and on Y_i, and f_i is a surjective G-map, it is easy to see that f_i maps each orbit in X_i one-to-one onto an orbit in Y_i. We define a G-map $\mu_i : Y_i \to X_i$ by choosing, for each orbit in Y_i, one of its f_i-preimages and reversing f_i. Then clearly $\mu = \mu_0 * \ldots * \mu_n$ is the required G-map (see (1.2.5.2)). ∎

We now recall, in a modified form, a classical property of classifying spaces (cf. [11]). We use the following notation:

p — a prime number;

$H_q(\cdot)$ — qth homology group with coefficients in $\mathbb{Z}_p$;

$B(n) := (\mathbb{Z}_p)_{(n)}/\mathbb{Z}_p$.

(3.5) PROPOSITION. *Let $i^\sim : B(n) \to B(k)$ denote the mapping induced by the standard embedding $i : (\mathbb{Z}_p)_{(n)} \to (\mathbb{Z}_p)_{(k)}$ where $k \geq n$. Then:*

(i) *$H_q(B(n)) \neq 0$ for $q \leq n$;*

(ii) *$H_q(B(n)) = 0$ for $q > n$;*

(iii) *the induced homomorphism $(i^\sim)_* : H_q(B(n)) \to H_q(B(k))$ is nontrivial for $q \leq n \leq k$.*

(3.6) LEMMA. *Assume that $X_0, \ldots, X_n$ are $\mathbb{Z}_p$-sets, $\mathbb{Z}_p$ acts freely on each X_i and $f_i : X_i \to \mathbb{Z}_p$ $(i = 0, \ldots, n)$ is a surjective $\mathbb{Z}_p$-map. Then for all $q \leq n$ the homomorphism*

$$f_*^\sim : H_q(X_0 * \ldots * X_n/\mathbb{Z}_p) \to H_q(B(n))$$

induced by $f = f_0 * \ldots * f_n : X_0 * \ldots * X_n \to (\mathbb{Z}_p)_{(n)}$ is an epimorphism; in particular, $H_q(X_0 * \ldots * X_n/\mathbb{Z}_p) \neq 0$ for $q \leq n$.

Proof. From (3.4) it follows that there is a $\mathbb{Z}_p$-map $\mu : (\mathbb{Z}_p)_{(n)} \to X_0 * \ldots * X_n$ with $f \circ \mu = \mathrm{id}_{(\mathbb{Z}_p)_{(n)}}$. Hence $f_*^\sim : H_q(X_0 * \ldots * X_n/\mathbb{Z}_p) \to H_q(B(n))$

and $\mu_*^\sim : H_q(B(n)) \to H_q(X_0 * \ldots * X_n/\mathbb{Z}_p)$ also satisfy $f_*^\sim \circ \mu_*^\sim = \mathrm{id}$, and the assertion follows from (3.5.i). ■

(3.7) LEMMA. *Suppose that $X_0, \ldots, X_n$ are $\mathbb{Z}_p$-sets and $\mathbb{Z}_p$ acts freely on X_i for $0 \le i \le n$. If $\Phi : X_0 * \ldots * X_n \to (\mathbb{Z}_p)_{(k)}$ is a $\mathbb{Z}_p$-map then $k \ge n$.*

Proof. Assume that $k < n$. Let $i : (\mathbb{Z}_p)_{(k)} \to (\mathbb{Z}_p)_{(2n+1)}$ denote the standard embedding. Consider the induced mappings

$$(\Phi^\sim)_* : H_n(X_0 * \ldots * X_n/\mathbb{Z}_p) \to H_n(B(k))$$

and

$$(i^\sim)_* : H_n(B(k)) \to H_n(B(2n+1)) .$$

By (3.5.ii) the homomorphism

$$(3.7.1) \qquad (i^\sim \circ \Phi^\sim)_* = i_*^\sim \circ \Phi_*^\sim : H_n(X_0 * \ldots * X_n/\mathbb{Z}_p) \to H_n(B(2n+1))$$

is trivial.

Consider now the mapping $j : (\mathbb{Z}_p)_{(n)} \to (\mathbb{Z}_p)_{(2n+1)}$ given by

$$j(t_0, x_0, \ldots, t_n, x_n) = (0, x_0, \ldots, 0, x_n, t_0, x_0, \ldots, t_n, x_n) .$$

It is easy to see that j is a $\mathbb{Z}_p$-map. It is also easy to construct a surjective $\mathbb{Z}_p$-map $f_i : X_i \to \mathbb{Z}_p$, since the action of $\mathbb{Z}_p$ on X_i is free and we may identify each orbit with $\mathbb{Z}_p$. Hence $f = f_0 * \ldots * f_n : X_0 * \ldots * X_n \to (\mathbb{Z}_p)_{(n)}$ is also a $\mathbb{Z}_p$-map. From (3.6) it follows that the homomorphism $f_*^\sim : H_n(X_0 * \ldots * X_n/\mathbb{Z}_p) \to H_n(B(n))$ is an epimorphism.

Moreover, it is easily seen that the standard embedding $\bar{i} : (\mathbb{Z}_p)_{(n)} \to (\mathbb{Z}_p)_{(2n+1)}$ and j are $\mathbb{Z}_p$-homotopic. Hence $\bar{i}^\sim$ and $j^\sim$ are homotopic. Thus by (3.5.iii) the homomorphism $j_*^\sim : H_n(B(n)) \to H_n(B(2n+1))$ is nontrivial. Hence

$$(3.7.2) \qquad (j \circ f)_*^\sim \text{ is nontrivial.}$$

In view of (3.7.1) and (3.7.2) the desired contradiction follows if we observe that the mapping $\mathcal{H} : X \times I \to (\mathbb{Z}_p)_{(2n+1)}$, where $X = X_0 * \ldots * X_n$, defined by

$$\begin{aligned}
&\mathcal{H}(x,t)\\
&\quad = (t_{(0)}(\Phi(x)) \cdot (1-t), x_{(0)}(\Phi(x)), \ldots, t_{(k)}(\Phi(x)) \cdot (1-t), x_{(k)}(\Phi(x)),\\
&\qquad 0, x_{k+1}, \ldots, t_{(0)}(f(x)) \cdot t, x_{(0)}(f(x)), \ldots, t_{(n)}(f(x)) \cdot t, x_{(n)}(f(x)))
\end{aligned}$$

is a $\mathbb{Z}_p$-homotopy from $i \circ \Phi$ to $j \circ f$. ■

Proof of Theorem (3.3). Since by our assumption $G \neq \{e\}$, we can find a prime number p and a subgroup P of G such that $P \approx \mathbb{Z}_p$ (by Sylow's Theorem, see (1.3.1)). Clearly, for each $i = 0, \ldots, n$, X_i is a $\mathbb{Z}_p$-set. We also see that G is a $\mathbb{Z}_p$-set. Moreover, $\mathbb{Z}_p$ acts freely on G and on $X_0, \ldots, X_n$. In particular, we can construct a surjective $\mathbb{Z}_p$-map $\kappa : G \to \mathbb{Z}_p$ as in the proof of (3.7). Then $\kappa_{(k)} \circ \phi : X_0 * \ldots * X_n \to (\mathbb{Z}_p)_{(k)}$ is a $\mathbb{Z}_p$-map and Lemma (3.7) applies. ∎

4. Construction of G-maps φ and η. In this section we will keep the notation introduced earlier in this paper.

First we will prove the existence of a G-map $\varphi : G_{(n)} \to S^n$ (see the introduction).

A subset A of S^n will be called an *elementary m-dimensional set* if there exist $(m+1)$-dimensional linear subspaces $W_1, \ldots, W_k$ of $\mathbb{R}^{n+1}$ such that

$$A = S^n \cap \bigcup_{i=1}^{k} W_i .$$

(4.1) Theorem. *If G acts orthogonally on S^n then there exists a G-map $\varphi : G_{(n)} \to S^n$.*

Proof (by induction). We show that for each $l = 0, \ldots, n$

(4.1.1) there exist a G-invariant elementary l-dimensional set A_l and a G-map $\varphi_l : G_{(l)} \to A_l$.

This is obvious in case $l = 0$. Assume that (4.1.1) holds for $m < n$ (in case $m = n$ the proof is complete). Since $\dim A_m < n$ we can choose $x_{m+1} \in S^n \setminus A_m$. Let $A_m := S^n \cap \bigcup_{i=1}^{k_m} W_i^m$. Consider

$$A_{m+1} := S^n \cap \Big(\bigcup_{g\in G} \bigcup_{i=1}^{k_m} \mathrm{lin}(x_0^{(i)}, \ldots, x_m^{(i)}, gx_{m+1}) \Big),$$

where $\{x_0^{(i)}, \ldots, x_m^{(i)}\}$ denotes a base of W_i^m. Note that among the subspaces $\mathrm{lin}(x_0^{(i)} \ldots, x_m^{(i)}, gx_{m+1})$ (where $i = 1, \ldots, k_m$) there exist $(m+2)$-dimensional ones. Thus $A_{m+1} = S^n \cap \bigcup_{i=1}^{k_{m+1}} W_i^{m+1}$, where W_i^{m+1} for $i = 1, \ldots, k_{m+1}$ are $(m+2)$-dimensional linear subspaces of $\mathbb{R}^{n+1}$. Since G acts orthogonally on S^n and A_m is G-invariant it follows that A_{m+1} is G-invariant. Now we define $\varphi_{m+1} : G_{(m+1)} \to A_{m+1}$ by

$$\varphi_{m+1}(y, t, g) = \frac{(1-t)\varphi_m(y) + t \cdot gx_{m+1}}{|(1-t)\varphi_m(y) + t \cdot gx_{m+1}|}$$

for $y \in G_{(m)}$, $g \in G$ and $t \in [0, 1]$. Clearly φ_{m+1} is a G-map. ∎

(4.2) Corollary. *If G acts orthogonally on $H \setminus \{0\}$, where H denotes a Hilbert space, then for every $n \in \mathbb{N}$ there exists a G-map $\varphi : G_{(n)} \to S$.*

Proof. Since the proof is much the same as that given for Theorem (4.1) we only sketch the essential steps.

Take any $x_0 \in H \setminus \{0\}$ and define $H_0 = \mathrm{span}(Gx_0)$. Define $\varphi_0 : G_{(0)} \to S$ by setting $\varphi_0(g) = gx_0$ for any $g \in G$.

Suppose $H_m \subset H$ and $\varphi_m : G_{(m)} \to S$ have been constructed for all $m < n$. Let $x_{m+1} \in (H_0 \oplus \ldots \oplus H_m)^\perp \setminus \{0\}$ and $H_{m+1} = \operatorname{span}(Gx_{m+1})$. Defining $\varphi_{m+1} : G_{(m+1)} \to S$ by

$$\varphi_{m+1}(y, t, g) = \frac{(1-t)\varphi_m(y) + t \cdot gx_{m+1}}{|(1-t)\varphi_m(y) + t \cdot gx_{m+1}|}$$

for $y \in G_{(m)}$, $g \in G$ and $t \in [0, 1]$ we are done. ∎

Now we construct a G-map η (see the introduction). Let S denote the sphere (either in $\mathbb{R}^{n+1}$ or in H). Suppose that a finite group G acts freely on S, the set C of critical points of the G-invariant functional $f \in C^1(S, \mathbb{R})$ is finite and that $f(C) = \{\lambda_0, \ldots, \lambda_k\}$. We denote by R_j the set $f^{-1}(\lambda_j) \cap C$ for $j = 0, \ldots, k$.

(4.3) THEOREM. *There exists a G-map $\eta : R_0 * \ldots * R_k \to G_{(k)}$.*

P r o o f. It suffices to observe that each R_i is a (free) G-set, construct a G-map $\eta_i : R_i \to G$ as before (see the proof of (3.7)), and take $\eta := \eta_0 * \ldots * \eta_k$. ∎

5. Construction of a G-map ψ. We keep the previous notation. Let us introduce:

CONDITION (P–S). If $\{f(v_n)\}$ is bounded and $|df(v_n)| \to 0$ as $n \to \infty$, then $\{v_n\}$ has a convergent subsequence.

In this section we prove

(5.1) THEOREM. *Suppose that a finite group G acts freely and orthogonally on $H \setminus \{0\}$, $f \in C^1(S, \mathbb{R})$ is bounded, G-invariant and condition* (P–S) *is satisfied. Moreover, let the set C of critical points of f be finite and $f(C) = \{\lambda_0, \ldots, \lambda_k\}$. Then there exists a G-map*

$$\psi : S \to R_0 * \ldots * R_k\,,$$

where R_j denotes the set $f^{-1}(\lambda_j) \cap C$ for $j = 0, \ldots, k$.

Since by our assumption C is finite, we can find $\varepsilon > 0$ such that

$$\forall x \in C \qquad B(x, 3\varepsilon) \cap (C \setminus \{x\}) = \emptyset\,.$$

Let $\gamma := \inf\{|\nabla f(y)| \mid y \in S \setminus B(C, \varepsilon)\}$. Since f is bounded and condition (P–S) is satisfied it is easy to show that $\gamma > 0$. Now, since by our assumption f is continuous and C is finite it follows that there exists $\varepsilon_1 > 0$ such that

(5.2.i) if $x \in B(c, 2\varepsilon_1)$ then $|f(x) - f(c)| < \gamma\varepsilon/4$ for $x \in S$, $c \in C$.

We can assume without loss of generality that $2\varepsilon_1 \le \varepsilon$.

From (2.4) it follows that for $N = B(C, \varepsilon_1)$ there exists a G-map $\widetilde{W} : S \to TS$ which satisfies the properties (2.4.i)–(2.4.v). Thus by (2.5) there exists a global flow $\Phi : \mathbb{R} \times S \to S$ satisfying (2.5.i) and (2.5.ii).

For every $x \in C$ we define

$$A_x := \overline{B(x, \varepsilon_1)} \quad \text{and} \quad B_x := B(x, 2\varepsilon_1)\,. \tag{5.2.ii}$$

We will use the notations introduced here throughout this section.

A. *Construction of an open covering $\mathcal{V}$ of S*

(5.3) Remark. It is easily seen that if $a \in C$ and $t > 0$ then

$$\mathfrak{V}(a,t) := \{y \in S \mid \exists s \in [0,t]\ \exists b \in C \setminus \{a\}\ \Phi(s,y) \in A_b\}$$

is a closed set.

(5.4) Lemma. *Let $a \in C$ and*

$$V_a := \{y \in S \mid \exists t > 0\ \Phi(t,y) \in B_a \wedge y \notin \mathfrak{V}(a,t)\}\,.$$

Then $\mathcal{V} = \{V_a\}_{a \in C}$ is an open covering of S satisfying

$$\forall a \in C\ \forall g \in G \quad V_{ga} = gV_a\,.$$

Proof. Since $V_a = \bigcup_{t>0} \Phi_t^{-1}(B_a) \setminus \mathfrak{V}(a,t)$, by (5.3) it follows that each V_a is open.

In order to show that $\{V_a\}_{a \in C}$ is a covering of S it suffices to prove that for any $x \in S$ there exists $t > 0$ such that $\Phi(t,x) \in B(C, 2\varepsilon_1)$. On the contrary, assume

$$\exists x \in S\ \forall \tau > 0 \quad \Phi(\tau, x) \in S \setminus B(C, 2\varepsilon_1)\,.$$

Note that

$$\partial f(\Phi(t,x))/\partial t = df(\Phi(t,x))(\widetilde{W}(\Phi(t,x))) \geq 0\,.$$

This implies that $f(\Phi(\cdot,x)) : \mathbb{R} \to \mathbb{R}$ is non-decreasing. Hence

$$\lim_{t\to\infty} f(\Phi(t,x)) = c \leq +\infty\,.$$

Moreover, for $s \leq t$

$$\begin{aligned} f(\Phi(t,x)) - f(\Phi(s,x)) &= \int_s^t df(\Phi(\tau,x))(\widetilde{W}(\Phi(\tau,x)))\,d\tau \\ &\geq \int_s^t |df(\Phi(\tau,x))|^2\,d\tau\,. \end{aligned} \tag{5.4.i}$$

Since f is bounded, $\int_0^\infty |df(\Phi(\tau,x))|^2\,d\tau < +\infty$. Thus

$$\forall \varepsilon > 0\ \exists r \in \mathbb{R}\ \forall x > r\ \forall x' > r \quad \Big|\int_x^{x'} |df(\Phi(\tau,x))|^2\,d\tau\Big| < \varepsilon\,. \tag{5.4.ii}$$

For $n \in \mathbb{N}$ we take $\varepsilon = 1/n$. There exists $r_n \in \mathbb{R}$ such that for $x > r_n$ and $x' > r_n$ condition (5.4.ii) is satisfied. We can assume without loss of generality that $r_n > n$ and take $x = r_n + 1$, $x' = r_n + 2$. Since $|df(\Phi(\cdot,x))| : \mathbb{R} \to \mathbb{R}$

is continuous there exists $t_n \in [r_n+1, r_n+2]$ such that

$$\int_x^{x'} |df(\Phi(\tau,x))|^2 d\tau = 1 \cdot |df(\Phi(t_n,x))|^2 < \frac{1}{n}\,.$$

We obtain a sequence $t_n \to +\infty$ such that $|df(\Phi(t_n,x))|^2 \to 0$. By condition (P–S) and the assumption that f is bounded we conclude that there exists a convergent subsequence $\{\Phi(t_{n_k},x)\}$ of $\{\Phi(t_n,x)\}$. Let $\Phi(t_{n_k},x) \to q$. The continuity of $|df|$ implies that $q \in C$ contrary to the assumption. Note that $A_{ga} = gA_a$ and $B_{ga} = gB_a$. By (2.5.ii), $\Phi(s,ga) = g\Phi(s,a)$ for any $s \in \mathbb{R}$. Moreover, by (2.3) the set C is G-invariant. Since G acts freely on $H\setminus\{0\}$, we conclude that $g\mathfrak{V}(a,s) = \mathfrak{V}(ga,s)$ for $s \in \mathbb{R}$. Thus our assertion follows. ∎

B. *The properties of the covering* $\mathcal{V}$

(5.5) PROPOSITION. *If $a_1, a_2 \in R_j$ and $a_1 \neq a_2$ then $V_{a_1} \cap V_{a_2} = \emptyset$.*

Proof. We need to prove that if $a, b \in C$, $a \neq b$ and $f(a) = f(b)$ then $V_a \cap V_b = \emptyset$. Indeed, assuming that there exists $x \in V_a \cap V_b$ from the definition of V_a and V_b we find:

(i) $t_a > 0$ such that $\Phi(t_a,x) \in B_a$; and
(ii) $t_b > 0$ such that $\Phi(t_b,x) \in B_b$.

Since $B_a \cap B_b = \emptyset$ it follows that $t_a \neq t_b$. Now it suffices to prove that if a and b are distinct points of C, $x \in S$ and there exist $t_1, t_2 \in \mathbb{R}$ such that $t_1 < t_2$, $\Phi(t_1,x) \in B_a$ and $\Phi(t_2,x) \in B_b$ then $f(a) \neq f(b)$. Observe that

$$|f(a) - f(b)| \geq |f(\Phi(t_1,x)) - f(\Phi(t_2,x))| - (|f(\Phi(t_1,x)) - f(a)| \\ + |f(\Phi(t_2,x)) - f(b)|)\,.$$

Note that by (5.2.i),

$$|f(\Phi(t_1,x)) - f(a)| < \gamma\varepsilon/4\,, \quad |f(\Phi(t_2,x)) - f(b)| < \gamma\varepsilon/4\,.$$

Now it is sufficient to show that $|f(\Phi(t_1,x)) - f(\Phi(t_2,x))| \geq \gamma\varepsilon/2$. We will use the properties of the mapping $\widetilde{W}$. We have

$$\begin{aligned} &|f(\Phi(t_1,x)) - f(\Phi(t_2,x))| \\ &= \Big|\int_{t_1}^{t_2} \frac{\partial}{\partial t}(f(\Phi(t,x)))\,dt\Big| = \Big|\int_{t_1}^{t_2} df(\Phi(t,x))(\Phi'_t(t,x))\,dt\Big| \\ &= \Big|\int_{t_1}^{t_2} df(\Phi(t,x))(\widetilde{W}(\Phi(t,x)))\,dt\Big| = \int_{t_1}^{t_2} df(\Phi(t,x))(\widetilde{W}(\Phi(t,x)))\,dt\,. \end{aligned}$$

Let $t_3 := \sup\{t \in [t_1,t_2] \mid \Phi(t,x) \in B(a,\varepsilon)\}$. Note that since $B_a \subset B(a,\varepsilon)$ it follows that t_3 is well-defined. Furthermore, by the continuity of Φ, $t_3 \neq t_2$.

We define

$$t_4 := \inf\{t \in [t_3, t_2] \mid \Phi(t,x) \in B(C,\varepsilon)\}\,.$$

Then by a similar argument t_4 is well-defined. Moreover, $t_3 \neq t_4$. It is obvious that $\Phi(t,x) \in S \setminus B(C,\varepsilon)$ for $t \in]t_3, t_4[$. Hence

$$\begin{aligned}\int_{t_3}^{t_4} df(\Phi(t,x))(\widetilde{W}(\Phi(t,x)))\,dt &\geq \int_{t_3}^{t_4} |df(\Phi(t,x))|^2\,dt \\ \geq \gamma \int_{t_3}^{t_4} |df(\Phi(t,x))|\,dt &\geq \frac{\gamma}{2}\int_{t_3}^{t_4} |\widetilde{W}(\Phi(t,x))|\,dt \geq \frac{\gamma}{2}\Big|\int_{t_3}^{t_4} \widetilde{W}(\Phi(t,x))\,dt\Big| \\ = \frac{\gamma}{2}\Big|\int_{t_3}^{t_4} \frac{\partial}{\partial t}(\Phi(t,x))\,dt\Big| &= \frac{\gamma}{2}|\Phi(t_4,x) - \Phi(t_3,x)|\,.\end{aligned}$$

Note that $\Phi(t_4,x) \in \overline{B(C\setminus\{a\},\varepsilon)}$ and $\Phi(t_3,x) \in \overline{B(a,\varepsilon)}$. This implies $|\Phi(t_4,x) - \Phi(t_3,x)| \geq \varepsilon$ as required. ∎

C. *Partition of unity subordinate to the covering $\mathcal{V}$*

It is not difficult to show

(5.6) LEMMA. *There exists a partition of unity $\{\theta_a\}_{a\in C}$ subordinate to the covering $\{V_a\}_{a\in C}$ such that*

(5.6.i) $$\forall x \in S\ \forall a \in C\ \forall g \in G \quad \theta_{ga}(x) = \theta_a(g^{-1}x)\,.$$

D. *Construction of a mapping $\psi : S \to R_0 * \ldots * R_k$*

(5.7) Remark. By (5.5) for every $x \in S$ and $j \in \{0,\ldots,k\}$ there exists at most one element $a_x(j) \in R_j$ such that $x \in V_{a_x(j)}$. Clearly,

$$\forall x \in S \quad \sum_{j=0}^{k} \theta_{a_x(j)}(x) = 1\,.$$

Now, it is easy to complete the

Proof of Theorem (5.1). It suffices to prove that the map $\psi : S \to R_0 * \ldots * R_k$ defined by

$$\psi(x) = [\theta_{a_x(0)}(x), a_x(0), \ldots, \theta_{a_x(k)}(x), a_x(k)]$$

is a G-map. By the definition of the topology of join and the continuity of θ_a for $a \in C$, ψ is continuous. For $x \in S$ and $g \in G$ we have

$$\psi(gx) = (\theta_{a_{gx}(0)}(gx), a_{gx}(0), \ldots, \theta_{a_{gx}(k)}(gx), a_{gx}(k))\,,$$

where $a_{gx}(j)$ is the unique element of R_j such that $gx \in V_{a_{gx}(j)}$ for $j \in \{0, \ldots, k\}$. According to (5.6.i),

$$\begin{aligned}\psi(gx) &= (\theta_{g^{-1}a_{gx}(0)}(x), a_{gx}(0), \ldots, \theta_{g^{-1}a_{gx}(k)}(x), a_{gx}(k))\\ &= g \cdot (\theta_{g^{-1}a_{gx}(0)}(x), g^{-1}a_{gx}(0), \ldots, \theta_{g^{-1}a_{gx}(k)}(x), g^{-1}a_{gx}(k))\,.\end{aligned}$$

Note that (5.5) and (5.7) imply that $g^{-1}a_{gx}(j) = a_x(j)$ for $j \in \{0, \ldots, k\}$. Thus $\psi(gx) = g\psi(x)$. ∎

The results proved so far have the following simple consequence:

(5.8) THEOREM. *Suppose that a finite group G acts freely and orthogonally on $\mathbb{R}^{n+1} \setminus \{0\}$, $f \in C^1(S^n, \mathbb{R})$ is G-invariant and that the set C of critical points of f is finite and $f(C) = \{\lambda_0, \ldots, \lambda_k\}$. Then there exists a G-map*

$$\psi : S^n \to R_0 * \ldots * R_k\,,$$

where $R_j := f^{-1}(\lambda_j) \cap C$ for $j = 0, \ldots, k$. ∎

6. Critical Point Theorems. Let S denote the unit sphere in a Hilbert space H.

(6.1) THEOREM. *Suppose that a finite nontrivial group G acts freely and orthogonally on $H \setminus \{0\}$, $f \in C^1(S, \mathbb{R})$ is bounded, G-invariant and condition* (P–S) *is satisfied. Then f has an infinite number of critical points.*

Proof. Assume that the set C of critical points of f is finite. Then (5.1) applies: there exists a G-map $\psi : S \to R_0 * \ldots * R_k$, where $k+1$ is the number of critical values of f. By (4.2) there exists a G-map $\varphi : G_{(n)} \to S$ for any $n \in \mathbb{N}$. By (4.3) there exists a G-map $\eta : R_0 * \ldots * R_k \to G_{(k)}$. Then $\eta \circ \psi \circ \varphi : G_{(n)} \to G_{(k)}$ is a G-map. Now we apply the generalized Borsuk Antipodal Theorem to obtain $k+1 \geq n+1$ for any n, contradicting our hypothesis. ∎

The next theorem may be proved like Theorem (6.1), by means of Theorem (5.8).

(6.2) THEOREM. *If a finite nontrivial group G acts freely and orthogonally on S^n and $f \in C^1(S^n, \mathbb{R})$ is G-invariant then f has at least $n+1$ critical orbits. Moreover, if the number of critical orbits is finite then there exist at least $n+1$ critical values.* ∎

(6.3) Remark. Let the assumptions of Theorem (6.2) hold. Since the G-action is free and since if x is a critical point of f then so is gx (comp. (2.3)), it follows that f has at least $(n+1)|G|$ critical points.

(6.4) THEOREM. *Suppose that G is a nontrivial finite group which acts freely and orthogonally on $H \setminus \{0\}$ and that $f \in C^1(S, \mathbb{R})$ is G-invariant.*

Then for every $n \in \mathbb{N}$ there exists a finite-dimensional sphere such that the restriction of f to it has at least $n+1$ critical orbits.

Proof. We choose an element $x_0 \in H \setminus \{0\}$ and consider the linear span of its orbit, $H_0 := \operatorname{span}(Gx_0)$. It is easily seen that H_0 is a G-invariant subset of H and $H_0 \cap S$ is a finite-dimensional sphere. Set $S^{m_0} := H_0 \cap S$. Now suppose that H_i has been constructed for all $i < n$. Choose $x_n \in (H_0 \oplus \ldots \oplus H_{n-1})^{\perp} \setminus \{0\}$ and let $H_n := \operatorname{span}(Gx_n) \cap S$. Since H_i is G-invariant for each $i = 1, \ldots, n$, so is $H_0 \oplus \ldots \oplus H_n$. Clearly, $(H_0 \oplus \ldots \oplus H_n) \cap S$ is a finite-dimensional sphere. Let $S^{m_n} := (H_0 \oplus \ldots \oplus H_n) \cap S$. Note that $m_n > m_{n-1}$. Thus $m_n \geq n$. We may assume that the set C_n of critical points of $f|S^{m_n}$ is finite. Let $f(C_n) = \{\lambda_0^n, \ldots, \lambda_{k_n}^n\}$. According to Theorem (5.8) there exists a G-map

$$\psi_n : S^{m_n} \to R_0 * \ldots * R_k ,$$

where $R_i := (f|S^{m_n})^{-1}(\lambda_i^n) \cap C_n$ for $i = 0, \ldots, k_n$.

By similar arguments to the proof of Theorem (6.1) we conclude that $k_n \geq m_n$. This implies that $k_n + 1 \geq n + 1$ and the assertion follows. ■

References

[1] T. Bartsch, *Critical orbits of invariant functionals and symmetry breaking*, Forschungsschwerpunkt Geometrie, Universität Heidelberg, Heft Nr 34, 1988.

[2] V. Benci and F. Pacella, *Morse theory for symmetric functionals on the sphere and an application to a bifurcation problem*, Nonlinear Anal. 9 (1985), 763–771.

[3] G. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, 1972.

[4] S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer, 1982.

[5] K. Deimling, *Nonlinear Functional Analysis*, Springer, 1985.

[6] E. Fadell and P. H. Rabinowitz, *Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems*, Invent. Math. 45 (1978), 139–174.

[7] A. Granas and J. Dugundji, *Fixed Point Theory*, PWN, Warszawa 1982.

[8] M. Hall, *The Theory of Groups*, Macmillan, New York 1959.

[9] W. Krawcewicz and W. Marzantowicz, *Lusternik–Schnirelmann method for functionals invariant with respect to a finite group action*, preprint, 1988.

[10] —, —, *Some remarks on the Lusternik–Schnirelmann method for non-differentiable functionals invariant with respect to a finite group action*, preprint, 1988.

[11] J. Milnor, *Construction of universal bundles, II*, Ann. of Math. 63 (1956), 430–436.

[12] R. S. Palais, *Critical point theory and the minimax principle*, in: Proc. Sympos. Pure Math. 15, Amer. Math. Soc., 1970, 185–212.

[13] —, *Lusternik–Schnirelmann theory on Banach manifolds*, Topology 5 (1966), 115–132.

[14] P. H. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math. 65, Amer. Math. Soc., 1984.

[15] V. A. Rokhlin and D. B. Fuks, *An Introductory Course in Topology. Geometric Chapters*, Nauka, Moscow 1977 (in Russian).
[16] J. A. Wolf, *Spaces of Constant Curvature*, University of California, Berkeley, Calif., 1972.

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