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On one-dimensional diffusion processes living in a bounded space interval

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Abstract. We prove that under some assumptions a one-dimensional Itô equation has a strong solution concentrated on a finite spatial interval, and the pathwise uniqueness holds.

Introduction. In the present paper we will consider a diffusion satisfying the stochastic integral Itô equation

$$X(t) = X(0) + \int_0^t a(s, X(s))\, ds + \int_0^t b(s, X(s))\, dW(s) \tag{1}$$

where $W(t)$ is a given one-dimensional Wiener process on a probability space $(\Omega, \mathcal{F}, P)$.

It is known ([1], p. 372) that if $b(t, r_i) = 0 \le (-1)^i a(t, r_i)$, $i = 0, 1$, $t \ge 0$, and if a and b are sufficiently regular, then (1) has a unique solution $X(t)$ concentrated on the interval $[r_0, r_1]$.

In this paper we consider strong solutions of (1) ([3], p. 149). An example of a stochastic integral equation which has a solution but has no strong solution is due to H. Tanaka ([3], p. 152). We will give some sufficient conditions in order that (1) has a unique (in the sense of pathwise uniqueness) strong solution $X(t)$, satisfying $X(t) \in (\alpha(t), \beta(t))$ for $t \ge 0$, where α and β are given sufficiently regular real-valued functions defined for $t \ge 0$.

Existence and pathwise uniqueness of the strong solution of equation (1) on a finite spatial interval. First we give some sufficient conditions in order that a strong solution $X(t)$ of the stochastic equation

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(1) exists and satisfies the additional condition

$$|X(t)| < 1 \quad \text{for } t \geq 0 .$$

We will need the following theorem ([1], Theorem 3.11, p. 300 in the case $d = 1$):

THEOREM 1. *Let $a : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ and $b : [0,\infty) \times \mathbb{R} \to \mathbb{R}$ be locally bounded and Borel measurable. Suppose that for each $T > 0$ and $N \geq 1$ there exist constants K_T and $K_{T,N}$ such that*

1) $|b(t,x)|^2 \leq K_T(1 + x^2)\,, \quad xa(t,x) \leq K_T(1 + x^2)\,,$
$$0 \leq t \leq T\,,\ x \in \mathbb{R}\,,$$
2) $|b(t,x) - b(t,y)| \vee |a(t,x) - a(t,y)| \leq K_{T,N}|x - y|\,,$
$$0 \leq t \leq T\,,\ |x| \vee |y| \leq N\,.$$

Given a 1-dimensional Brownian motion W and an independent $\mathbb{R}$-valued random variable ξ on a probability space $(\Omega, \mathcal{F}, P)$ such that $\mathbb{E}[|\xi|^2] < \infty$, there exists a process X with $X(0) = \xi$ a.s. such that $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}, W, X)$ is a solution of the stochastic integral equation (1), where $\mathcal{F}_t = \mathcal{F}_t^W \vee \sigma(\xi)$ ($\sigma(\xi)$ denotes the minimal σ-algebra with respect to which ξ is measurable).

Let $\Phi(t,x)$ be a monotone (in x) continuous function, defined for $t \in [0,T], x \in (-1,1)$, for which the derivatives $\Phi_t(t,x)$, $\Phi_x(t,x)$ and $\Phi_{xx}(t,x)$ exist and are continuous. For each $t \in [0,T]$ there exists a function $\Psi(t,x)$ inverse to $\Phi(t,x)$, i.e. $\Phi(t,\Psi(t,x)) = x$, $\Psi(t,\Phi(t,x)) = x$. If $\xi(t)$ satisfies (1) and $|\xi(t)| < 1$ for $t \in [0,T]$, then applying Itô's formula ([2], Theorem 4, p. 24) we conclude that the process $X(t) = \Phi(t,\xi(t))$ satisfies the equation

$$dX(t) = m(t,X(t))\,dt + \sigma(t,X(t))\,dW(t)\,,$$

where

$$m(t,x) = \frac{\partial \Phi}{\partial t}(t,\Psi(t,x)) + \frac{\partial \Phi}{\partial x}(t,\Psi(t,x))a(t,\Psi(t,x)) + \frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2}(t,\Psi(t,x))b^2(t,\Psi(t,x))\,, \tag{2}$$

$$\sigma(t,x) = \frac{\partial \Phi}{\partial x}(t,\Psi(t,x))b(t,\Psi(t,x))\,. \tag{3}$$

Let

$$p(x) = \int_0^x \frac{ds}{\sqrt{1+s^2}}\,, \tag{4}$$

$$\Phi(x) = p^{-1}\left(\ln\frac{1+x}{1-x}\right). \tag{5}$$

Note that Φ is an increasing one-to-one mapping from $(-1,1)$ onto $\mathbb{R}$. Define

$$\Psi(x) = \Phi^{-1}(x) = \frac{e^{p(x)}-1}{e^{p(x)}+1}\,. \tag{6}$$

THEOREM 2. *Assume that a* 1*-dimensional Wiener process* $W(t)$ *and an independent* $\mathbb{R}$*-valued random variable* X_0 *on a probability space* $(\Omega,\mathcal{F},P)$ *are given*, $|X_0|<1$ *with probability* 1. *Let the coefficients* $a(t,x)$ *and* $b(t,x)$ *of* (1) *be defined*, *Borel measurable and locally bounded for* $t\ge 0$, $|x|\le 1$. *Suppose further that*

1) *for each* $T>0$ *there exists a constant* K_T *such that*

$$|a(t,x)-a(t,y)|+|b(t,x)-b(t,y)|\le K_T|x-y|$$

$t\in[0,T]$, $|x|\le 1$, $|y|\le 1$,

2) $b(t,\mp 1)=0$ *for* $0\le t\le T$,

3) $a(t,1)\le 0$, $a(t,-1)\ge 0$ *for* $0\le t\le T$,

4) $\mathbb{E}(\Phi(X_0))^2<\infty$.

Then there exists a process $X(t)$ *with* $X(0)=X_0$ *a.s. such that* $(\Omega,\ \mathcal{F},\ P,\ \{\mathcal{F}_t\},\ W,\ X(t))$ *is a solution of the stochastic integral equation* (1), *where* $\mathcal{F}_t=\mathcal{F}_t^W\vee\sigma(X_0)$, *and* $|X(t)|<1$ *for* $0\le t\le T$ *a.s. If* $X_1(t)$ *and* $X_2(t)$ *are two solutions of* (1) *with* $P(X_i(0)=X_0)=1$ *and* $|X_i(t)|<1$ *a.s. for* $i=1,2$ *and for* $t\in[0,T]$, *then*

$$P\{\sup_{0\le t\le T}|X_1(t)-X_2(t)|=0\}=1\,.$$

P r o o f. By 1) and 2) we have $|b(t,x)|=|b(t,x)-b(t,1)|\le K_T|x-1|$. Thus

$$\left|\frac{b(t,x)}{x-1}\right|\le K_T \quad \text{for } 0\le t\le T\,, |x|<1\,. \tag{7}$$

Analogously

$$\left|\frac{b(t,x)}{x+1}\right|\le K_T \quad \text{for } 0\le t\le T\,, |x|<1\,. \tag{8}$$

From 1) and 3) we have

$$\begin{aligned}\frac{a(t,x)}{x+1} &= \frac{a(t,x)-a(t,-1)}{x+1}+\frac{a(t,-1)}{x+1}\\ &\ge \frac{a(t,x)-a(t,-1)}{x+1}\ge\frac{-|a(t,x)-a(t,-1)|}{x+1}\,.\end{aligned}$$

Hence

$$\frac{a(t,x)}{x+1}\ge -K_T \quad \text{for } 0\le t\le T\,, |x|<1\,. \tag{9}$$

Analogously

$$\frac{a(t,x)}{1-x} \le \frac{a(t,x)-a(t,1)}{1-x} \le K_T \quad \text{for } 0 \le t \le T\,, |x| < 1\,. \tag{10}$$

Consider the equation (1) with the drift coefficient $m(t,x)$ and the diffusion coefficient $\sigma(t,x)$ given by the formulas (2) and (3); Φ and Ψ are given by (5) and (6). We will prove that they satisfy all assumptions of Theorem 1. By (6)

$$\Psi'(x) = \frac{2e^{p(x)}}{\sqrt{1+x^2}(e^{p(x)}+1)^2}\,,$$

$$\Psi''(x) = \frac{2e^{p(x)}[(1-e^{p(x)})\sqrt{1+x^2} - x(e^{p(x)}+1)]}{(1+x^2)^{3/2}[e^{p(x)}+1]^3}\,.$$

Since $\Phi \circ \Psi = \mathrm{id}$, we have

$$\Phi'(\Psi(x)) = \frac{\sqrt{1+x^2}(e^{p(x)}+1)^2}{2e^{p(x)}}\,.$$

Differentiating the identity $\Phi'(\Psi(x))\Psi'(x) = 1$, we obtain $\Phi''(\Psi(x)) = -\Psi''(x)\{\Psi'(x)\}^{-3}$. Thus

$$\begin{aligned} m(t,x) &= a(t,\Psi(x))\frac{\sqrt{1+x^2}(e^{p(x)}+1)^2}{2e^{p(x)}} \\ &\quad - \frac{1}{2}b^2(t,\Psi(x))\left(\frac{b(t,\Psi(x))}{\Psi'(x)}\right)^2 \frac{\Psi''(x)}{\Psi'(x)}\,, \end{aligned} \tag{11}$$

$$\sigma(t,x) = b(t,\Psi(x))\frac{\sqrt{1+x^2}(e^{p(x)}+1)^2}{2e^{p(x)}}\,. \tag{12}$$

If $x \ge 0$, then $p(x) \ge 0$ and by (7) and (12) we obtain

$$\begin{aligned} |\sigma(t,x)| &= \left|\frac{b(t,\Psi(x))}{1-\Psi(x)}\right| |1-\Psi(x)| \frac{\sqrt{1+x^2}(e^{p(x)}+1)^2}{2e^{p(x)}} \\ &\le K_T \frac{e^{p(x)}+1}{e^{p(x)}}\sqrt{1+x^2} \le 2K_T\sqrt{1+x^2}\,. \end{aligned}$$

If $x \le 0$, then $p(x) \le 0$ and by (8) and (12) we have

$$\begin{aligned} |\sigma(t,x)| &= \left|\frac{b(t,\Psi(x))}{1+\Psi(x)}\right| |1+\Psi(x)| \frac{\sqrt{1+x^2}(e^{p(x)}+1)^2}{2e^{p(x)}} \\ &\le K_T\sqrt{1+x^2}(e^{p(x)}+1) \le 2K_T\sqrt{1+x^2}\,. \end{aligned}$$

Thus $\sigma(t,x)$ satisfies Condition 1) of Theorem 1.

If $x \geq 0$, then by (10)

$$xa(t,\Psi(x))\frac{\sqrt{1+x^2}(e^{p(x)}+1)^2}{2e^{p(x)}} = \frac{a(t,\Psi(x))}{1-\Psi(x)}x\sqrt{1+x^2}(1+e^{-p(x)}) \leq 2K_T(1+x^2)\,. \tag{13}$$

If $x \leq 0$, then by (9)

$$xa(t,\Psi(x))\frac{\sqrt{1+x^2}(e^{p(x)}+1)^2}{2e^{p(x)}} = \frac{a(t,\Psi(x))}{1+\Psi(x)}x\sqrt{1+x^2}(e^{p(x)}+1) \tag{14}$$

$$\leq -K_T x\sqrt{1+x^2}(e^{p(x)}+1) = K_T(-x)\sqrt{1+x^2}(e^{p(x)}+1) \leq 2K_T(1+x^2)\,.$$

Next

$$-\frac{1}{2}\frac{\Psi''(x)}{\Psi'(x)} = \frac{1}{2}\frac{\Psi(x)}{\sqrt{1+x^2}} + \frac{x}{2(1+x^2)}\,. \tag{15}$$

Since $b(t,\Psi(x))/\Psi'(x) = \sigma(t,x)$ satisfies Condition 1) of Theorem 1, by (13)–(15) we conclude that $m(t,x)$ satisfies Condition 1) of Theorem 1. Condition 2) of Theorem 1 also holds.

Thus, there exists a process $Y(t)$ satisfying (1) with the coefficients $m(t,x)$ and $\sigma(t,x)$ with the initial condition $Y(0) = \Phi(0,X_0)$. Using Itô's formula, we prove that the process $X(t) = \Psi(t,Y(t))$ satisfies the equation

$$\begin{aligned} dX(t) &= a_1(t,X(t))dt + b_1(t,X(t))dW(t)\,, \quad \text{where} \\ a_1(t,x) &= \Psi'(\Phi(x))m(t,\Phi(x)) + \tfrac{1}{2}\Psi''(\Phi(x))\sigma^2(t,\Phi(x))\,, \\ b_1(t,x) &= \Psi'(\Phi(x))\sigma(t,\Phi(x))\,. \end{aligned}$$

Applying formulas (2), (3) and the identity $\Psi \circ \Phi = \mathrm{id}$, we obtain

$$a_1(t,x) = a(t,x)(\Psi\circ\Phi)'(x) + \tfrac{1}{2}b^2(t,x)(\Psi\circ\Phi)''(x) = a(t,x)\,.$$

Analogously,

$$b_1(t,x) = b(t,x)(\Psi\circ\Phi)'(x) = b(t,x)\,.$$

Thus $X(t)$ is a strong solution of (1) with the initial condition $X(0) = \Psi(0,Y(0)) = \Psi(0,\Phi(0,X_0)) = X_0$. Moreover, $|X(t)| < 1$ for $t \geq 0$ a.s. Let $X_1(t)$ and $X_2(t)$ be two solutions of (1) with $P(X_i(0) = X_0) = 1$ and $|X_i(t)| < 1$ for $t \in [0,T]$, $i = 1,2$. Extend b to be zero outside $[-1,1]$ and set $a(t,x) = a(t,-1)$, $x < -1$, and $a(t,x) = a(t,1)$, $x > 1$. Then from Theorem 3.7 of [1], p. 297, we conclude that $P\{X_1(t) = X_2(t)$ for $0 \leq t \leq T\} = 1$, that is to say, the pathwise uniqueness holds. The proof is finished.

If the coefficients of (1) satisfy the assumptions of Theorem 2 and additionally $a(t,x)$ and $b(t,x)$ are continuous in both arguments, then ([2], Theorem 2, p. 68 and [2], p. 66) the solution of (1) is a diffusion with diffusion coefficient $b^2(t,x)$ and drift coefficient $a(t,x)$.

Let $f(t,x)$ be a real function defined in $G=\{(t,x):0\le t\le T,\ \alpha(t)\le x\le\beta(t)\}$, where $\alpha,\ \beta\in C^1[0,T]$. Assume that $f(t,x)$ is C^3 in some open neighbourhood of G and $(\partial f/\partial x)(t,x)>0$ in G. Moreover, suppose $f(t,\cdot)$ is a one-to-one mapping from $(\alpha(t),\beta(t))$ onto $(-1,1)$ for $t\in[0,T]$. Let $g(t,\cdot)$ denote the inverse of $f(t,\cdot)$, i.e.,

$$g(t,f(t,x))\equiv x\equiv f(t,g(t,x))\qquad\text{for } t\in[0,T]\,.$$

From Theorem 2 follows:

COROLLARY 1. *Assume that a* 1*-dimensional Wiener process* $W(t)$ *and an independent* $\mathbb{R}$*-valued random variable* X_0 *on a probability space* $(\Omega,\mathcal{F},P)$ *are given, and* $X_0\in(\alpha(0),\beta(0))$ *a.s. Let* $a(t,x)$ *and* $b(t,x)$ *be measurable in* G. *Suppose the following assumptions are satisfied*:

1) $|a(t,x)-a(t,y)|+|b(t,x)-b(t,y)|\le K|x-y|$ *for* $(t,x),\ (t,y)\in G$,
2) $b(t,\alpha(t))=b(t,\beta(t))=0$ *for* $t\in[0,T]$,
3) $a(t,\alpha(t))\ge\alpha'(t),\ a(t,\beta(t))\le\beta'(t)$ *for* $t\in[0,T]$,
4) $\mathbb{E}(\Phi[f(0,X_0)])^2<\infty$.

Then there exists a process $X(t)$ *satisfying the conditions*:

(A) $X(t)=X_0$ *for* $t=0$,
(B) $X(t)\in(\alpha(t),\beta(t))$ *a.s. for* $t\in[0,T]$.
(C) $(\Omega,\mathcal{F},P,\{\mathcal{F}_t\},W,X(t))$ *is a solution of* (1), *where* $\mathcal{F}_t=\mathcal{F}_t^W\vee\sigma(X_0)$.

If $X(t)$ *and* $\overline{X}(t)$ *are two solutions of* (1) *satisfying* (A)–(C), *then*

$$P\{\sup_{0\le t\le T}|X(t)-\overline{X}(t)|=0\}=1\,.$$

P r o o f. Define

$$a_1(t,x)=\frac{\partial f}{\partial t}(t,g(t,x))+\frac{\partial f}{\partial x}(t,g(t,x))a(t,g(t,x))+\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t,g(t,x))b^2(t,g(t,x))\,,\tag{16}$$

$$b_1(t,x)=\frac{\partial f}{\partial x}(t,g(t,x))b(t,g(t,x))\,.\tag{17}$$

We will show that $a_1(t,x)$ and $b_1(t,x)$ satisfy all the assumptions of Theorem 2.

Since f and g are C^3, by 1) the coefficients $a_1(t,x)$ and $b_1(t,x)$ satisfy Condition 1) of Theorem 2. Since $g(t,-1)\equiv\alpha(t)$, $g(t,1)\equiv\beta(t)$, $f(t,\beta(t))\equiv 1$ and $f_x(t,x)>0$, 2)–4) imply Conditions 2)–4) of Theorem 2, respectively.

Thus, by Theorem 2, there exists a solution $X_1(t)$ of (1) with the coefficients $a_1(t,x)$ and $b_1(t,x)$ satisfying $X_1(0)=f(0,X_0)$, $|X_1(t)|<1$ a.s. for $t\in[0,T]$. In the same way as in Theorem 2 we prove that the process $X(t)=g(t,X_1(t))$ is a solution of (1) with the coefficients $a(t,x)$ and $b(t,x)$. Moreover, $X(t)$ satisfies Conditions (A)–(C).

If $X(t)$ and $\overline{X}(t)$ are two solutions of (1) satisfying (A)–(C), then by Theorem 2

$$P\{\sup_{0\le t\le T}|X(t)-\overline{X}(t)|=0\}=P\{\sup_{0\le t\le T}|f(t,X(t))-f(t,\overline{X}(t))|=0\}=1\,.$$

The corollary is proved.

If the conditions of Corollary 1 are fulfilled and additionally $a(t,x)$ and $b(t,x)$ are continuous in both arguments, then ([2], Theorem 2, p. 68 and [2], p. 66) $X(t)$ is a diffusion with diffusion coefficient $b^2(t,x)$ and drift coefficient $a(t,x)$.

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