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Generalized Schwarzian derivatives for generalized fractional linear transformations

by JOHN RYAN (Sydney)

Abstract. Generalizations of the classical Schwarzian derivative of complex analysis have been proposed by Osgood and Stowe [12, 13], Carne [5], and Ahlfors [3]. We present another generalization of the Schwarzian derivative over vector spaces.

Introduction. Our approach is to define an analogue of the Schwarzian derivatives in $\mathbb{R} \cup \{\infty\}$ using the Clifford algebra generated from $\mathbb{R}^n$. More precisely, we use Vahlen's group of Clifford matrices to construct a "derivative" which in appearance bears an extremely close resemblance to the classical Schwarzian derivative. As conformal transformations in dimensions greater than two correspond to Möbius transformations we are forced to introduce a family of Schwarzians in higher dimensions. We show that a C^3 diffeomorphism annihilated by this family of Schwarzian derivatives is, up to a linear isomorphism, a Möbius transformation. We also show that these generalized Schwarzian derivatives possess a conformal invariance under Möbius transformations, and contain the generalized Schwarzian derivatives described by Ahlfors [3]. Unfortunately, this work also tells us that the method used for obtaining the chain rule for the classical Schwarzian derivative (see [10]) breaks down in higher dimensions.

Motivated by the fact that the analogue of Vahlen's group of Clifford matrices over Minkowski space is $U(2,2)$ we show that the fractional linear transformations associated with $U(2,2)$, $\mathrm{Sp}(n,\mathbb{R})$, the real symplectic group, and $H(n,n)$, the quaternionic unitary group, all have Schwarzian derivatives associated with them. These transformations have previously been described in [7, 9], and elsewhere. We also show that the conformal group over $\mathbb{R}^{p,q}$ has a generalized Schwarzian derivative.

Preliminaries. From $\mathbb{R}^n$ we may construct a Clifford algebra A_n. This can be done [4, 14] by taking an orthonormal basis $\{e_j\}_{j=1}^n$ of $\mathbb{R}^n$ and

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introducing the basis

$$1, e_1, \ldots, e_n, \ldots, e_{j_1} \ldots e_{j_r}, \ldots, e_1 \ldots e_n \tag{1}$$

of A_n, where 1 is the identity and $j_1 < \ldots < j_r$ with $1 \le r \le n$. Moreover, the elements $e_1, \ldots, e_n$ satisfy the identity

$$e_i e_j + e_j e_i = -2\delta_{ij} 1 \tag{2}$$

within A_n, where δ_{ij} is the Kronecker delta. We now have $\mathbb{R}^n \subseteq A_n$ and each non-zero vector $x \in \mathbb{R}^n \setminus \{0\}$ has a multiplicative inverse $x^{-1} = -x/|x|^2 \in \mathbb{R}^n$, which corresponds to the Kelvin inverse of a vector.

Writing x as $x_1 e_1 + \ldots + x_n e_n$ we may obtain

$$e_1(x_1 e_1 + \ldots + x_n e_n) e_1 = -x_1 e_1 + x_2 e_2 + \ldots + x_n e_n,$$

which describes a reflection along the line spanned by e_1. In greater generality, for each $y \in S^{n-1}$ the element yxy is a vector, and this action describes a reflection along the line spanned by y. By induction, for $y_1, \ldots, y_k \in S^{n-1}$ the element $y_1 \ldots y_k x y_k \ldots y_1$ is a vector and this action describes an orthogonal transformation of $\mathbb{R}^n$. The element $y_1 \ldots y_k$ is an element lying in A_n. This group is called $\mathrm{Pin}(n)$ (see [4]). More formally, we have

$$\mathrm{Pin}(n) = \{a \in A_n : a = y_1 \ldots y_k \text{ where } k \in \mathbb{N} \text{ and } y_j \in S^{n-1} \text{ for } 1 \le j \le k\}.$$

In [4] it is shown that $\mathrm{Pin}(n)$ is a double covering of $O(n)$, the orthogonal group (i.e. there is a surjective group homomorphism $\Theta : \mathrm{Pin}(n) \to O(n)$ such that $\ker \Theta \cong \mathbb{Z}_2$).

We also need the antiautomorphism $\sim : A_n \to A_n$, $e_{j_1} \ldots e_{j_r} \mapsto e_{j_r} \ldots e_{j_1}$. It is usual to write $\widetilde{X}$ for $\sim(X)$, where $X \in A_n$ (see [14]). If $a = y_1 \ldots y_k \in \mathrm{Pin}(n)$ then $y_k \ldots y_1 = \widetilde{a}$.

Besides $\sim$ we need the antiautomorphism $- : A_n \to A_n$, $e_{j_1} \ldots e_{j_r} \mapsto (-1)^r e_{j_r} \ldots e_{j_1}$. Again, it is usual [14] to write $\overline{X}$ for $-(X)$. If we write X as $x_0 + \ldots + x_{1\ldots n} e_1 \ldots e_n$ then we can easily deduce that the identity part of $X\overline{X}$ is $x_0^2 + \ldots + x_{1\ldots n}^2$. So A_n is a trace algebra.

Following Vahlen [15] and Mass [11], Ahlfors [1, 2] has used Clifford algebras to describe properties of Möbius transformations in $\mathbb{R}^n \cup \{\infty\}$.

We shall now briefly redescribe these transformations.

The transformations

(a) $T : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$, $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal transformation and $T(\infty) = \infty$,

(b) $R : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$, $\begin{matrix} x \mapsto x + v \text{ for } x \in \mathbb{R}^n \text{ and } v \in \mathbb{R}^n, \\ \infty \mapsto \infty, \end{matrix}$

(c) $D : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$, $\begin{array}{l} x \mapsto \lambda x \text{ for } x \in \mathbb{R}^n \text{ and } \lambda \in \mathbb{R}, \\ \infty \mapsto \infty, \end{array}$

(d) $\mathrm{In} : \mathbb{R}^n \cup \{\infty\} \to \mathbb{R}^n \cup \{\infty\}$, $\begin{array}{l} x \mapsto x^{-1} \text{ for } x \in \mathbb{R}^n \setminus \{0\}, \\ \infty \mapsto 0, \\ 0 \mapsto \infty, \end{array}$

are all special examples of Möbius transformations.

Definition 1. The group of diffeomorphisms of $\mathbb{R}^n \cup \{\infty\}$ generated by the transformations (a)–(d) is called the *Möbius group*, and is denoted by $\mathrm{M\ddot{o}b}(n)$. An element of $\mathrm{M\ddot{o}b}(n)$ is called a *Möbius transformation*.

When $n = 1$ the Clifford algebra is the complex field, and in this case it is extremely well known that a sense preserving Möbius transformation in two real dimensions can be written as $(az+b)(cz+d)^{-1}$ where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{C})$ and $z \in \mathbb{C} \cup \{\infty\}$.

In higher dimensions we have:

Definition 2. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in A_n$ and

(i) $a = a_1 \ldots a_{n_1}$, $b = b_1 \ldots b_{n_2}$, $c = c_1 \ldots c_{n_3}$, $d = d_1 \ldots d_{n_4}$, with $n_1, n_2, n_3, n_4 \in \mathbb{N}$ and $a_i, b_j, c_k, d_l \in \mathbb{R}^n$ for $1 \le i \le n_1$, $1 \le j \le n_2$, $1 \le k \le n_3$, $1 \le l \le n_4$,

(ii) $a\widetilde{c}, \widetilde{c}d, d\widetilde{b}, \widetilde{b}a \in \mathbb{R}^n$,

(iii) $a\widetilde{d} - b\widetilde{c} \in \mathbb{R} \setminus \{0\}$,

is called a *Vahlen matrix*.

From (2) and (i) we see that if $a\widetilde{c}$ is in $\mathbb{R}^n$ then so is $\widetilde{c}(a\widetilde{c})c = \widetilde{c}a(\widetilde{c}c)$. But $\widetilde{c}c \in \mathbb{R}$, and so $\widetilde{c}a \in \mathbb{R}^n$, Consequently, (ii) is equivalent to saying $\widetilde{c}a, d\,\widetilde{c}, \widetilde{b}d, a\widetilde{b} \in \mathbb{R}^n$.

As $\widetilde{c}d \in \mathbb{R}^n$ we have $\widetilde{c}cx + \widetilde{c}d \in \mathbb{R}^n$ for each $x \in \mathbb{R}^n$, so if $c \neq 0$ then $cx + d$ is invertible in A_n for all but one value of $x \in \mathbb{R}^n \cup \{0\}$. If $c = 0$ then it follows from Definition 2 that d is invertible in A_n. Consequently, $(ax+b)(cx+d)^{-1}$ is a well defined element of A_n for all but one value of $x \in \mathbb{R}^n \cup \{0\}$.

When $c \neq 0$ we have

$$(ax+b)(cx+d)^{-1} = ac^{-1} + \lambda(cx\widetilde{c} + d\widetilde{c})^{-1} \tag{3}$$

where $\lambda \in \mathbb{R} \setminus \{0\}$, and when $c = 0$,

$$(ax+b)(cx+d)^{-1} = axd^{-1} + bd^{-1}. \tag{4}$$

Both (3) and (4) are Möbius transformations.

From (3) and (4) we have

LEMMA 1 [1]. *Each Vahlen matrix can be expressed as a finite product of the special Vahlen matrices*

$$\begin{pmatrix} a & 0 \\ 0 & \widetilde{a}^{-1} \end{pmatrix}, \quad \begin{pmatrix} \lambda^{1/2} & 0 \\ 0 & \lambda^{-1/2} \end{pmatrix}, \quad \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $a \in \mathrm{Pin}(n)$, $\lambda \in \mathbb{R}^+$, *and* $v \in \mathbb{R}^n$. ■

These special Vahlen matrices transform into special Möbius transformations (a)–(d). Using this fact, the identities (3) and (4), and Lemma 1 it is straightforward to deduce

PROPOSITION 1 [1]. *The set* $V(n)$ *of Vahlen matrices over* $\mathbb{R}^n$ *forms a group under matrix multiplication*, *and the projection*

$$p : V(n) \to \mathrm{M\ddot{o}b}(n), \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (ax+b)(cx+d)^{-1},$$

is a surjective group homomorphism. ■

By trying to determine the Vahlen matrices for which the equation

$$x = (ax+b)(cx+d)^{-1}$$

holds for all $x \in \mathbb{R}^n$ we may use (3) and (4) to obtain

PROPOSITION 2.

$$\mathrm{Ker}(p) = \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \begin{pmatrix} \lambda e_1 \dots e_n & 0 \\ 0 & -\lambda(e_1 \dots e_n)^{-1} \end{pmatrix} : \lambda \in \mathbb{R} \setminus \{0\} \right\}. \ ■$$

Consequently, the group $V(n) \setminus \mathbb{R}^+$ is a four-fold covering group of $\mathrm{M\ddot{o}b}(n)$. Now,

$$V(n) \setminus \mathbb{R}^+ \cong \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(n) : a\widetilde{d} - b\widetilde{c} = \pm 1 \right\}.$$

The subgroup

$$V_+(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in V(n) : a\widetilde{d} - b\widetilde{c} = 1 \right\}$$

of $V(n) \setminus \mathbb{R}^+$ is a natural generalization of $\mathrm{SL}(2, \mathbb{R})$.

The Vahlen matrices introduced here are not quite the same as those described in [1]. We now introduce those matrices:

DEFINITION 3. A matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with $a, b, c, d \in A_n$ and

(i) $a = a_1 \dots a_{n_1}$, $b = b_1 \dots b_{n_2}$, $c = c_1 \dots c_{n_3}$, $d = d_1 \dots d_{n_4}$, with $a_i, b_j, c_k, d_l \in \mathbb{R} + \mathbb{R}^n$,

(ii) $\overline{a}c, \overline{c}d, \overline{d}b, \overline{b}a \in \mathbb{R} + \mathbb{R}^n$,

(iii) $a\widetilde{d} - b\widetilde{c} \in \mathbb{R} \setminus \{0\}$,

where $\mathbb{R} + \mathbb{R}^n$ is spanned by $1, e_1, \dots, e_n$, is called a *refined Vahlen matrix*.

We denote the set of refined Vahlen matrices over $\mathbb{R}+\mathbb{R}^n$ by $V_0(n)$. By similar arguments to those given above we find [1] that $V_0(n)$ is a group. The subgroup

$$V_{0,+}(n)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in V_0(n): a\widetilde{d}-b\widetilde{c}=1\right\}$$

is a generalization of $\mathrm{SL}(2,\mathbb{C})$. Indeed, $V_{0,+}(1)=\mathrm{SL}(2,\mathbb{C})$.

Other properties of these types of matrices can be found in [6].

1. Now suppose that A is a real normed algebra with an identity, and $U(A)$ is the open set of invertible elements in A. Suppose that V is a domain in $\mathbb{R}^n$ and $f:V\to U(A)$ is a C^1 function. For $y\in S^{n-1}$ we shall let $f(x)_y$ denote the partial derivative of f at x in the direction of y.

The following simple result is crucial to all that follows:

PROPOSITION 3. *Suppose that $f(x)^{-1}$ denotes the algebraic inverse of $f(x)$. Then $(f(x)^{-1})_y=-f(x)^{-1}f(x)_yf(x)^{-1}$.*

Proof.

$$\begin{aligned}\frac{1}{h}(f(x+hy)^{-1}-f(x)^{-1})&=\frac{1}{h}f(x+hy)^{-1}(f(x)-f(x+hy))f(x)^{-1}\\&=-f(x+hy)^{-1}\left(\frac{f(x+hy)-f(x)}{h}\right)f(x)^{-1}.\end{aligned}$$

So

$$\lim_{h\to 0}\frac{1}{h}(f(x+hy)^{-1}-f(x)^{-1})=-f(x)^{-1}f(x)_yf(x)^{-1}.\ \blacksquare$$

This result is an elementary generalization of the basic result that for $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}\setminus\{0\}$, $f(x)=1/x$, we have $(df/dx)(x)=-1/x^2$.

2. From Proposition 3 and (3) and (4) we have

LEMMA 2. *Suppose that $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in V(n)\backslash\mathbb{R}^+$ and $\Phi(z)=(az+b)(cx+d)^{-1}$. Then for each $y\in S^{n-1}$ we have*

$$\Phi(x)_y=\begin{cases}-\lambda\widetilde{c}^{-1}(x+c^{-1}d)^{-}y(x+c^{-1}d)^{-1}c^{-1} & \text{if } c\neq 0,\\ ayd^{-1} & \text{otherwise}.\end{cases}\ \blacksquare$$

From Lemma 2 and Proposition 3 it is now easy to deduce the following formula:

$$\Phi(x)_{yyy}\Phi(x)_y^{-1}-\tfrac{3}{2}\{\Phi(x)_{yy}\Phi(x)_y^{-1}\}^2=0\,. \tag{5}$$

Here $\Phi(x)_{yyy}$ and $\Phi(x)_{yy}$ mean respectively the third and second partial derivatives of Φ at x in the direction of y. Moreover, $\Phi(x)_y^{-1}$ denotes the Kelvin inverse of the vector $\Phi(x)_y$. (From the expressions appearing in Lemma 2 it is straightforward to see that $\Phi(x)_y$ is a non-zero vector.)

Expression (5) is very similar in appearance to the classical Schwarzian derivative of a Möbius transformation in $\mathbb{C} \cup \{\infty\}$ (see for example [10]).

LEMMA 3. *Suppose that* $w : V \hookrightarrow \mathbb{R}^n$ *is a* C^1 *diffeomorphism. Then* $w(x)_y$ *is a non-zero vector for each* $x \in V$. ■

Using Lemma 3 we can now make the following definition:

DEFINITION 4. Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism. Then we define $\{S, w\}_y$ to be $w_{yyy}w_y^{-1} - \frac{3}{2}(w_{yy}w_y^{-1})^2$, and we call $\{S, w\}_y$ the *Schwarzian derivative of* w *in the direction of* $y \in S^{n-1}$.

$\{S, w\}_y$ takes its values in the Lie subalgebra of A_n spanned by $\{1, e_ie_j, e_ie_je_ke_l : 1 \leq i < k < l \leq n\}$.

From Proposition 3 we have

LEMMA 4. *Suppose that* $w : V \hookrightarrow \mathbb{R}^n$ *is a* C^3 *diffeomorphism. Then*

$$(w(x)_{yy}w(x)_y^{-1})_y = w(x)_{yyy}w(x)_y^{-1} - (w_{yy}(x)w(x)_y^{-1})^2,$$

where $(w(x)_{yy}w(x)_y^{-1})_y$ *denotes the partial derivative of* $w(x)_{yy}w(x)_y^{-1}$ *at* x *in the direction of* y. ■

As a consequence of Lemma 4 we have

PROPOSITION 4. *Suppose that* $w : V \hookrightarrow \mathbb{R}^n$ *is a* C^3 *diffeomorphism. Then*

$$\{S, w\}_y = (w_{yy}w_y^{-1})_y - \tfrac{1}{2}(w_{yy}w_y^{-1})^2. \quad ■ \tag{6}$$

Expression (6) is completely analogous to the other well known form of the classical Schwarzian (see [10]).

We shall now try to determine solutions to the equation

$$\{S, w\}_y = 0.$$

First we note

LEMMA 5. *Suppose that* $L : \mathbb{R}^n \to \mathbb{R}^n$ *is an isomorphism. Then* $\{S, L\}_y = 0$ *for all* $y \in S^{n-1}$. ■

The fact that L is a solution to our generalized Schwarzian represents a departure from the results in complex analysis, and is a consequence of the fact that the Schwarzian presented here is dependent on our choice of y.

Bearing this in mind we are led to the following result:

PROPOSITION 5. *Suppose that* $w : V \hookrightarrow \mathbb{R}^n$ *is a* C^3 *diffeomorphism and* $\{S, w\}_{e_1} = 0$. *Suppose also that* $w_{e_1e_1} \neq 0$. *Then there exist* C^3 *maps* $a(x_2, \ldots, x_n)$, $b(x_2, \ldots, x_n)$, $c(x_2, \ldots, x_n)$ *and* $d(x_2, \ldots, x_n)$ *such that*

$$w(x) = (a(x_2, \ldots, x_n) + x_1)^{-1}b(x_2, \ldots, x_n) + c(x_2, \ldots, x_n). \tag{7}$$

Proof. First we set $w(x)_{e_1e_1}w(x)_{e_1}^{-1} = v(x)$. So the equation $\{S, w\}_{e_1} = 0$ becomes

$$\frac{\partial v}{\partial x_1} = \frac{1}{2}v^2. \tag{8}$$

As $w(x)_{e_1e_1} \neq 0$ we find that v is invertible in the Clifford algebra. So (8) is equivalent to

$$v^{-1}\frac{\partial v}{\partial x_1}v^{-1} = \frac{1}{2},$$

or

$$-v^{-1}\frac{\partial v}{\partial z_1}v^{-1} = \frac{1}{2}.$$

But from Proposition 3 we have

$$v^{-1}\frac{\partial v}{\partial x_1}v^{-1} = \frac{\partial}{\partial x_1}(v^{-1}).$$

So $(\partial/\partial x_1)(v^{-1}) = -1/2$. Consequently,

$$v(x)^{-1} = -\tfrac{1}{2}(x_1 + a(x_2, \ldots, x_n)).$$

As $v(x)$ is invertible in A_n, $x_1 + a(x_2, \ldots, x_n)$ must be invertible in A_n. So

$$-2(x_1 + a(x_2, \ldots, x_n))^{-1} = v(x).$$

We now set $\partial w/\partial x_1 = u(x)$. So we have

$$\frac{\partial u}{\partial x_1}(x) = -2(x_1 + a(x_2, \ldots, x_n))^{-1}u(x). \tag{9}$$

Equation (9) tells us that $u(x)$ is a C^∞ function in the variable x_1. It also enables us to deduce that $u(x)$ is a real-analytic function in x_1.

Explicitly working out the Taylor expansion of $u(x)$ about one fixed value $x_1 = x_1'$ we have

$$u(x) = -2(a(x_2, \ldots, x_n) + x_1)^{-2}b(x_1', x_2, \ldots, x_n).$$

So

$$w(x) = (a(x_2, \ldots, x_n) + x_1)^{-1}b(x_1', x_2, \ldots, x_n) + c(x_2, \ldots, x_n),$$

where a, b and c are A_n-valued functions. ∎

We may also easily deduce

PROPOSITION 6. *Suppose that* $w : V \hookrightarrow \mathbb{R}^n$ *is a* C^3 *diffeomorphism and* $(\partial^2 w/\partial x_1^2)(x) = 0$ *on some neighbourhood of* $x_0 \in V$. *Then on that neighbourhood we have*

$$w(x) = x_1 a'(x_2, \ldots, x_n) + b'(x_2, \ldots, x_n), \tag{10}$$

where a' *and* b' *are* A_n*-valued functions.* ∎

Now using elementary continuity arguments we have, from Propositions 5 and 6,

PROPOSITION 7. *Suppose that* $w : V \hookrightarrow \mathbb{R}^n$ *is a* C^3 *diffeomorphism satisfying* $\{S, w\}_{e_1} = 0$ *for all* $x \in V$. *If* $(\partial^2 w/\partial x_1^2)(x_0) \neq 0$ *for some* $x_0 \in V$, *then* $(\partial^2 w/\partial x_1^2)(x) \neq 0$ *for any* $x \in V$. ∎

We now deduce

LEMMA 6. *The function* $c(x_2, \ldots, x_n)$ *appearing in* (7) *is a vector-valued function.*

Outline proof. The result follows immediately from allowing the term x_1, on the right hand side of (7), to vary. ∎

We now see that

$$w(x) - c(x_2, \ldots, x_n) = (a(x_2, \ldots, x_n) + x_1)^{-1} b(x_2, \ldots, x_n)$$

is a vector. As we can take the Kelvin inverse of the left hand side of (11), we see that $b(x_2, \ldots, x_n)$ is invertible in A_n. By now allowing x_1 to vary we have, from (11),

LEMMA 7. $b(x_2, \ldots, x_n)^{-1} a(x_2, \ldots, x_n)$ *is a vector*, *and so is* $b(x_2, \ldots$ $\ldots, x_n)$. ∎

As a consequence of Lemma 7 we have

LEMMA 8. *The function* $a(x_2, \ldots, x_n)$ *lies in the subspace of* A_n *spanned by the set* $\{1, e_i e_j : 1 \leq i < j \leq n\}$.

As a consequence of all this we can rewrite (7) as

$$w(x) = (\lambda_1(x_2, \ldots, x_n) + x_1 \mu_1(x_2, \ldots, x_n))^{-1} + \gamma_1(x_2, \ldots, x_n) \tag{12}$$

where λ_1, μ_1, and γ_1 are all vectors.

Similar calculations tell us that the functions $a'(x_2, \ldots, x_n)$ and $b'(x_2, \ldots, x_n)$ appearing in (10) are vectors.

(10) and (12) give us

THEOREM 1. *Suppose that* $w : V \hookrightarrow \mathbb{R}^n$ *is a* C^3 *diffeomorphism satisfying* $\{S, w\}_y = 0$ *for each* $y \in S^{n-1}$. *Then for any line* $l \subseteq \mathbb{R}^n$ *with* $l \cap V \neq \emptyset$, *on each connected line segment of* $V \cap l$ *the diffeomorphism* w *is the restriction of a Möbius transformation on* $\mathbb{R}^n \cup \{\infty\}$.

In fact, elementary geometry and continuity arguments give us

THEOREM 2. *Suppose that* $w : V \hookrightarrow \mathbb{R}^n$ *is a* C^3 *diffeomorphism satisfying* $\{S, w\}_y = 0$ *for each* $y \in S^{n-1}$. *Then for any line* $l \subseteq \mathbb{R}^n$ *with* $l \cap V \neq \emptyset$, $w|_{V \cap l}$ *is the restriction of a Möbius transformation on* $\mathbb{R}^n \cup \{\infty\}$. ∎

It might initially be suspected that if $w : V \hookrightarrow \mathbb{R}^n$ is C^3 diffeomorphism and $\{S, w\}_{e_j} = 0$ for $j = 1, \ldots, n$ then $w(x) = (a(Lx) + b)(c(Lx) + d)^{-1}$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is a Vahlen matrix and $L : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Unfortunately, this is not true.

Consider $w(x_1e_1 + x_2e_2) = (1/x_1)e_1 + (1/x_2)e_2$. Then $\{S, w\}_{e_1} = \{S, w\}_{e_2} = 0$, but $w(x_1e_1 + x_2e_2)$ is not a Möbius transformation. Bearing the example in mind we shall continue to look at C^3 diffeomorphisms whose generalized Schwarzian vanishes at all points in V and in all directions. First we prove:

PROPOSITION 8. *Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism and $\{S, w\}_y = 0$ for $y \in S^{n-1}$. Suppose also that on each line l with $V \cap l \neq \emptyset$ we have*

$$w(x) = (\lambda_l(x_2^{\perp}) + x_l\mu_l(x_l^{\perp}))^{-1} + \gamma_l(x_l^{\perp}), \tag{13}$$

where $x_l^{\perp}$ is a variable independent of x_l, and x_l is a parametrization of l. Then $\gamma_l(x_l^{\perp})$ is a constant.

Proof. Choose a point $x_0 \in V$, and a ball $B(x_0, r)$. For each ray r_{x_0} passing through x_0 we have

$$\begin{aligned} w(x) = (\lambda(x_0)(\theta_1, \ldots, \theta_{n-1}) + |r_{x_0}|\mu(x_0)(\theta_1, \ldots, \theta_{n-1}))^{-1} \\ + \gamma_{x_0}(\theta_1, \ldots, \theta_{n-1}), \end{aligned} \tag{14}$$

where $\theta_1, \ldots, \theta_{n-1}$ is a parametrization of S^{n-1}. So on each ray $w(x)$ has a unique continuation.

From (14) we have $\lim_{|r_{x_0}| \to \infty} w(x) = \gamma_{x_0}(\theta'_1, \ldots, \theta'_{n-1})$, where $(\theta'_1, \ldots, \theta'_{n-1}) \in \gamma_{x_0} \cap S^{n-1}$. Similarly, for $x_1 \in B(x_0, r) \setminus \{x_0\}$ we have

$$w(x) - (\lambda_{x_1}(\theta_1, \ldots, \theta_{n-1}) + |r_{x_1}|\mu_{x_1}(\theta_1, \ldots, \theta_{n-1}))^{-1} + \gamma_{x_1}(\theta_1, \ldots, \theta_{n-1})$$

and therefore $\lim_{|r_{x_1}| \to \infty} w(x) = \gamma_{x_1}(\theta'_1, \ldots, \theta'_{n-1})$.

Now choose a continuous function $z : (0, \infty) \to \mathbb{R}^n$ so that $z(0) = x_0$ and $z(t)$ is asymptotic to the ray r_{x_1}. As λ_l, μ_l and γ_l are continuous we obtain $\lim_{t\to\infty} w(z(t)) = \gamma_{x_0}(\theta'_1, \ldots, \theta'_{n-1})$. Consequently, $\gamma_{x_1}(\theta'_1, \ldots, \theta'_1) = \gamma_{x_0}(\theta'_1, \ldots, \theta'_{n-1})$. As this is true for each $x_1 \in B(x_0, r)$, $\gamma_l(x_l^{\perp})$ is a constant. ∎

We shall denote this constant vector by γ. Trivially we have:

LEMMA 9. *Suppose that $w(x)$ is as in Proposition 8. Then the C^3 diffeomorphism $w(x) - \gamma$ also has the generalized Schwarzian zero for all $y \in S^{n-1}$. Moreover, on each line l we have*

$$w(x) - \gamma = (\lambda_l(x_l^{\perp}) + x_l\mu_l(x_l^{\perp}))^{-1}. \ \blacksquare$$

Via direct computation we may deduce

PROPOSITION 9. *Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism and $\{S, w(x)\}_y = 0$ for all $x \in V$ and all $y \in S^{n-1}$. Then $\{S, w(x)^{-1}\}_y = 0$ for all $x \in V$ and all $y \in S$.* ∎

On taking the Kelvin inverse of $w(x)-\gamma$ it follows from Proposition 6 that on any two-dimensional hyperspace of $\mathbb{R}^n$ spanned by e_i and e_j and intersecting V we have

$$\begin{aligned}(w(x)-\gamma)^{-1} = {} & v_1(x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_n)\\ & + x_i v_i(x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_n)\\ & + x_j v_j(x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_n)\\ & + x_i x_j v_{ij}(x_1,\ldots,\widehat{x}_i,\ldots,\widehat{x}_j,\ldots,x_n),\end{aligned}$$

where v_1, v_i, v_j and v_{ij} are vectors. On setting $x_i = u_i - u_j$ and $x_j = u_i + u_j$ it now follows from Propositions 6 and 9 that $v_{ij} = 0$. Consequently, we have

THEOREM 3. *Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism satisfying $\{S, w\}_y = 0$ for each $y \in S^{n-1}$. Then there is an isomorphism $L : \mathbb{R}^n \to \mathbb{R}^n$ and a Vahlen matrix $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$ such that $w(x) = (a(Lx)+b)(c(Lx)+d)^{-1}$.* ∎

We now turn to look at other properties of this generalized Schwarzian. We begin with

THEOREM 4. *Suppose that $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism, and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in V(n) \setminus \mathbb{R}^n_+$. Then*

$$\{S, (aw+b)(cw+d)^{-1}\}_y = (w\widetilde{c} + \widetilde{d}\,)^{-1}\{S, w\}_y (w\widetilde{c} + \widetilde{d}\,). \tag{15}$$

Outline proof. When $c = 0$, the result follows from (4). When $c \neq 0$ we have $(aw+b)(cw+d)^{-1} = ac^{-1} + \lambda(cw\widetilde{c} + d\widetilde{c})^{-1}$ where $\lambda \neq 1$. The result now follows from Proposition 3.

As $cw\widetilde{c} + d\widetilde{c}$ is a vector in $\mathbb{R}^n$, $cw+d$ can be expressed as a product of vectors in $\mathbb{R}^n$. Consequently, (15) can be rewritten as

$$\{S, (aw+b)(cw+d)^{-1}\}_y = \operatorname{sgn}(cw+d)\frac{(cw+d)\{S, w\}_y(c\widetilde{w}+d)}{|cw+d|^2} \tag{16}$$

where $\operatorname{sgn}(cw+d)$ is the sign of $(cw+d)(c\widetilde{w}+d)$.

If we dictate that the basis (1) is an orthonormal basis for A_n then (16) yields

PROPOSITION 10. *If $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism and $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in V(n) \setminus \mathbb{R}^+$ then for each $y_1, y_2 \in S^{n-1}$ we have*

$$\begin{aligned}&\langle \{S, w\}_{y_1}, \{S, w\}_{y_2}\rangle\\ &\qquad = \langle \{S, (aw+b)(cw+d)^{-1}\}_{y_1}, \{S, (aw+b)(cw+d)^{-1}\}_{y_2}\rangle.\end{aligned}$$ ∎

If $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism we shall let $\{S, w\}_{y,0}$ denote the identity component of $\{S, w\}_y$, while $\{S, w\}_{y,ij}$ denotes the bivector component of $\{S, w\}_y$, that is, the component spanned by $\{e_i e_j : 1 \le i < j \le n\}$.

Moreover, $\{S,w\}_{y,ijkl}$ denotes the four-vector component of $\{S,w\}_y$, spanned by $\{e_ie_je_ke_l : 1 \le i < j < k < l \le n\}$. As

$$(cw+d)e_ie_j(c\widetilde{w}+d) = \frac{(cw+d)e_i(c\widetilde{w}+d)(cw+d)e_j(c\widetilde{w}+d)}{(cw+d)(c\widetilde{w}+d)},$$

we have from (16)

Proposition 11. *Suppose $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in V(n) \setminus \mathbb{R}^+$. Then*

$$\{S,(aw+b)(cw+d)^{-1}\}_{y,ij} = \operatorname{sgn}(cw+d)\frac{(cw+d)\{S,w\}_{y,ij}(c\widetilde{w}+d)}{|cw+d|^2},$$

$$\{S,(aw+b)(cw+d)^{-1}\}_{y,ijkl} = \operatorname{sgn}(cw+d)\frac{(cw+d)\{S,w\}_{y,ijkl}(c\widetilde{w}+d)}{|cw+d|^2}. \ \blacksquare$$

We also have

Proposition 12. *Suppose $w : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in V(n) \setminus \mathbb{R}^+$. Then*

$$\{S,(aw+b)(cw+d)^{-1}\}_{y,0} = \{S,w\}_{y,0}. \ \blacksquare$$

Propositions 11 and 12 give us

$$\langle \{S,(aw+b)(cw+d)^{-1}\}_{y_1,ij}, \{S,(aw+b)(cw+d)^{-1}\}_{y_2,ij}\rangle = \langle \{S,w\}_{y_1,ij}, \{S,w\}_{y_2,ij}\rangle,$$

and

$$\langle \{S,(aw+b)(cw+d)^{-1}\}_{y_1,ijkl}, \{S,(aw+b)(cw+d)^{-1}\}_{y_2,ijkl}\rangle = \langle \{S,w\}_{y_1,ijkl}, \{S,w\}_{y_2,ijkl}\rangle.$$

Explicitly computing $\{S,w\}_{y,0}$ we get

$$\langle w_{yyy}, w_y\rangle |w_y|^{-2} - \tfrac{3}{2}\langle w_{yy}, w_y\rangle^2 |w_y|^{-4} + \tfrac{3}{2}|w_{yy}|^2|w_y|^{-2}.$$

This expression corresponds to one of the generalizations of the Schwarzian derivative given in [3].

Using differential forms we find that $\{S,w\}_{y,ij}$ is equivalent to

$$w_y \wedge w_{yyy} - 3\langle w_y, w_{yy}\rangle (w_y \wedge w_{yy})|w_y|^{-4},$$

where w_y, w_{yyy} are all regarded as 1-forms. This expression is identical to the second generalized Schwarzian derivative appearing in [3].

We now show that the usual method of obtaining a chain rule for the Schwarzian in one complex variable breaks down.

Suppose now $g(w) : V \hookrightarrow \mathbb{R}^n$ is a C^3 diffeomorphism. Ideally we would like to obtain an expression for $\{S, g(w)\}_y$ in terms of $\{S,g\}_{w_y}$ and $\{S,w\}_y$. First we note that $g(w)_{yyy}$ contains the term $Dg_{w(x)}w_{yyy}$, while $g(w)_{yy}$

contains the term $Dg_{w(x)}w_{yy}$, and $g(w)_y$ is equal to $Dg_{w(x)}w_y$. We could re-express $Dg_{w(x)}w_{yyy}$, $Dg_{w(x)}w_{yy}$ and $Dg_{w(x)}w_y$ as $a_1(x,y)w_{yyy}\widetilde{a}_1(x,y)$, $a_2(x,y)w_{yy}\widetilde{a}_2(x,y)$ and $a_3(x,y)w_y\widetilde{a}_3(x,y)$, respectively, where $a_j(x,y) = b_{j,1}(x,y)\dots b_{j,n_j}(x,y)$ with $b_{i,j}(x,y) \in \mathbb{R}^n\backslash\{0\}$ for $j = 1,2,3$ and $1 \le i \le n_j$.

In general $a_j(x,y) = a_k(x,y)$ only for $j = k$ so we are unable to use this approach to extend the chain rule given in Theorem 4 to obtain a generalization of the Schwarzian chain rule described in [10].

3. Besides A_n we can also construct [14] the Clifford algebra $A_{p,q}$ from the vector space $\mathbb{R}^{p,q}$. The space $\mathbb{R}^{p,q}$ is spanned by the elements $f_1,\dots,f_p$, $e_{p+1},\dots,e_{p+q}$, and it is endowed with the quadratic form $\langle\,,\rangle$, where

$$\langle x,x\rangle = x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_{p+q}^2$$

for $x = x_1f_1 + \dots + x_pf_p + x_{p+1}e_{p+1} + \dots + x_{p+q}e_{p+q}$. To construct $A_{p,q}$ we define the relations

$$e_if_j = -f_je_i, \quad e_ie_j + e_je_i = -2\delta_{ij}, \quad f_if_j + f_jf_i = 2\delta_{ij}.$$

It may now be deduced that $A_{p,q}$ has dimension 2^{p+q}. When $p = 0$ and $q = n$ we have $A_{0,n} = A_n$. It is straightforward to extend the antiautomorphisms $\sim$ and $-$ to $A_{p,q}$ (see [14]). Also, we have the following extension of the Pin group:

$$\mathrm{Pin}(p,q) = \{a \in A_{p,q} : a = a_1\dots a_k,\ k \in \mathbb{N} \text{ and } a_j \in \mathbb{R}^{p,q} \text{ where } a_j^2 = \pm 1 \text{ for } 1 \le j \le k\}.$$

Moreover [14], $\langle ax\widetilde{a}, ax\widetilde{a}\rangle = \langle x,x\rangle$ for each $a \in \mathrm{Pin}(p,q)$. It may easily be verified that $\mathrm{Pin}(p,q)$ is a covering group of

$$O(p,q) = \{T : \mathbb{R}^{p,q} \to \mathbb{R}^{p,q} : T \text{ is linear and } \langle Tx,Tx\rangle = \langle x,x\rangle \text{ for all } x \in \mathbb{R}^{p,q}\}.$$

If we take the closure, within the algebra $A_{p,q}(2)$ (of 2×2 matrices with coefficients in $A_{p,q}$), of the group generated by

$$\left\{\begin{pmatrix} a & 0 \\ 0 & \widetilde{a}^{-1}\end{pmatrix}, \begin{pmatrix} 1 & v \\ 0 & 1\end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0\end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1}\end{pmatrix} : a \in \mathrm{Pin}(p,q),\ v \in \mathbb{R}^{p,q},\ \lambda \in \mathbb{R}^+\right\}$$

we obtain a new group which we denote by $V(p,q)$. Again, when $p = 0$ and $q = n$ we obtain $V(n) \setminus \mathbb{R}^+$.

We could also take the closure, within $A_{p,q}(2)$, of the group generated by

$$\left\{ \begin{pmatrix} a & 0 \\ 0 & \widetilde{a}^{-1} \end{pmatrix}, \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : a = a_1 \ldots a_r, r \in \mathbb{N}, \right.$$
$$\left. a_j \in \mathbb{R} + \mathbb{R}^{p,q} \text{ with } a_j^2 = \pm 1 \text{ for } 1 \le j \le r, v \in \mathbb{R} + \mathbb{R}^{p,q}, \lambda \in \mathbb{R}^+ \right\}$$

where $\mathbb{R} + \mathbb{R}^{p,q}$ is spanned by $1, f_1, \ldots, f_p, e_{p+1}, \ldots, e_{p+q}$. We denote this group by $V_0(p,q)$. When $p = 0$ and $q = n$ we have $V_0(p,q) = V_0(n)/\mathbb{R}^+$.

For $x = x_0 + x_1 f_1 + \ldots + x_p f_p \in \mathbb{R} + \mathbb{R}^{p,0}$ we have $x\overline{x} = x_0^2 - x_1^2 - \ldots - x_p^2$, so $\mathbb{R} + \mathbb{R}^{3,0}$ inherits the same structure as the four-dimensional Minkowski space. On making the identifications

$$1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad f_1 \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{17}$$
$$f_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad f_3 \mapsto \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

we see [8] that $\mathbb{R} + \mathbb{R}^{3,0}$ is identified with H_2, the space of 2×2 Hermitean matrices. Also, for

$$A = \begin{pmatrix} x_0 + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & x_0 - x_1 \end{pmatrix} \in H_2$$

we have $\det A = x_0^2 - x_1^2 - x_2^2 - x_3^2$. Using the identifications (17) it is straightforward calculation to see that $A_{3,0}$ is isomorphic to $\mathbb{C}(2)$, the algebra of 2×2 complex matrices.

Via this isomorphism it may now be deduced from the description of $V_0(p,q)$ that

$$V_0(3,0) \cong U(2,2) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathbb{C}(2) \text{ and} \right.$$
$$\left. \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \begin{pmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & \overline{D}^T \end{pmatrix} = \pm \begin{pmatrix} 0 & I_2 \\ -I_2 & 0 \end{pmatrix} \right\},$$

where $I_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

In greater generality, we have the group

$$U(n,n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathbb{C}(n) \text{ and} \right.$$
$$\left. \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & \overline{D}^T \end{pmatrix} = \pm \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\},$$

where I_n is the $n \times n$ identity matrix.

We shall let H_n denote the space of $n \times n$ Hermitean matrices.

As $U(n,n)$ is the closure of the subgroup of $\mathbb{C}(2n)$ generated by the set

$$\left\{\begin{pmatrix} A & 0 \\ 0 & (\overline{A}^T)^{-1}\end{pmatrix}, \begin{pmatrix} I_n & B \\ 0 & I_n\end{pmatrix}, \begin{pmatrix} 0 & \pm I_n \\ I_n & 0\end{pmatrix} : A \in \mathbb{C}(n),\ B \in H(n)\right\} \tag{18}$$

we can deduce that for each $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in U(n,n)$ the function

$$\det_{C,D} : H_n \to \mathbb{C}, \quad X \mapsto \det(CX + D)$$

is non-zero on an open, dense subset of H_n. Hence $(AX+B)(CX+D)^{-1}$ is well defined on this open, dense set. Moreover, using (18) we see that $(AX+B)(CX+D)^{-1} \in H_n$ whenever $(CX+D)^{-1}$ is defined.

The fractional linear transformation $(AX+B)(CX+D)^{-1}$ has previously been described in [7, 9], and elsewhere.

4. From the previous section we may deduce:

PROPOSITION 13. *Suppose that* $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in U(n,n)$, *and* $z \in H_n \setminus \{0\}$. *Let* $\Phi(X) = (AX+B)(CX+D)^{-1}$. *Then*

$$\Phi(X)_{zzz}\Phi(X)_z^{-1} - \tfrac{3}{2}\{\Phi(X)_{zz}\Phi(X)_z^{-1}\}^2 = 0,$$

where $\Phi(X)_z$ *denotes the partial derivative of* $\Phi(X)$ *in the direction of* z. ∎

In particular, Proposition 13 tells us that the group $U(2,2)$, used to describe Möbius transformations in Minkowski space, has a generalized Schwarzian derivative associated with it.

Proposition 13 leads us to the following definition.

DEFINITION 5. Suppose that V is a domain in H_n and $h : V \hookrightarrow H_n$ is a C^3 diffeomorphism, and for some direction $z \in H \setminus \{0\}$ the element $h(X)_z$ is invertible. Then

$$h(X)_{zzz}h(X)_z^{-1} - \tfrac{3}{2}\{h(X)_{zz}h(X)_z^{-1}\}^2$$

is called the $U(n,n)$ *Schwarzian derivative of* $h(X)$ *in the direction of* z. We denote it by

$$\{S_{U(n,n)}, h(X)\}_z.$$

By similar arguments to those used to deduce Theorem 4 we have

THEOREM 5. *Suppose that* $\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right) \in U(n,n)$, V *is a domain in* H_n *and* $h : V \hookrightarrow H_n$ *is a* C^3 *diffeomorphism. Suppose that for some direction* $z \in H_n \setminus \{0\}$ *the element* $h(X)_z$ *is invertible. Then*

$$\begin{aligned}\{S_{U(n,n)}, &(Ah(X)+B)(h(X)+D)^{-1}\} \\ &= (h(X)\overline{C}^T + D^T)^{-1}\{S_{U(n,n)}, h(X)\}_z(h(X)\overline{C}^T + \overline{D}^T).\end{aligned}$$ ∎

5. Besides the groups $V(n)$ and $U(n,n)$ we can also associate a Schwarzian with the real symplectic group

$$\mathrm{Sp}(n,\mathbb{R}) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} : A, B, C, D \in \mathbb{R}(n) \text{ and } \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} A^T & C^T \\ B^T & D^T \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\},$$

described in [7, 9], and elsewhere. $\mathrm{Sp}(n,\mathbb{R})$ can be seen as the closure of the subgroup of $\mathbb{R}(2n)$ with generators the set

$$\left\{ \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}, \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : A, B \in \mathbb{R}(n) \right\}.$$

By similar arguments to those used in Section 3 we find that for $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{R})$ the matrix $CX + D$ is invertible on an open, dense subset of $S_n = \{X \in \mathbb{R}(n) : X^T = X\}$. Moreover, $(AX + B)(CX + D)^{-1} \in S_n$ on this set.

Definition 6. Suppose that V is a domain in S_n and $h : V \hookrightarrow S_n$ is a C^3 diffeomorphism. Suppose also for some direction $z \in S_n \setminus \{0\}$ the element $h(X)_z$ is invertible. Then

$$h(X)_{zzz} h(X)_z - \tfrac{3}{2}\{h(X)_{zz} h(X)_z^{-1}\}^2$$

is called the $\mathrm{Sp}(n,\mathbb{R})$ *Schwarzian derivative of* $h(X)$ *in the direction of* z. We denote it by $\{S_{\mathrm{Sp}(n,\mathbb{R})}, h(X)\}_z$.

Theorem 6. *Suppose that* $\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}(n,\mathbb{R})$. *Then*

$$\begin{aligned} &\{S_{\mathrm{Sp}(n,\mathbb{R})}, (Ah(X) + B)(Ch(X) + D)^{-1}\}_z \\ &\qquad = (h(X)C^T + D^T)^{-1} \{S_{\mathrm{Sp}(n,\mathbb{R})}, h(X)\}_z (h(X)C^T + D^T). \end{aligned}$$

If $h(X) = X$ *for all* $X \in S_n$ *then*

$$\{S_{\mathrm{Sp}(n,\mathbb{R})}, (AX + B)(CX + D)^{-1}\}_z = 0. \ \blacksquare$$

By similar arguments we may introduce a Schwarzian derivative and an analogue of Theorems 5 and 6 for the quaternionic group

$$H(n,n) = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{H}(2n) : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \begin{pmatrix} \overline{A}^T & \overline{C}^T \\ \overline{B}^T & \overline{D}^T \end{pmatrix} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\},$$

where $-$ here denotes quaternionic conjugation.

6. In this final section we briefly describe how the results of the previous two sections carry through to the group $V(p,q)$.

First suppose that $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in V(p,q)$. Then it follows from the description of $V(p,q)$ given in Section 3 that $(cx + d)(\widetilde{x} + d)$ is real-valued, non-zero on an open dense subset of $\mathbb{R}^{p,q}$. Consequently, $(ax+b)(cx+d)^{-1}$ is well defined on this set. Moreover, it follows from our characterization of $V(p,q)$ that $(ax+b)(cx+d)^{-1}$ is a Möbius transformation on $\mathbb{R}^{p,q}$. It is now straightforward to construct a Schwarzian derivative on $\mathbb{R}^{p,q}$ and to obtain an analogue of Theorems 5 and 6 in this setting.

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DEPARTMENT OF PURE MATHEMATICS
UNIVERSITY OF SYDNEY
SYDNEY, NEW SOUTH WALES 2006
AUSTRALIA

Current address:
DEPARTMENT OF MATHEMATICAL SCIENCES
UNIVERSITY OF ARKANSAS
FAYETTEVILLE, ARKANSAS 72701
U.S.A.

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