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Second order semilinear Volterra integrodifferential equation in Banach space

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Abstract. By using the theory of strongly continuous cosine families of linear operators in Banach space the existence of solutions of some semilinear second order Volterra integrodifferential equations in Banach spaces is proved. The results are applied to some integro-partial differential equations.

1. Introduction. We consider the abstract semilinear second order initial value problem

$$
(1) \qquad \begin{cases} \dfrac{d^2u}{dt^2} = Au + \displaystyle\int_0^t g(t,s,u(s),u'(s))\,ds + f(t)\,, \quad t \in \mathbb{R}\,, \\ u(0) = x, \quad \dfrac{du}{dt}(0) = y\,, \end{cases}
$$

where A is a linear operator from a real Banach space X into itself, u is a mapping from $\mathbb{R}$ to X, g is a nonlinear mapping from $\mathbb{R}\times\mathbb{R}\times X\times X$ into X, and f is a function from $\mathbb{R}$ to X; $x, y \in X$. In this paper we discuss the problem of existence and uniqueness of solutions of (1). We extend the results by C. C. Travis and G. F. Webb [4], using the author's results [1]. In particular, we consider the classical solutions of (1) under more general hypotheses on g and f than in [4]. Our main tool is the theory of strongly continuous cosine families of linear operators in Banach space. The basic ideas and results of this theory can be found for example in [5].

2. Preliminaries. Let A be the linear operator defined in Section 1. We make the following assumption on A.

(Z_1) A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators from the Banach space X into itself.

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Recall that the infinitesimal generator of a strongly continuous cosine family $C(t)$ is the operator $A : X \supset D(A) \to X$ defined by

$$Ax := (d^2/dt^2)C(t)x|_{t=0}, \quad x \in D(A)\,, \tag{2}$$

where

$$D(A) := \{x \in X : C(t)x \text{ is twice continuously differentiable in } t\}\,. \tag{3}$$

Let

$$E := \{x \in X : C(t)x \text{ is once continuously differentiable in } t\}\,.$$

It is known (see [5, Proposition 2.2]) that $D(A)$ is dense in X and A is a closed operator in X.

We define the associated sine family $S(t)$, $t \in \mathbb{R}$, by

$$S(t)x := \int_0^t C(s)x\,ds\,, \quad x \in X,\ t \in \mathbb{R}\,. \tag{4}$$

From assumption (Z_1) it follows (see [5, (2.11) and (2.12)]) that there are constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|C(t)\| \leq Me^{\omega|t|} \text{ and } \|S(t)\| \leq Me^{\omega|t|} \quad \text{for } t \in \mathbb{R}\,. \tag{5}$$

Remark that $S(t)X \subset E$ and $S(t)E \subset D(A)$ for $t \in \mathbb{R}$, $(d/dt)C(t)x = AS(t)x$ for $x \in E$ and $t \in \mathbb{R}$, and $(d^2/dt^2)C(t)x = AC(t)x = C(t)Ax$ for $x \in D(A)$ and $t \in \mathbb{R}$ (see [5, (2.17)–(2.19)]). For $x \in X$ and $s, r \in \mathbb{R}$, we have (see [5])

$$\int_s^r S(t)x\,dt \in D(A)\,, \tag{6}$$

$$A\int_s^r S(t)x\,dt = [C(r) - C(s)]x\,. \tag{7}$$

Note that the adjoint operator $A^* : X^* \to X^*$ is well defined, for $\overline{D(A)} = X$.

We make the following assumption on A^*.

(Z_2) The adjoint operator A^* is densely defined in X^*, i.e. $\overline{D(A^*)} = X^*$.

3. Existence of solutions in a special case. In this section we consider a special case of the problem (1):

$$\begin{cases} \dfrac{d^2u}{dt^2} = Au + \displaystyle\int_0^t g(t,s,u(s))\,ds + f(t), & t \in \mathbb{R}\,, \\ u(0) = x\,, \quad \dfrac{du}{dt}(0) = y\,. \end{cases} \tag{8}$$

The motivation for this comes from [4], where (8) is considered under the assumption that g is continuously differentiable in its first variable and

$f : \mathbb{R} \to X$ is continuously differentiable. We replace this assumption by the local Lipschitz condition.

Without loss of generality we can assume that A is boundedly invertible. It is proved in [3, §6] that for $0 \le \alpha \le 1$ the fractional powers $(-A)^\alpha$ exist as closed linear operators in X, $D((-A)^\alpha) \subset D((-A)^\beta)$ for $0 \le \beta \le \alpha \le 1$, and $(-A)^\alpha(-A)^\beta = (-A)^{\alpha+\beta}$ for $0 \le \alpha + \beta \le 1$. We assume in addition (cf. [4])

(Z_3) For $0 < \alpha \le 1$, $(-A)^\alpha$ maps $D((-A)^\alpha)$ onto X and is 1-1, so that $D((-A)^\alpha)$ is a Banach space when endowed with the norm $\|x\|_\alpha := \|(-A)^\alpha x\|$, $x \in D((-A)^\alpha)$. We denote this Banach space by X_α. We further assume that A^{-1} is compact.

We need the following lemmas:

LEMMA 1 ([4, Lemma 2.1]). *Suppose* (Z_1) *and* (Z_3) *hold. Then*

(9) *for* $0 < \alpha < 1$, $(-A)^{-\alpha}$ *is compact if and only if* A^{-1} *is compact,*

(10) *for* $0 < \alpha < 1$ *and* $t \in \mathbb{R}$, $(-A)^{-\alpha}C(t) = C(t)(-A)^{-\alpha}$ *and* $(-A)^{-\alpha}S(t) = S(t)(-A)^{-\alpha}$.

LEMMA 2. *Suppose* (Z_1)–(Z_3) *hold. Let* $v : [-T,T] \to X$, *for* $T > 0$, *be a Lipschitzian mapping with Lipschitz constant* $L > 0$, *and let*

$$q(t) := \int_0^t S(t-s)v(s)\,ds \quad \textit{for } t \in [-T,T]. \tag{11}$$

Then q *is twice continuously differentiable in* $[-T,T]$ *with* $q(t) \in D(A)$, *and*

$$q'(t) = \int_0^t C(t-s)v(s)\,ds, \quad t \in [-T,T], \tag{12}$$

$$q''(t) = Aq(t) + v(t), \quad t \in [-T,T]; \tag{13}$$

$$(-A)^{\alpha-1}q'(t) \in E \quad \textit{for } 0 \le \alpha \le 1,\ t \in [-T,T]. \tag{14}$$

Proof. This follows from Lemmas 1–4 of [1].

We make the following assumptions on the functions g and f:

(Z_4) $g : \mathbb{R} \times \mathbb{R} \times D \to X$ is continuous, where D is an open subset of X_α for some $\alpha \in [0,1)$,

(Z_5) f is locally Lipschitz, and g is locally Lipschitz with respect to the first variable (i.e. for given $t_0 \in \mathbb{R}$ there exist $T > 0$, an open bounded neighborhood $D_x \subset D$ of x and a constant $a > 0$ such that $\|g(s_1, r, x_1) - g(s_2, r, x_1)\| \le a|s_1 - s_2|$ for all $s_1, s_2, r \in (t_0 - T, t_0 + T)$ and $x_1 \in D_x$).

THEOREM 1. *Let assumptions* (Z_1)–(Z_5) *hold. Let* $x \in D$ *and* $(-A)^{\alpha-1}y \in E$. *There exist* $T > 0$ *and a continuous function* $u : [-T,T] \to X_\alpha$

satisfying

$$u(t) = C(t)x + S(t)y + \int_0^t S(t-s) \int_0^s g(s,r,u(r))\,dr\,ds + \int_0^t S(t-s)f(s)\,ds\,, \quad t \in [-T,T]\,. \tag{15}$$

If, in addition, $x \in D(A)$ and $y \in E$, then u is twice continuously differentiable, $u(t) \in D(A)$ for $t \in [-T,T]$, and u satisfies (8).

Proof. Let

$$\Phi(t) = C(t)x + S(t)y + \int_0^t S(t-s)f(s)\,ds \tag{16}$$

and observe that $\Phi : \mathbb{R} \to X_\alpha$ is continuous by virtue of Lemma 2. For $\gamma > 0$ let

$$N_\gamma(x) := \{x_1 \in X_\alpha : \|x - x_1\|_\alpha < \gamma\}\,.$$

Now choose $\gamma > 0$ and $T > 0$ such that

$$N_\gamma(x) \subset D\,, \tag{17}$$

$$\|\Phi(t) - x\|_\alpha < \gamma/2\,, \quad t \in [-T,T]\,. \tag{18}$$

Let K be the closed bounded convex subset of $C := C([-T,T];X_\alpha)$ defined by

$$K := \{\eta \in C : \|\eta - \Phi\|_C \le \gamma/2\}\,,$$

where $\|\cdot\|_C$ denotes the supremum norm in C. Notice that $\eta(t) \in D$ for $\eta \in K$ and $t \in [-T,T]$, since

$$\begin{aligned}\|\eta(t) - x\|_\alpha &\le \|\eta(t) - \Phi(t)\|_\alpha + \|\Phi(t) - x\|_\alpha \\ &\le \|\eta - \Phi\|_C + \|\Phi(t) - x\|_\alpha < \gamma/2 + \gamma/2 = \gamma\,.\end{aligned}$$

Define the transformation G on K by

$$(G\eta)(t) := \Phi(t) + \int_0^t S(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds\,, \quad t \in [-T,T]\,. \tag{19}$$

Observe that the function $v : [T,T] \to X$ defined by

$$v(s) := \int_0^s g(s,r,\eta(r))\,dr \tag{20}$$

satisfies the Lipschitz condition by virtue of (Z_5). For $t \in [-T,T]$ we have

$$\|(G\eta)(t) - \Phi(t)\|_\alpha = \Big\|(-A)^{\alpha-1}\Big[-A \int_0^t S(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds\Big]\Big\| < \gamma/2$$

for sufficiently small $T > 0$, since the function

$$(21) \qquad [-T,T] \ni t \to A \int_0^t S(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds \in X_\alpha$$

is continuous by Lemma 2. Further, $G\eta$ is continuous as a function from $[-T,T]$ to X_α, and thus G maps K into K.

We next show that $G : K \to K$ is continuous. By (Z_4), given $\varepsilon > 0$ there exists $\delta > 0$ such that for $\eta_1, \eta_2 \in K$ with $\|\eta_1 - \eta_2\|_C < \delta$ and $s \in [-T,T]$, we have

$$\sup\{\|g(s,r,\eta_1(r)) - g(s,r,\eta_2(r))\| : -T \le r \le T\} < \varepsilon\,.$$

On the other hand, for every $x^* \in D(A^*)$ we have

$$\begin{aligned}
&\Big|\Big\langle A \int_0^t S(t-s) \int_0^s [g(s,r,\eta_1(r)) - g(s,r,\eta_2(r))]\,dr\,ds, x^*\Big\rangle\Big| \\
&\quad = \Big|\Big\langle \int_0^t S(t-s) \int_0^s [g(s,r,\eta_1(r)) - g(s,r,\eta_2(r))]\,dr\,ds, A^*x^*\Big\rangle\Big| \\
&\quad \le \Big| \int_0^t Me^{\omega|t-s|} \Big| \int_0^s \varepsilon\,dr\Big|\,ds\Big|\,\|A^*x^*\|\,.
\end{aligned}$$

Since $D(A^*)$ is dense in X^* and the mapping (21) is bounded for $\eta \in K$, we see from the above that the mapping

$$K \ni \eta \to A \int_0^t S(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds \in X$$

is continuous in the weak topology on X.

Now, for $\eta_1, \eta_2 \in K$, $t \in [-T,T]$,

$$\begin{aligned}
&\|(G\eta_1)(t) - (G\eta_2)(t)\|_\alpha \\
&\quad = \Big\|(-A)^{\alpha-1}\Big\{-A \int_0^t S(t-s) \int_0^s [g(s,r,\eta_1(r)) - g(s,r,\eta_2(r))]\,dr\,ds\Big\}\Big\|
\end{aligned}$$

and the continuity of G follows, by the compactness of the operator $(-A)^{\alpha-1}$.

We next show that the set $\{G\eta : \eta \in K\}$ is equicontinuous as a collection of functions in C. For $\eta \in K$ and $-T \le t \le t+h \le T$, by (19), we have

$$\begin{aligned}
&\|(G\eta)(t+h) - (G\eta)(t)\|_\alpha \\
&\qquad \le \|\Phi(t+h) - \Phi(t)\|_\alpha + \|(-A)^{\alpha-1}[(w\eta)(t+h) - (w\eta)(t)]\|
\end{aligned}$$

where

$$(22) \qquad (w\eta)(t) := (-A) \int_0^t S(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds\,.$$

By Lemma 2, $\Phi : [-T,T] \to X_\alpha$ is continuous, and so uniformly continuous, i.e. $\|\Phi(t+h) - \Phi(t)\|_\alpha \to 0$ as $h \to 0$.

On the other hand, we have (cf. [1, Lemma 4])

$$
\begin{aligned}
&(-A)^{\alpha-1}[(w\eta)(t+h) - (w\eta)(t)] \\
&\quad = (-A)S(h)\Big[(-A)^{\alpha-1} \int_0^t C(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds\Big] \\
&\qquad + [C(h) - I](-A)^{\alpha-1}\Big[(-A) \int_0^t S(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds\Big] \\
&\qquad + (-A)^{\alpha-1}\Big[(-A) \int_t^{t+h} S(t+h-s) \int_0^s g(s,r,\eta(r))\,dr\,ds\Big] \\
&\quad = w_1 + w_2 + w_3\,.
\end{aligned}
$$

We prove that $w_i \to 0$ as $h \to 0$, uniformly in $\eta \in K$. Indeed, $w_1 = (-A)S(h)(y\eta)(t)$, where

$$
(y\eta)(t) := (-A)^{\alpha-1} \int_0^t C(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds \in E
$$

for $t \in [-T,T]$ and $\eta \in K$. Since $\{y\eta : \eta \in K\}$ is a compact subset of E we have $w_1 \to 0$ as $h \to 0$, uniformly in $\eta \in K$. Analogously we have $w_2 = [C(h) - I](-A)^{\alpha-1}(w\eta)(t) \to 0$ as $h \to 0$, uniformly in $\eta \in K$.

Since (see [1, (12)])

$$
\|w_3\| \le \|(-A)^{\alpha-1}\| L(Me^{\omega T} + 1)h\,,
$$

where $L > 0$ is the Lipschitz constant for the function v defined by (20), we obtain $w_3 \to 0$ as $h \to 0$, uniformly in $\eta \in K$.

The claimed equicontinuity of $\{G\eta : \eta \in K\}$ now follows.

Lastly, we show that for each $t \in [-T,T]$ the set $\{(G\eta)(t) : \eta \in K\}$ is precompact in X_α (see [4]).

Since $(-A)^{-\beta} : X \to X_\alpha$ is compact for $\alpha < \beta$, it suffices to show that $\{(-A)^\beta[(G\eta)(t) - \Phi(t)] : \eta \in K\}$ is bounded in X for $\alpha < \beta \le 1$. We have

$$
\begin{aligned}
&\|(-A)^\beta(G\eta - \Phi)(t)\| \\
&\qquad = \Big\|(-A)^{\beta-1}\Big[(-A) \int_0^t S(t-s) \int_0^s g(s,r,\eta(r))\,dr\,ds\Big]\Big\| \\
&\qquad \le \|(-A)^{\beta-1}\| L(Me^{\omega T} + 1)|t| \quad \text{(see [1, (12)])}.
\end{aligned}
$$

By Schauder's fixed point theorem, G has a fixed point in K, which is a solution of (15). If $x \in D(A)$ and $y \in E$, then the solution of (15) is a solution of (8) by [1, Theorem 1].

THEOREM 2 (cf. [4]). *Under the assumptions of Theorem* 1, *if in addition* g *maps each closed, bounded set in* $\mathbb{R}\times\mathbb{R}\times D$ *into a bounded set in* X, *and if* u *is a solution of* (15) *noncontinuable to the right on* $[0,d]$, *then either* $d=+\infty$, *or given any closed, bounded set* U *in* D, *there is a sequence* $t_k\to d^-$ *such that* $u(t_k)\notin U$. *An analogous result holds for a solution noncontinuable to the left.*

Proof. Assume that $d<+\infty$ and the conclusion of the theorem is false. Then there is a closed bounded set U in D such that $u(t)\in U$ for $t_1\le t<d$, where $0\le t_1<d$. Set

$$\widehat{x}:=C(d)x+S(d)y+\int_0^d S(d-s)f(s)\,ds+\int_0^d S(d-s)\int_0^s g(s,r,u(r))\,dr\,ds\,.$$

Then we have

$$\|u(t)-\widehat{x}\|_\alpha\le\|\Phi(t)-\Phi(d)\|_\alpha+\|(-A)^{\alpha-1}[(wu)(t)-(wu)(d)]\|\,,$$

where Φ is defined by (16) and w by (22). Let $h:=d-t$. Arguing as in the proof of Theorem 1, we have

$$\begin{aligned}\|u(t)-\widehat{x}\|_\alpha&\le\|\Phi(d)-\Phi(d-h)\|_\alpha+\|(-A)^{\alpha-1}[(wu)(d)-(wu)(d-h)]\|\\&\le\|\Phi(d)-\Phi(d-h)\|_\alpha\\&\quad+\Big\|-AS(h)\Big[(-A)^{\alpha-1}\int_0^d C(d-s)\int_0^s g(s,r,u(r))\,dr\,ds\Big]\Big\|\\&\quad+\Big\|[C(h)-I](-A)^{\alpha-1}\Big[(-A)\int_0^d S(d-s)\int_0^s g(s,r,u(r))\,dr\,ds\Big]\Big\|\\&\quad+\Big\|(-A)^{\alpha-1}\Big[(-A)\int_{d-h}^d S(d-h-s)\int_0^s g(s,r,u(r))\,dr\,ds\Big]\Big\|.\end{aligned}$$

Since g is bounded on $[0,d]\times[0,d]\times U$, $(-A)^{\alpha-1}$ is compact and $\Phi:\mathbb{R}\to X_\alpha$ is continuous, we get

$$\lim_{h\to0^+}\|u(t)-\widehat{x}\|_\alpha=0\,.$$

Since

$$\begin{aligned}u'(t)&=S(t)Ax+C(t)y\\&\quad+\int_0^t C(t-s)\int_0^s g(s,r,u(r))\,dr\,ds+\int_0^t C(t-s)f(s)\,ds\end{aligned}$$

for $0\le t<d$, we see that

$$\lim_{t\to d^-}\|u'(t)-\widehat{y}\|_\alpha=0\,,$$

where

$$\widehat{y} := S(d)Ax + C(d)y + \int_0^d C(d-s) \int_0^s g(s,r,u(r))\,dr\,ds + \int_0^d C(d-s)f(s)\,ds\,.$$

By Lemma 2 we have $(-A)^{\alpha-1}\widehat{y} \in E$.

Now arguing as in the proof of [4, Proposition 2.2], we may extend u to $[0, d+d_1]$, where $d_1 > 0$, in such a way that u satisfies (15) for $0 \le t \le d+d_1$. But this contradicts the noncontinuability assumption and the proof is complete.

COROLLARY 1 (cf. [4]). *Let the hypotheses of Theorem 2 hold and, in addition, let $D = D((-A)^\alpha)$. If u is a solution of* (15) *noncontinuable to the right on* $[0,d)$, *then either $d = +\infty$ or $\limsup_{t\to d^-} \|u(t)\|_\alpha = +\infty$. An analogous result holds for a solution noncontinuable to the left.*

4. Existence of solutions of (1) in the Lipschitz case. In this section we consider the problem (1) for $t \in (0,T]$, $T > 0$, i.e.

$$\left\{ \begin{array}{l} \dfrac{d^2u}{dt^2} + Au + \displaystyle\int_0^t g(t,s,u(s),u'(s))\,ds + f(t), \quad t \in (0,T]\,, \\ u(0) = x\,, \quad u'(0) = y\,, \end{array} \right. \tag{23}$$

where A is a linear operator defined in the introduction, satisfying assumptions (Z_1) and (Z_2) with $g : [0,T] \times [0,T] \times X \times X \to X$, $f : [0,T] \to X$, $x, y \in X$.

DEFINITION 1. A function $u : [0,T] \to X$ is said to be a *solution* of the problem (23) if it is of class C^1 in $[0,T]$, C^2 in $(0,T]$ and satisfies (23).

We make the following assumptions on g and f:

(Z_6) $g : [0,T] \times [0,T] \times X \times X \to X$ is continuous.

(Z_7) $f : [0,T] \to X$ and $g(\cdot, s, u, v) : [0,T] \to X$ satisfy the Lipschitz condition for $s \in [0,T]$, $u, v \in X$.

Similarly to the case of differential equations we have the following theorem (cf. [1, Theorem 2]).

THEOREM 3. *Let* (Z_6) *hold and let $f : [0,T] \to X$ be continuous. If u is a solution of the problem* (23), *then u is a solution of the integral equation*

$$\begin{aligned} u(t) &= C(t)x + S(t)y \\ &\quad + \int_0^t S(t-s) \int_0^s g(s,r,u(r),u'(r))\,dr\,ds + \int_0^t S(t-s)f(s)\,ds\,. \end{aligned} \tag{24}$$

THEOREM 4. *Suppose* (Z_1), (Z_2), (Z_6) *and* (Z_7) *hold and let* $x \in D(A)$ *and* $y \in E$. *If* $u \in C^1([0,T],X)$ *is a solution of* (24), *then* u *is a solution of the problem* (23).

Proof. First we remark that the function $v : [0,T] \to X$ defined by

$$v(t) := \int_0^t g(t,s,u(s),u'(s))\,ds, \qquad t \in [0,T], \tag{25}$$

satisfies the Lipschitz condition. Indeed, let t and $t+h$ be any two points in $[0,T]$. We have

$$\begin{aligned} v(t+h) - v(t) = &\int_0^t [g(t+h,s,u(s),u'(s)) - g(t,s,u(s),u'(s))]\,ds \\ &+ \int_t^{t+h} g(t+h,s,u(s),u'(s))\,ds. \end{aligned}$$

Hence

$$\|v(t+h) - v(t)\| \le \int_0^t a|h|\,ds + K|h| \le (aT+K)|h| = L|h|,$$

where $a > 0$ is the Lipschitz constant for g, $K := \sup\{\|g(t,s,u(s),u'(s))\| : s,t \in [0,T]\}$, and $L := aT + K$. This implies, by (Z_7), that the mapping $[0,T] \ni t \to v(t) + f(t)$ also satisfies the Lipschitz condition, where v is defined by (25). Thus, by [1, Theorem 1], u is a solution of the equation

$$\frac{d^2z}{dt^2} = Az + \int_0^t g(t,s,u(s),u'(s))\,ds + f(t), \qquad t \in (0,T],$$

with $z(0) = x$, $z'(0) = y$. This means that u is a solution of the problem (23).

THEOREM 5. *Suppose* (Z_1), (Z_2) *and* (Z_6) *hold. Let* $f : [0,T] \to X$ *be continuous. Suppose that there exists* $L > 0$ *such that*

$$\|g(t,s,x,y) - g(t,s,u,v)\| \le L(\|x-u\| + \|y-v\|) \qquad \text{for } t,s \in [0,T],\ x,y,u,v \in X.$$

Then for every $x \in E$ *and* $y \in X$ *there exists exactly one solution of the integral equation* (24) *belonging to* $C^1([0,T],X)$.

The proof of this theorem is omitted, as it is a slight modification of the proof of [1, Theorem 4].

As a consequence of Theorems 4 and 5 we get

THEOREM 6. *If*

(i) *assumptions* (Z_1), (Z_2) *and* (Z_6) *hold*,

(ii) $f : [0,T] \to X$ *and* $g(\cdot, s, \cdot, \cdot) : [0,T] \times X \times X \to X$ *satisfy the Lipschitz condition for* $s \in [0,T]$,

(iii) $x \in D(A)$ *and* $y \in E$,

then the problem (23) *has exactly one solution which is a unique solution of the integral equation* (24).

5. Examples. We consider the following two integro-partial differential equations (see [4]).

EXAMPLE 1.

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) + \int\limits_0^t \delta(t,s,w(x,s))\,ds + h(x,t), \quad 0 < x < \pi, \; t \in \mathbb{R}, \\ w(0,t) = w(\pi,t) = 0, \quad t \in \mathbb{R}, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad 0 < x < \pi. \end{cases}$$

Assume that $\delta : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous, and satisfies the Lipschitz condition with respect to its first variable. Let $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and let it satisfy the Lipschitz condition with respect to its second variable. Suppose that w_0 and w_1 are continuous in $[0,\pi]$.

Analogously to [4] we take $X := L^2([0,\pi])$ and let $A : X \to X$ be defined by $Az := z''$, where $D(A) := \{z \in X : z, z'$ are absolutely continuous, $z'' \in X$, $z(0) = z(\pi) = 0\}$. Since $X = L^2([0,\pi])$ is a reflexive Banach space, A satisfies assumption (Z_2) (see [3, Theorem 5.29]). In [4] it is proved that A also satistfies (Z_1) and (Z_3) with $\alpha = 1/2$. Based on the considerations in Example 4.1 of [4], we may demonstrate that (26) satisfies the hypotheses of Theorems 1 and 2, and hence we get the local existence of solution to this integro-partial differential equation under more general assumptions than in [4, Example 4.1].

EXAMPLE 2. Consider

$$\begin{cases} w_{tt}(x,t) = w_{xx}(x,t) + \int\limits_0^t \delta(t,s,w(x,s),w_t(x,s))\,ds + h(x,t), \\ w(0,t) = w(\pi,t) = 0, \\ w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \end{cases}$$

for $x \in (0,\pi)$ and $t \in (0,T)$, $T > 0$.

Let $\delta : [0,T] \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : (0,\pi) \times [0,T] \to \mathbb{R}$ be continuous and suppose there exists a constant $L > 0$ such that for $s \in [0,T]$, $x \in (0,\pi)$, $t_1, t_2 \in [0,T]$ and $p_1, p_2, q_1, q_2 \in \mathbb{R}$

$$|\delta(t_1,s,p_1,q_1) - \delta(t_2,s,p_2,q_2)| \le L(|t_1 - t_2| + |p_1 - p_2| + |q_1 - q_2|),$$
$$|h(x,t_1) - h(x,t_2)| \le L|t_1 - t_2|.$$

Let X and A be as in Example 1. Then it is readily verified that the hypotheses of Theorem 6 are satisfied. From this we get the existence and uniqueness for the problem (27).

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