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A finiteness theorem for Riemannian submersions

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Abstract. Given some geometric bounds for the base space and the fibres, there is a finite number of conjugacy classes of Riemannian submersions between compact Riemannian manifolds.

Introduction. Last years brought a number of finiteness theorems for compact Riemannian manifolds satisfying some geometric bounds ([C], [GP], [GPW], [P], [A], [AC] and others). On the other hand, in the theory of foliations ([M], [R2], etc.) one can find a number of results establishing some rigidity of Riemannian foliations under different circumstances. These facts led the author to the following

PROBLEM (*). Given some bounds on the geometry of the foliated manifolds and geometry of leaves prove that there exist finitely many conjugacy classes of Riemannian foliations of compact manifolds with bundle-like Riemannian metrics satisfying the bounds and some extra topological conditions if necessary. (Two foliations $\mathcal{F}_1$ and $\mathcal{F}_2$ of M_1 and M_2 are *conjugate* if there exists a diffeomorphism $\Psi : M_1 \to M_2$ which maps the leaves of $\mathcal{F}_1$ to the leaves of $\mathcal{F}_2$.)

In this paper, we get a finiteness theorem for Riemannian foliations of the simplest type, i.e. for those given by global Riemannian submersions. For any non-negative constants D, V, κ and τ, and any integers p and $n \geq 1$ denote by $\mathcal{R}(D, V, \kappa, \tau, p, n)$ the collection of all Riemannian submersions

$$f : M \to B \tag{1}$$

satisfying the following conditions:

(a) $\dim M = n$ and $\dim B = n - p$.

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(b) B and the fibres F are compact and their sectional curvatures K_B and K_F satisfy $|K_B|, |K_F| \leq \kappa^2$.

(c) $\operatorname{vol} B \geq V$, $\operatorname{diam} B \leq D$ and there exists a fibre F such that $\operatorname{vol} F \geq V$ and $\operatorname{diam} F \leq D$.

(d) The norms of the second fundamental tensors of the fibres are uniformly bounded by τ.

With this notation our result can be stated as follows.

THEOREM. *For any D, V, κ, τ, n and p the collection $\mathcal{R}(D, V, \kappa, \tau, p, n)$ contains finitely many conjugacy classes.*

The proof of the Theorem is given in Section 2. The line of the proof is similar to that of [C], [P] and [GP]. In particular, the notion of centre of mass [BK] plays an essential role. In Section 1, we prove some technical lemmas and show how to adapt this notion to our situation. Section 3 contains a more detailed motivation for Problem (*) and some discussion of the general case. Also, we provide the reader with an example which shows that the assumption (d) on the second fundamental tensor of the fibres cannot be dropped.

Lemma 1 of Section 1 implies that the volumes and diameters of all the fibres of any submersion of class $\mathcal{R}(D, V, \kappa, \tau, p, n)$ are uniformly bounded by the constants $V \exp(\tau D(n-p))$ and $D \exp(\tau D)$, respectively. It follows that our assumptions (a)–(d) imply the analogous bounds for the geometry of M. Therefore, in the collection $\mathcal{R}(D, V, \kappa, \tau, p, n)$ one can find only finitely many diffeomorphism classes of the total spaces M. The significance of our result consists in constructing a diffeomorphism which maps the fibres of a Riemannian submersion to the fibres of another one.

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1. Technical lemmas

1.1. *Geodesic projection.* For a Riemannian submersion (1), TM decomposes into the direct sum $\mathcal{V} \oplus \mathcal{H}$ of the vertical subbundle $\mathcal{V}$ of vectors tangent to the fibres of f and of the horizontal subbundle $\mathcal{H}$ of vectors orthogonal to them. The second fundamental tensor T of f ([N], see also [E]) is defined by

$$T(X, Y) = h\nabla_{vX} vY + v\nabla_{vX} hY, \tag{2}$$

where $v : TM \to \mathcal{V}$ and $h : TM \to \mathcal{H}$ are the canonical projections. Recall also that any M-geodesic horizontal at a point remains horizontal all the time and projects to a geodesic on B. The horizontal lifts of a B-geodesic are geodesics on M. Assuming that the fibres of f are compact and taking a geodesic $c : [0,1] \to B$ we can define the map g_c of the fibre $F_0 = f^{-1}(c(0))$ to $F_1 = f^{-1}(c(1))$ as follows: $g_c(x) = c_x(1)$, where c_x is the horizontal lift of c satisfying $c_x(0) = x$. The maps of this form map fibres to fibres diffeomorphically and they are isometric if and only if the fibres are totally geodesic, i.e. iff $T = 0$ [E].

LEMMA 1. *If l is the length of a B-geodesic c, then the map $g = g_c$ satisfies*

$$d_f(g(x), g(y)) \le e^{\tau l} d_f(x, y) \tag{3}$$

for all x and y, where d_f is the distance function for the fibres of f and $\tau = \sup \|T\|$.

Proof. Define the map $H : [0,1] \times [0,1] \to M$ by $H(s,t) = c_{\gamma(s)}(t)$, where $\gamma : [0,1] \to F_0$ is a minimal fibre geodesic joining x to y. Let $V = \partial H/\partial s$ and $X = \partial H/\partial t$. Denote by $L(t)$ the length of the curve $H(\cdot, t)$. Then

$$L(t) = \int_0^1 \|V(s,t)\| \, ds$$

and

$$L'(t) = \int_0^1 \frac{\langle \nabla_X V, V\rangle}{\|V\|} \, ds = - \int_0^1 \frac{\langle X, \nabla_V V\rangle}{\|V\|} \, ds$$

because the fields V and X commute. Moreover, V is vertical and X horizontal, so $\langle X, \nabla_V V\rangle = \langle X, T(V,V)\rangle$ and $L'(t) \le \tau L(t)$, which implies (3).

If r is less than the injectivity radius of B, y, $z \in B$ and $d(y,z) < r$, then there exists a unique B-geodesic c of length $l < r$ joining y to z. Hereafter, the corresponding map g_c is referred to as the *geodesic projection* of the fibre $F_y = f^{-1}(y)$ onto F_z.

1.2. *Centre of mass.* Recall the notion of centre of mass as defined in [BK].

Let N be any Riemannian manifold of dimension n, and $x \in N$. Assume that

$$r < r_x := \min\{r(x), \tfrac{1}{4}\pi/\kappa\}, \tag{4}$$

where $r(x)$ is the injectivity radius of N at x and κ^2 bounds from above the sectional curvature of N on $B(x,r)$. For any points $x_1, \ldots, x_m$ of $B(x,r)$ and any non-negative numbers $\lambda_1, \ldots, \lambda_m$ satisfying $\sum \lambda_i = 1$ the function $\mu : B(x,r) \to \mathbb{R}$ defined by $\mu(x) = \sum \lambda_i d(x, x_i)^2$ is convex and admits a unique minimum point C. C is called the *centre of mass of x_i's w.r.t. the mass distribution* (λ_i). It depends smoothly on x_i's and λ_i's: If X is a manifold equipped with a smooth partition of unity $(\lambda_1, \ldots, \lambda_m)$ subordinate to an open covering $(U_1, \ldots, U_m)$ and smooth maps $f_i : U_i \to B(x,r)$, $i = 1, \ldots, m$, then the map

$$x \mapsto C(x) = \text{the centre of mass of } (f_i(x)) \text{ w.r.t. } (\lambda_i(x)) \tag{5}$$

is smooth. Note that $C(x) \in B(x,r)$ so $d(C(x), f_i(x)) < 2r$ for any i.

The following result can be extracted from the proof of the Lemma in [P].

LEMMA 2. *If $r < 6^{2-n} r_x$, $\delta < \frac{1}{70} r_x \kappa$, the partition (λ_i) is defined by $\lambda_i = \eta_i / \sum \eta_j$ for some functions $\eta_i : U_i \to [0,1]$, $\|d\eta_i\| \le 2/\operatorname{diam} U_i$, the maps f_i are immersions and $\|df_i(y) - P \circ df_j(y)\| < \delta^2$, where $P : T_{f_j(y)} N \to T_{f_i(y)} M$ is the parallel transport along the unique geodesic joining $f_j(y)$ to $f_i(y)$ in $B(x,r)$, then the map (5) is an immersion.*

2. Proof of the Theorem. Let

$$r_0 = \min\left\{\frac{\pi}{4\kappa}, \frac{\pi V}{\omega_p}\left(\frac{\kappa}{D\sinh\kappa}\right)^{p-1}, \frac{\pi V}{\omega_{n-p}}\left(\frac{\kappa}{D\sinh\kappa}\right)^{n-p-1}\right\},$$

where ω_m denotes the volume of the unit sphere of dimension m. The results of [HK] show that for any Riemannian submersion (1) of class $\mathcal{R}(D, V, \kappa, \tau, p, n)$ the injectivity radii of B and of the fibres F are not less than $2r_0$ and the balls of radius r_0 in B and F are convex.

Fix $r < 6^{2-n} r_0$ such that $8r \exp(8r\tau) < r_0$.

The finiteness theorem of [P] implies that in any infinite sequence $f_i : M_i \to B_i$ of Riemannian submersions of class $\mathcal{R}(D, V, \kappa, \tau, p, n)$ one can find two, say $f : M \to B$ and $\bar{f} : \overline{M} \to \overline{B}$, such that

(i) B and $\overline{B}$ are diffeomorphic and can be covered by the same number, say m, of convex balls $B(y_i, r)$ and $B(\overline{y}_i, r)$ which have the same intersection pattern, i.e. $B(y_i, r) \cap B(y_j, r) \neq \emptyset$ iff $B(\overline{y}_i, r) \cap B(\overline{y}_j, r) \neq \emptyset$,

(ii) the fibres $F_i = f^{-1}(y_i)$ and $\overline{F}_j = \bar{f}^{-1}(\overline{y}_j)$ are diffeomorphic and there are diffeomorphisms $h_i : F_i \to \overline{F}_i$ satisfying

$$\max(\operatorname{dil}(h_i), \operatorname{dil}(h_i^{-1})) \le 2\,, \tag{6}$$

$$d_f(\overline{g}_{ji} \circ h_i \circ g_{ij}, h_j) < \varepsilon := r\exp(-9r\tau), \tag{7}$$

where $g_{ij} : F_j \to F_i$ and $\overline{g}_{ij} : \overline{F}_j \to \overline{F}_i$ are the geodesic projections.

The existence of maps h_i satisfying (6) and (7) follows from the geometry bounds for the fibres, Ascoli's Theorem and the estimates in the proof of inequality (4.1) in [P].

Define a map $\Psi : M \to \overline{M}$ as follows.

Take any smooth partition of unity (λ_i) on M subordinate to the covering $(B(y_i, r))$ and satisfying the conditions of Lemma 2. For any y in B denote by $C(y)$ the centre of mass of the points $(\Phi_i(y))$ w.r.t. the mass distribution

$(\lambda_i(y))$. Here, Φ_i, $i = 1, \dots, m$, are suitably chosen diffeomorphisms between the balls $B(y_i, r)$ and $B(\overline{y}_i, r)$,

$$\Phi_i = \exp^{\overline{M}}_{y_i} \circ \overline{p}_i \circ (\exp^M_{y_i} \circ p_i)^{-1} \tag{8}$$

for some isometries $p_i : \mathbb{R}^{n-p} \to T_{y_i}B$ and $\overline{p}_i : \mathbb{R}^{n-p} \to T_{\overline{y}_i}\overline{B}$, so that $C : M \to \overline{M}$ is a diffeomorphism [P].

Note that the maps (8) extend to maps between the corresponding balls of radius r_0.

For any i define a diffeomorphism $\Psi_i : U_i \to \overline{U}_i$, $U_i = f^{-1}(B(y_i, r))$ and $\overline{U}_i = \overline{f}^{-1}(B(\overline{y}_i, r))$, in the following way: Given x, find its geodesic projection z to the fibre F_i, and denote by $\Psi_i(x)$ the geodesic projection of $h_i(z)$ to the fibre of $\overline{f}$ over $\Phi_i(f(x))$.

Finally, take $x \in M$ and let $y = f(x)$. Take all the points $\overline{z}_i = \Psi_i(x)$ and project them to the fibre $\overline{F}$ of $\overline{f}$ over $\overline{y} = C(y)$. Denote by $\overline{u}_i$ the points of $\overline{F}$ obtained in this way. If $y \in B(y_i, r) \cap B(y_j, r)$, then $d(\Phi_i(y), \Phi_j(y)) < 4r$, $d(\overline{y}, \Phi_i(y)) < 8r$ and $d(\overline{y}_i, \overline{y}) < 9r$. Lemma 1 and the estimates of (ii) imply that $d(\overline{u}_i, \overline{u}_j) < \varepsilon \exp(9r\tau) = r$, so the centre of mass $\Psi(x)$ of the points (u_i) w.r.t. to the mass distribution $(\lambda_i(y))$ is well defined.

Obviously, the map Ψ is smooth and $\overline{f} \circ \Psi = C \circ f$. Since C is a diffeomorphism, $\operatorname{im} \Psi_{*x}$ is transverse to the fibre of $\overline{f}$. Lemma 2 implies that $\Psi|F$ has maximal rank for any fiber F and therefore the vertical subspace $\mathcal{V}_{\Psi(x)}$ of $T_{\Psi(x)}\overline{M}$ is contained in $\operatorname{im} \Psi_{*x}$. Therefore, $\Psi : M \to \overline{M}$ has maximal rank and, because of compactness of M, defines a covering map.

A similar construction provides us with a smooth fibred map $\Xi : \overline{M} \to M$.

The corresponding map $\overline{C} : \overline{B} \to B$ satisfies $d(\overline{C}(C(y)), y) < 8r$ for all $y \in B$. In fact, $\overline{C}(\overline{y})$, $\overline{y} = C(y)$, is defined as the centre of mass of the points $\Phi_i^{-1}(\overline{y})$ w.r.t. the mass distribution $(\overline{\lambda}_i(\overline{y}_i))$, $\overline{\lambda}_i = \lambda_i \circ \Phi_i$, and $d(\overline{y}, \Phi_i(y)) < 2r$, so $d(y, \Phi_i^{-1}(\overline{y})) < 4r$ and $d(\overline{C}(\overline{y}), y) < 8r$.

Lemma 1 and the dilatation estimates (6) imply that

$$d(x, \Xi \circ \Psi(x)) < 8r \exp(8r\tau) < r_0 .$$

This implies that $\Xi \circ \Psi$ is homotopic to the identity. In the same way $\Psi \circ \Xi \sim \mathrm{id}_{\overline{M}}$. Consequently, Ψ is a diffeomorphism.

3. Final comments. (A) Definitely, the assumptions on the intrinsic geometry of B and F are necessary. Dropping some of them one could construct infinitely many Riemannian submersions with non-diffeomorphic fibres and/or base spaces. The example below shows that the bound for the norm of the second fundamental tensors of the fibres is also essential.

For any matrix $A \in \mathrm{SL}(2, \mathbb{Z})$ consider the manifold $T^3_A = (T^2 \times \mathbb{R})/A$. Then T^3_A fibres over the circle and it is easy to show that there exist infinitely many pairwise non-conjugate fibrations of this form. In fact, two fibrations $T^3_A \to S^1$ and $T^3_B \to S^1$ are conjugate if and only if the matrices A and B are conjugate in $\mathrm{SL}(2,\mathbb{Z})$.

Each of the manifolds T^3_A can be equipped with a Riemannian structure g_A for which the natural projection $T^3_A \to S^1$ becomes a Riemannian submersion and all the fibres are flat tori of the same volume V. If $A = \exp(B)$ for a matrix B with $\mathrm{tr}(B) = 0$ and $A_t = \exp(tB)$, then the Riemannian structure $\widetilde{g}_A$ on $\mathbb{R}^3$ given in the canonical frame $(\partial/\partial x, \partial/\partial y, \partial/\partial t)$ at (x, y, t) by the matrix

$$\begin{pmatrix} (A_t^{-1})^\top \cdot A_t^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

is A-invariant and projects to a metric on T^3_A. The metric obtained in this way has the properties listed above and, moreover, T^3_A has a fibre of diameter $1/\sqrt{2}$.

An elementary calculation involving the structural equations for Riemannian submersions shows that the norm of the second fundamental tensor of the fibres of T^3_A is bounded from below by $C\|A\|$ for a universal constant C, so for any τ the class $\mathcal{R}(D, V, \kappa, \tau, p, n)$ contains only a finite number of Riemannian submersions of this form.

In the same way, suspending some homomorphisms $\pi_1(\Sigma) \to \mathrm{SL}(2,\mathbb{Z})$, one could construct an infinite family of pairwise non-conjugate Riemannian submersions onto a closed surface Σ equipped with a hyperbolic Riemannian metric. Also in this case, the submersions would satisfy all the geometry bounds except the one for the norm of the second fundamental tensor of the fibres.

(B) Let us now recall some results on Riemannian foliations which could motivate Problem $(*)$.

A Riemannian foliation is called *transversely hyperbolic*, *transversely elliptic* or *transversely Euclidean* if its holonomy pseudogroup consists of local isometries of a space form of curvature -1, 1 or 0, respectively.

David Epstein [Ep] proved that for any transversely hyperbolic flow (i.e. transversely hyperbolic foliation of dimension 1) of a compact manifold M one has the following dichotomy: Either all the orbits are closed and constitute a Seifert fibration, or none of the orbit is closed, $n = \dim M = 3$ or 4 and the flow is conjugate to the eigenspace-foliation of an eigenvalue of a matrix $A \in \mathrm{SL}(n,\mathbb{Z})$ acting on T^n_A.

Similar results for transversely Euclidean and transversely elliptic flows were obtained by Yves Carrière in Appendix A of [M]. In both cases, the orbits could constitute orbits of a Seifert fibration. If not, in the Euclidean case, the foliation is conjugate to a flow on $T^k \times P$, P being a flat manifold, such that its restriction to the first factor is linear. In the elliptic case, the flow is obtained by suspension of an isometry of a compact Riemannian manifold of curvature 1.

Our problem is also related to the following conjecture formulated by Etienne Ghys in Appendix E of [M]: The conjugacy classes of Riemannian foliations close to a given Riemannian foliation only depend on finitely many parameters. If some finiteness theorems for Riemannian foliations holded, then—up to a finite number of possible classes—the conjugacy classes would be determined by the geometric bounds.

Also, note that Molino's First Structure Theorem ([M], Theorem 5.1 and Proposition 5.2) says that the closures of the leaves of a Riemannian foliation $\mathcal{F}$ of a compact manifold M form a singular foliation $\overline{\mathcal{F}}$ of M. If all the leaves of $\overline{\mathcal{F}}$ are of the same dimension, then the space $M/\overline{\mathcal{F}}$ admits the structure of a Satake manifold. Extending our results to the category of Satake manifolds one could get a finiteness result for Riemannian foliations with regular closures.

(C) As pointed out by M. T. Anderson [A], at the core of some finiteness and convergence results for Riemannian manifolds there is the proof of the existence of harmonic coordinates and some estimates of the metric tensor in these coordinates. Therefore, the existence of f-related harmonic coordinates should lead to a finiteness result for Riemannian submersions (1) under some bounds for Ricci curvatures replacing our bounds for sectional curvatures.

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