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Algebras of Borel measurable functions

by

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Abstract. We show that, for each $0 < \alpha < \omega_1$, in the αth class in the Baire classification of Borel measurable real functions defined on some uncountable Polish space there is a function which cannot be expressed as a countable union of functions which are (on their domains) in the αth class in Sierpiński's classification. This, in particular, solves positively a problem of Kempisty who asked whether for $1 < \alpha < \omega_1$ the αth Baire and Sierpiński classes are different. We also show that, for every α, in the αth class of Sierpiński's classification there is a function which cannot be expressed as a countable union of functions each of which is on its domain in one of the two αth classes of Young's classification (we refer here to the classical numbering of Baire's, Young's and Sierpiński's classes and not to the one used in the paper).

1. Introduction. In [CM] and [CMPS] the following diagram was considered:

$$
\begin{array}{ccccc}
 & & \mathbf{U}_\alpha & & \\
 & \nearrow & & \searrow & \\
\mathbf{B}_\alpha & & & & \mathbf{L}_\alpha \cup \mathbf{U}_\alpha \to \mathbf{B}_{\alpha+1}\,, \\
 & \searrow & & \nearrow & \\
 & & \mathbf{L}_\alpha & &
\end{array} \tag{1}
$$

where $\mathbf{B}_\alpha$ is the αth class in the Baire classification of real functions defined on $[0,1]$ and $\mathbf{L}_\alpha$ and $\mathbf{U}_\alpha$ are the classes of limits of, respectively, nondecreasing and nonincreasing sequences of functions from $\mathbf{B}_\alpha$; the arrows stand for proper inclusions. It was shown there that in every class of (1) there is a function which cannot be expressed as a union of countably many partial functions from lower classes. In the present paper, considering the algebra $\mathbf{L}_\alpha + \mathbf{U}_\alpha$ of all algebraic sums of functions from $\mathbf{L}_\alpha$ and $\mathbf{U}_\alpha$, we add to (1) the following diagram (cl stands for closure in the uniform convergence

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topology):

$$\mathbf{L}_\alpha \cup \mathbf{U}_\alpha \to \mathbf{L}_\alpha + \mathbf{U}_\alpha \to \mathrm{cl}(\mathbf{L}_\alpha + \mathbf{U}_\alpha) = \mathbf{B}_{\alpha+1} \tag{2}$$

(the equality in (2) was proved by Sierpiński in [S$_2$] for $\alpha = 0$ and the proof remains the same for all $\alpha < \omega_1$). In fact, we consider algebras of bounded functions and then the diagram (2) gets more subtle (b stands for the bounded functions in the given class):

$$\begin{aligned} \mathrm{b}\mathbf{L}_\alpha \cup \mathrm{b}\mathbf{U}_\alpha &\to \mathrm{b}\mathbf{L}_\alpha + \mathrm{b}\mathbf{U}_\alpha \\ &\to \mathrm{cl}(\mathrm{b}\mathbf{L}_\alpha + \mathrm{b}\mathbf{U}_\alpha) \to \mathrm{b}(\mathrm{cl}(\mathbf{L}_\alpha + \mathbf{U}_\alpha)) = \mathrm{b}\mathbf{B}_{\alpha+1}\,. \end{aligned} \tag{3}$$

Again we show that in every class displayed in (3) there exists a function which cannot be expressed as a sum of countably many partial functions from lower classes. This, in particular, implies that the second inclusion from (2), $\mathbf{L}_\alpha + \mathbf{U}_\alpha \subset \mathrm{cl}(\mathbf{L}_\alpha + \mathbf{U}_\alpha) = \mathbf{B}_{\alpha+1}$, is proper. This solves a problem of Kempisty [Ke] (for $\alpha = 0$ the inclusion was shown to be proper by Sierpiński [S$_1$]).

We work in a more general setting enabling us to obtain, for example, analogous results for functions measurable with respect to the projective classes Σ^1_n.

2. Notation, definitions and basic facts. We use standard set-theoretical notation. $\mathbb{N}$ is the set of positive integers, $\mathbb{R}$ the set of reals, and $P(A)$ the family of all subsets of a set A. If A is fixed and $\mathcal{A} \subseteq P(A)$ then $\mathcal{A}^{\mathrm{c}} = \{A \setminus B : B \in \mathcal{A}\}$. $\mathcal{A}_\delta$ will stand for all countable intersections of elements of $\mathcal{A}$. A family $\mathcal{A} \subseteq P(A)$ is a *partition* of A if $\bigcup \mathcal{A} = A$ and for all $X, Y \in \mathcal{A}$, $X \neq Y$, we have $X \cap Y = \emptyset$. If $\mathcal{A} \subseteq P(A)$ and $X \subseteq A$, then $\mathcal{A}|X = \{Y \cap X : Y \in \mathcal{A}\}$. We denote by $r(\mathcal{A})$ the *ring of sets* generated by $\mathcal{A}$, i.e. the smallest family containing $\mathcal{A}$ and closed under taking complements and finite unions. Suppose that $\mathcal{A} \subseteq P(A)$ is a family of sets. We say that $\mathcal{A}$ is a *σ-class* if $\{\emptyset, A\} \subseteq \mathcal{A}$ and $\mathcal{A}$ is closed under finite intersections and countable unions. If $\mathcal{A} \subseteq P(A)$, then we denote by $\mathcal{A}'$ the minimal σ-class containing $r(\mathcal{A})$. The symbol χ_A will denote the characteristic function of A. The domain of a function f will be denoted by $\mathrm{dom}\, f$ and its range by $\mathrm{Rg}\, f$. If A and B are sets, then ${}^A B$ is the set of all functions with domain A and range contained in B. If $f \in {}^A B$ and $C \subseteq A$, then $f|C$ denotes the restriction of f to C. We write ${}^{\underset{\smile}{A}} B$ for the set of all partial functions from A to B, i.e. ${}^{\underset{\smile}{A}} B = \{f \in {}^C B : C \subseteq A\}$. Let f be a real function defined on some set A; then $\inf f = \inf\{f(x) : x \in A\}$, $\sup f = \sup\{f(x) : x \in A\}$ and if f is bounded $\|f\| = \sup |f|$. If $\mathcal{H}$ is any class of real functions, we denote by $\mathrm{b}\mathcal{H}$ the class of all bounded functions from $\mathcal{H}$, and by $\mathrm{cl}\,\mathcal{H}$ the class of all uniform limits of functions from $\mathcal{H}$. For $\mathcal{G} \subseteq {}^Z\mathbb{R}$ and $\mathcal{H} \subseteq {}^Z\mathbb{R}$ let $\mathcal{G} + \mathcal{H} = \{g + h : g \in \mathcal{G} \text{ and } h \in \mathcal{H}\}$. Let $A \subseteq \mathbb{R}$.

We denote by $VB(A)$ the family of all real functions of bounded variation on A, and by $C(A)$ the continuous functions on A.

We write $\mathcal{N}$ and $\mathcal{C}$ for the spaces ${}^{\mathbb{N}}\mathbb{N}$ and ${}^{\mathbb{N}}\{0,1\}$, respectively, with the product topology. The first space is homeomorphic to the irrational numbers and the second to the Cantor set.

Let Z be any set. Let $\mathcal{F} \subseteq {}^{Z}\mathbb{R}$ and $\mathcal{G} \subseteq {}^{\underset{\smile}{Z}}\mathbb{R}$. We denote by $\operatorname{dec}(\mathcal{F},\mathcal{G})$ the least cardinal κ such that for every $f \in \mathcal{F}$ one can find a family $\{g_\alpha : \alpha < \kappa\} \subseteq \mathcal{G}$ such that $\{\operatorname{dom} g_\alpha : \alpha < \kappa\}$ is a partition of Z and $f = \bigcup\{g_\alpha : \alpha < \kappa\}$. We shall only use this definition when it makes sense, i.e., when such subfamilies of $\mathcal{G}$ exist.

Suppose $\mathcal{A}$ is a σ-class. We denote by $\overline{\mathbf{M}}\mathcal{A}$ the family of all functions $f \in {}^{Z}\mathbb{R}$ such that $f^{-1}((-\infty,c)) \in \mathcal{A}$ for every $c \in \mathbb{R}$. Similarly, $\underline{\mathbf{M}}\mathcal{A}$ is the family of all $f \in {}^{Z}\mathbb{R}$ such that $f^{-1}((c,\infty)) \in \mathcal{A}$ for every $c \in \mathbb{R}$. Note that $f \in \underline{\mathbf{M}}\mathcal{A}$ if and only if $-f \in \overline{\mathbf{M}}\mathcal{A}$. We put $\mathbf{M}\mathcal{A} = \underline{\mathbf{M}}\mathcal{A} \cap \overline{\mathbf{M}}\mathcal{A}$. We denote by $\mathbf{M}_B\mathcal{A}$, $\underline{\mathbf{M}}_B\mathcal{A}$ and $\overline{\mathbf{M}}_B\mathcal{A}$ the functions from $\mathbf{M}\mathcal{A}$, $\underline{\mathbf{M}}\mathcal{A}$ and $\overline{\mathbf{M}}\mathcal{A}$, respectively, for which $\operatorname{Rg} f \subseteq B$. Note that if $\mathcal{A}$ is a σ-class and B is closed then $\mathbf{M}_B\mathcal{A}$, $\underline{\mathbf{M}}_B\mathcal{A}$ and $\overline{\mathbf{M}}_B\mathcal{A}$ are complete metric spaces (in the uniform convergence topology) and the same is true for $\mathrm{b}\mathbf{M}\mathcal{A}$, $\mathrm{b}\underline{\mathbf{M}}\mathcal{A}$ and $\mathrm{b}\overline{\mathbf{M}}\mathcal{A}$. Let $R\underline{\mathbf{M}}_B\mathcal{A} = \bigcup\{\underline{\mathbf{M}}_B(\mathcal{A}|X) : X \in P(Z)\}$, $R\overline{\mathbf{M}}_B\mathcal{A} = \bigcup\{\overline{\mathbf{M}}_B(\mathcal{A}|X) : X \in P(Z)\}$, $R(\underline{\mathbf{M}}_B\mathcal{A} + \overline{\mathbf{M}}_B\mathcal{A}) = \bigcup\{\underline{\mathbf{M}}_B(\mathcal{A}|X) + \overline{\mathbf{M}}_B(\mathcal{A}|X) : X \in P(Z)\}$ and $R(\operatorname{cl}(\mathrm{b}\underline{\mathbf{M}}_B\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}_B\mathcal{A})) = \bigcup\{\operatorname{cl}(\mathrm{b}\underline{\mathbf{M}}_B(\mathcal{A}|X) + \mathrm{b}\overline{\mathbf{M}}_B(\mathcal{A}|X)) : X \in P(Z)\}$.

We use standard notation from Descriptive Set Theory. For example, Σ^0_α (Π^0_α) denotes the αth additive (multiplicative, resp.) class in the hierarchy of Borel sets, and Σ^1_n is the nth projective class in the hierarchy of projective sets.

For X a Polish space and $\alpha < \omega_1$, let $\mathbf{B}_\alpha(X) = \{f \in {}^{X}\mathbb{R} : f^{-1}(G) \in \Sigma^0_{1+\alpha}(X)$ for each G open in $\mathbb{R}\}$. If $X = \mathbb{R}$ we write briefly $\mathbf{B}_\alpha(\mathbb{R}) = \mathbf{B}_\alpha$. We have $\mathbf{B}_\alpha(X) = \mathbf{M}\Sigma^0_{1+\alpha}(X)$. We also write $\mathbf{L}_\alpha(X)$ and $\mathbf{U}_\alpha(X)$ to denote $\underline{\mathbf{M}}\Sigma^0_{1+\alpha}(X)$ and $\overline{\mathbf{M}}\Sigma^0_{1+\alpha}(X)$, respectively. For $X = \mathbb{R}$ we write $\mathbf{L}_\alpha(\mathbb{R}) = \mathbf{L}_\alpha$ and $\mathbf{U}_\alpha(\mathbb{R}) = \mathbf{U}_\alpha$. Obviously, $\mathbf{L}_0(X)$ and $\mathbf{U}_0(X)$ are the classes of lower and upper semicontinuous functions on X with values in $\mathbb{R}$.

R e m a r k. In the classical notation the class $\mathbf{B}_\alpha$ for $\alpha < \omega$ and $\mathbf{B}_{\alpha+1}$ for $\alpha \ge \omega$ is called the αth class in the Baire classification and the classes $\mathbf{L}_\alpha$, $\mathbf{U}_\alpha$ ($\mathbf{L}_\alpha + \mathbf{U}_\alpha$) for $\alpha < \omega$ have the number $\alpha + 1$ and for $\alpha \ge \omega$ the number α in Young's (Sierpiński's, resp.) classification (compare for instance [L]).

We say that a class $\mathcal{A}$ has the *reduction property* if for any $A, B \in \mathcal{A}$ there are $A^*, B^* \in \mathcal{A}$ such that $A^* \subseteq A$ and $B^* \subseteq B$, $A^* \cap B^* = \emptyset$ and $A^* \cup B^* = A \cup B$. Note that if $1 < \alpha < \omega_1$, then Σ^0_α has the reduction property. The same is true of Σ^1_n, $n \in \mathbb{N}$. Moreover, if Z is zero-dimensional, then $\Sigma^0_1(Z)$ also has the reduction property.

Following the idea used in [Mo] we deal in this paper with a certain fixed family $\mathcal{T}$ of Polish spaces such that:

(i) if $X \subseteq Z \in \mathcal{T}$ and X is a closed subset of Z, then also $X \in \mathcal{T}$;

(ii) $\mathcal{T}$ is closed under finite Cartesian products;

(iii) $\mathcal{N}, \mathbb{R} \in \mathcal{T}$.

As in [Mo] the idea is to include in $\mathcal{T}$ any Polish space one wants to consider.

Now assume that to each $Z \in \mathcal{T}$ we have assigned a certain family $\mathcal{A}(Z)$ of subsets of Z. Denote by $\mathcal{A}$ the collection of all these families. We say that $\mathcal{A}$ is *closed under continuous substitutions* if for each $X, Y \in \mathcal{T}$ and for every continuous function $f \in {}^X Y$ we have $f^{-1}(A) \in \mathcal{A}(X)$ for every $A \in \mathcal{A}(Y)$. We shall call $\mathcal{A}$ a *hereditary σ-class* if $\mathcal{A}$ is closed under continuous substitutions and if for each $Z \in \mathcal{T}$ the following two conditions are satisfied:

(I) $\mathcal{A}(Z)$ is a σ-class;

(II) $\mathcal{A}(Z)|X = \mathcal{A}(X)$ for each closed $X \subseteq Z$.

Obviously, Σ^0_α, $\alpha < \omega_1$, and Σ^1_n, $n \in \mathbb{N}$, are examples of hereditary σ-classes.

Let X and Y be any sets. For $A \subseteq X \times Y$ and $x \in X$ let $A_x = \{y \in Y : (x, y) \in A\}$. If $\mathcal{A} \subseteq P(Y)$, a set $A \subseteq X \times Y$ is called a *universal set* for $\mathcal{A}$ if $\mathcal{A} = \{A_x : x \in X\}$. Recall that if X and Y are Polish spaces and X is uncountable, then for any $\alpha < \omega_1$ there is a universal set for $\Sigma^0_\alpha(Y)$ in the class $\Sigma^0_\alpha(X \times Y)$, and the same is true for the classes Σ^1_n, $n \in \mathbb{N}$ (see [Mo]).

Let $F \in {}^{X\times Y}\mathbb{R}$ and $(x, y) \in X \times Y$. We put $F_x(y) = F(x, y)$. A function $F \in {}^{X\times Y}\mathbb{R}$ is called a *universal function* for a class $\mathcal{H} \subseteq {}^Y\mathbb{R}$ if $\mathcal{H} = \{F_x : x \in X\}$.

We shall use the following known facts. Theorems 2.A and 2.B were formulated in [CMPS, Cor. 2.2 and Cor. 2.4] in a weaker form but, in fact, they are exactly the theorems proved there.

THEOREM 2.A. *If $\mathcal{A}$ is a σ-class of subsets of Z with the reduction property, then for every countable family of functions $\mathcal{H} \subseteq \mathbf{M}\mathcal{A}$ there exists $g \in \mathbf{M}\mathcal{A}$ such that* $\inf|f - g| > 0$ *for every $f \in \mathcal{H}$.* ∎

THEOREM 2.B. *If $\mathcal{A}$ is a σ-class of subsets of Z and $f \in \underline{\mathbf{M}}\mathcal{A}$, then the set $\{g \in \overline{\mathbf{M}}\mathcal{A} : \inf|f - g| > 0\}$ is open and dense in $\overline{\mathbf{M}}\mathcal{A}$.* ∎

THEOREM 2.C ([CM, Th. 2.1]). *If $Z \in \mathcal{T}$ and $\mathcal{A}$ is a hereditary σ-class such that $\mathcal{A}(Z)$ has a universal set in $\mathcal{A}(\mathcal{C} \times Z)$, then there exists a universal function for $\underline{\mathbf{M}}\mathcal{A}(Z)$ in $\underline{\mathbf{M}}\mathcal{A}(\mathcal{C} \times Z)$.* ∎

THEOREM 2.D (see, for instance, [CM, Prop. 1.1]). *If $n \in \mathbb{N}$ and $\mathcal{A}$ is a σ-class then $f \in R\underline{\mathbf{M}}_{[-n,n]}\mathcal{A}$ if and only if there exists $f^* \in \underline{\mathbf{M}}_{[-n,n]}\mathcal{A}$ such that $f = f^*|\operatorname{dom} f$. A similar result holds for functions from $\overline{\mathbf{M}}_{[-n,n]}\mathcal{A}$.* ∎

Now to formulate Theorem 2.E ([H, XIV, p. 277]) we introduce some notation used in [H].

A family $\mathcal{F}$ of real functions defined on a common domain D will be called an *ordinary function system* if

(i) every real function which is constant on D is in $\mathcal{F}$;

(ii) the maximum and minimum of two functions from $\mathcal{F}$ is in $\mathcal{F}$;

(iii) the sum, difference, product, and quotient (with nowhere vanishing denominator) of two functions from $\mathcal{F}$ is in $\mathcal{F}$.

An ordinary function system $\mathcal{F}$ is called *complete* if it also satisfies the following condition:

(iv) the limit of a uniformly convergent sequence of functions from $\mathcal{F}$ is in $\mathcal{F}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two families of functions. The function f is said to be of class $(\mathcal{A},\mathcal{B})$ if for each $c \in \mathbb{R}$ the set $f^{-1}((c,\infty))$ is in $\mathcal{A}$ and the set $f^{-1}([c,\infty))$ is in $\mathcal{B}$ ([H, p. 267]).

Let $\mathcal{F}$ be a given family of functions defined on a common domain. Let f range over $\mathcal{F}$, and let g and h range over all real functions which are pointwise limits of, respectively, nondecreasing and nonincreasing sequences of functions from $\mathcal{F}$. Then the sets of the form $f^{-1}([c,\infty))$, $g^{-1}((c,\infty))$, $h^{-1}([c,\infty))$ will be called N, P, Q *sets*, respectively ([H, p. 270]). Countable intersections of N sets will be called N_δ sets. $\mathcal{P}$ and $\mathcal{Q}$ will stand for the families of all P and Q sets respectively.

The functions forming the least complete ordinary function system over $\mathcal{F}$ will be called v *functions* ([H, VII, p. 272]).

The following theorem was proved in [H, XIV, p. 277].

THEOREM 2.E. *Let $\mathcal{F}$ be an ordinary function system. If Q_0 is a* Q *set*, *then each function $\phi : Q_0 \to \mathbb{R}$ which is of class $(\mathcal{P}|Q_0, \mathcal{Q}|Q_0)$ can be extended to a function of class $(\mathcal{P},\mathcal{Q})$, that is* (see [H, VII, p. 272]), *to a* v *function.* ∎

We now derive a corollary we shall use in the sequel.

COROLLARY 2.F. *If $\mathcal{A}$ is a σ-class of subsets of some set Z and if $\phi \in \mathbf{M}(\mathcal{A}'|S)$ where $S \in \mathcal{A}_\delta$, then ϕ can be extended to some $\phi^* \in \mathbf{M}\mathcal{A}'$.*

Proof. Notice that all $B \in \mathcal{A}$ are N sets for the ordinary function system $\mathbf{M}\mathcal{A}'$, because $\chi_B \in \mathbf{M}\mathcal{A}'$. Thus the sets from $\mathcal{A}_\delta$ are N_δ sets and therefore Q sets ([H, VI, p. 271]). Thus, by Theorem 2.E, the function ϕ can be extended to a v function ϕ^*. But, as $\mathbf{M}\mathcal{A}'$ is a complete ordinary function system ([H, III, p. 268]), every v function is in $\mathbf{M}\mathcal{A}'$. ∎

3. Algebras of measurable functions

LEMMA 3.1. *Let $\mathcal{A}$ be a σ-class of subsets of Z. Let $h \in \overline{\mathbf{M}}\mathcal{A}$, $|\mathrm{Rg}\, h| < \aleph_0$, $v \in \overline{\mathbf{M}}\mathcal{A}$. Then for each $\varepsilon > 0$ there exists $g \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}$ such that $\|g\| < \varepsilon$ and* $\inf |g + h - v| > 0$.

Proof. Let $\mathrm{Rg}\, h = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_1 < \ldots < \alpha_n$. Let $A_i = \{z \in Z : h(z) = \alpha_i\}$, $i \le n$. Assume $\varepsilon < 1$. By Theorem 2.B for each $i \le n$ there exists $g_i \in \underline{\mathbf{M}}_{[-1,1]}(\mathcal{A}|A_i)$ with $\inf |g_i + \alpha_i - v| > 0$ and $\|g_i - \varepsilon(n-i)/(2n)\| < \varepsilon/(4n)$. Observe that $\sup g_{i+1} < \inf g_i$, $i = 1, \ldots, n-1$. Define $g(z) = g_i(z)$ for $z \in A_i$, $i \le n$. To see that $g \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}$ we check that $g^{-1}((a, 1]) \in \mathcal{A}$ for each $a \in [0, 1]$. Assume first that

$$\min g_k \le a \le \max g_k \tag{1}$$

for some $k \le n$. Then

$$g^{-1}((a, 1]) = \bigcup_{i=1}^{k-1} A_i \cup g_k^{-1}((a, 1]) = \bigcup_{i=1}^{k-1} A_i \cup (B \cap A_k)$$

for some $B \in \mathcal{A}$. Further, we have

$$\bigcup_{i=1}^{k-1} A_i \cup (B \cap A_k) = \Big(\bigcup_{i=1}^{k-1} A_i\Big) \cup \Big(\Big(\bigcup_{i=1}^{k} A_i\Big) \cap B\Big) \in \mathcal{A}\,.$$

If (1) is not satisfied one can easily see that either

$$g^{-1}((a, 1]) = \bigcup_{i=1}^{k} A_i \in \mathcal{A}$$

for some $k \le n$, or $g^{-1}((a, 1]) = \emptyset$. ■

We now apply Lemma 3.1 to prove the following:

LEMMA 3.2. *Let $w \in \overline{\mathbf{M}}\mathcal{A}$. Then the set $\{(l, u) \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A} \times \overline{\mathbf{M}}_{[-1,1]}\mathcal{A} :$ $\inf |u + l - w| > 0\}$ is residual in $\underline{\mathbf{M}}_{[-1,1]}\mathcal{A} \times \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}$, in fact open and dense.*

Proof. Let $u \in \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}$, $l \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}$, $\varepsilon > 0$. Let $n \in \mathbb{N}$ and $n > \varepsilon^{-1}$. Define $A_i = u^{-1}([(i-1)n^{-1}, in^{-1}))$ for $i \in \{-n+1, \ldots, n-1\}$, $A_n = u^{-1}([(n-1)n^{-1}, 1])$ and $h = \sum_{i=-n+1}^{n} (i-1)n^{-1}\chi_{A_i}$. Obviously, $h \in \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}$, $\|u - h\| < \varepsilon$ and $|\mathrm{Rg}\, h| < \aleph_0$. Let $l' = \max(\min(l, 1-\varepsilon), -1+\varepsilon)$ and $-l' + w = v$. The functions h and v satisfy the conditions of the hypothesis of Lemma 3.1 and, by that lemma, there exists $g \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}$ such that $\|g\| < \varepsilon$ and $\delta = \inf |g + h + l' - w| > 0$. Of course $(l' + g, h) \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A} \times \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}$ and for any pair $(\widetilde{l}, \widetilde{u}) \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A} \times \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}$ such that $\|\widetilde{u} - h\| < \delta/2$ and $\|\widetilde{l} - (l' + g)\| < \delta/2$ we have $\inf |\widetilde{u} + \widetilde{l} - w| > 0$. ■

We also need the following dual lemma.

LEMMA 3.3. *Let $w \in \underline{\mathbf{M}}\mathcal{A}$. Then the set $\{(l,u) \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A} \times \overline{\mathbf{M}}_{[-1,1]}\mathcal{A} : \inf|u+l-w| > 0\}$ is residual in $\underline{\mathbf{M}}_{[-1,1]}\mathcal{A} \times \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}$, in fact open and dense.* ■

From Lemmas 3.2 and 3.3 and the Baire category theorem we derive the following corollary.

COROLLARY 3.4. *If $\mathcal{A}$ is a σ-class of subsets of Z, then for every countable family $\mathcal{H} \subseteq \underline{\mathbf{M}}\mathcal{A} \cup \overline{\mathbf{M}}\mathcal{A}$ there exists $f \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z) + \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z)$ such that $f(t) \neq g(t)$ for every $g \in \mathcal{H}$ and every $t \in Z$.* ■

We are now able to prove our first decomposition theorem. The scheme of the proof is, in fact, the same as for Theorem 3.2 of [CMPS].

THEOREM 3.5. *Let $\mathcal{A}$ be a hereditary σ-class, and let $Z \in \mathcal{T}$ be uncountable and such that $\mathcal{A}(Z)$ has a universal set in $\mathcal{A}(\mathcal{C} \times Z)$. Then there exists $f \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z) + \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z)$ such that there is no countable partition of Z, $Z = \bigcup\{Z_n : n \in \mathbb{N}\}$, such that $f|Z_n \in \underline{\mathbf{M}}(\mathcal{A}(Z)|Z_n) \cup \overline{\mathbf{M}}(\mathcal{A}(Z)|Z_n)$ for every $n \in \mathbb{N}$. In other words*,

$$\operatorname{dec}(\underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z) + \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z), R\underline{\mathbf{M}}\mathcal{A}(Z) \cup R\overline{\mathbf{M}}\mathcal{A}(Z)) > \aleph_0 .$$

Proof. Let $C \subseteq Z$ be homeomorphic to $\mathcal{C}$. Let $F \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(C \times Z)$ and $G \in \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(C \times Z)$ be universal functions for $\underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z)$ and $\overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z)$, respectively. Let $\pi = (\pi_1, \pi_2, \ldots) : C \to {}^{\mathbb{N}}C$ be a fixed homeomorphism. For every $n \in \mathbb{N}$ let $f_n \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z)$ and $g_n \in \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z)$ be such that $f_n(t) = F(\pi_n(t), t)$ and $g_n(t) = G(\pi_n(t), t)$ for every $t \in C$. By Corollary 3.4 there exists $f \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z) + \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z)$ such that $f(t) \neq g_n(t)$ and $f(t) \neq f_n(t)$ for each $t \in Z$.

Now assume that $f = \bigcup\{h_k : k \in \mathbb{N}\}$ and $h_k \in R\underline{\mathbf{M}}\mathcal{A}(Z) \cup R\overline{\mathbf{M}}\mathcal{A}(Z)$ for each $k \in \mathbb{N}$. Let $h_k^* \in \underline{\mathbf{M}}\mathcal{A}(Z) \cup \overline{\mathbf{M}}\mathcal{A}(Z)$ be an extension of h_k (see Theorem 2.D). There exists $c \in C$ such that for every $k \in \mathbb{N}$ and for every $t \in Z$ either $h_k^*(t) = F(\pi_k(c), t)$ or $h_k^*(t) = G(\pi_k(c), t)$. Thus $f(c) \in \{f_k(c) : k \in \mathbb{N}\} \cup \{g_k(c) : k \in \mathbb{N}\}$, which is impossible. ■

COROLLARY 3.6. *If Z is an uncountable Polish space, then for any $\alpha < \omega_1$*

$$\operatorname{dec}(\mathbf{L}_\alpha(Z) + \mathbf{U}_\alpha(Z), R\mathbf{L}_\alpha(Z) \cup R\mathbf{U}_\alpha(Z)) > \aleph_0 . \ ■$$

COROLLARY 3.7. *If Z is an uncountable Polish space, then for any $n \in \mathbb{N}$*

$$\operatorname{dec}(\underline{\mathbf{M}}\Sigma^1_n(Z) + \overline{\mathbf{M}}\Sigma^1_n(Z), R\underline{\mathbf{M}}\Sigma^1_n(Z) \cup R\overline{\mathbf{M}}\Sigma^1_n(Z)) > \aleph_0 . \ ■$$

We shall need the following lemma.

Lemma 3.8. *If $\mathcal{A}$ is a hereditary σ-class, $Z \in \mathcal{T}$ and $\mathcal{A}(Z)$ has a universal set in $\mathcal{A}(\mathcal{C}\times Z)$, then for every $n \in \mathbb{N}$ the class $\underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)+\overline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)$ has a universal function in $\underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(\mathcal{C}\times Z)+\overline{\mathbf{M}}_{[-n,n]}\mathcal{A}(\mathcal{C}\times Z)$.*

Proof. Let $\phi = (\phi_1,\phi_2) : \mathcal{C} \to \mathcal{C}^2$ be any homeomorphism. Let $F \in \underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(\mathcal{C}\times Z)$ and $G \in \overline{\mathbf{M}}_{[-n,n]}\mathcal{A}(\mathcal{C}\times Z)$ be universal functions for $\underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)$ and $\underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)$, respectively. Then $H(c,x) = F(\phi_1(c),x)+G(\phi_2(c),x)$ is a universal function for $\underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)+\overline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)$. ∎

Lemma 3.9. *Let $\mathcal{A}$ be a σ-class of subsets of some set Z. If $A \in r(\mathcal{A})$ then $\chi_A \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A}+\mathrm{b}\overline{\mathbf{M}}\mathcal{A}$.*

Proof. The family

$$\mathcal{S} = \{A \in r(\mathcal{A}) : \chi_A \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A}+\mathrm{b}\overline{\mathbf{M}}\mathcal{A}\}$$

is obviously closed under finite intersections and taking complements and at the same time $\mathcal{A} \subseteq \mathcal{S}$. Thus $\mathcal{S} = r(\mathcal{A})$. ∎

Lemma 3.10. *Let $n \in \mathbb{N}$, let $\mathcal{A}$ be a σ-class of subsets of some set Z and $g \in \underline{\mathbf{M}}_{[-N,N]}\mathcal{A}+\overline{\mathbf{M}}_{[-N,N]}\mathcal{A}$, $N \in \mathbb{N}$. Then there exists $w \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A}+\mathrm{b}\overline{\mathbf{M}}\mathcal{A}$ such that $\|g-w\| < 2^{-n+1}$ and*

$$w = \sum_{i=-2^{n+1}N}^{2^{n+1}N} i\cdot 2^{-n}\chi_{A_i}\,,$$

where the sets A_i are pairwise disjoint and, for each i, $A_i \in r(\mathcal{A})$ and therefore $\chi_{A_i} \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A}+\mathrm{b}\overline{\mathbf{M}}\mathcal{A}$.

Proof. Let $g = u+l$, $l \in \underline{\mathbf{M}}_{[-N,N]}\mathcal{A}$ and $u \in \overline{\mathbf{M}}_{[-N,N]}\mathcal{A}$. Let $B_i = l^{-1}((i\cdot 2^{-n},(i+1)\cdot 2^{-n}])$ and $C_i = u^{-1}([i\cdot 2^{-n},(i+1)\cdot 2^{-n}))$. The sets B_i and C_i belong to $r(\mathcal{A})$. Let

$$w = \sum_{i=-2^nN}^{2^nN} i\cdot 2^{-n}\chi_{B_{i-1}} + \sum_{i=-2^nN}^{2^nN} i\cdot 2^{-n}\chi_{C_i} = \sum_{j=-2^{n+1}N}^{2^{n+1}N} j\cdot 2^{-n}\chi_{A_j}\,,$$

where $A_j = \bigcup\{B_i\cap C_k : i+k+1 = j\}$. It follows from Lemma 3.9 that w is the function we need. ∎

Lemma 3.11. *Let $\mathcal{A}$ be a σ-class of subsets of some set Z. Let $f,g \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A}+\mathrm{b}\overline{\mathbf{M}}\mathcal{A}$. Let $\varepsilon > 0$. Then there exists $h \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A}+\mathrm{b}\overline{\mathbf{M}}\mathcal{A}$ such that $\|h-g\| < 3\varepsilon$ and $\inf|h-f| \ge \varepsilon/3$.*

Proof. By Lemma 3.10 there exist $\phi = \sum_{i=1}^N c_i\chi_{A_i}$ and $\psi = \sum_{j=1}^M d_j\chi_{B_j}$ such that $A_i, B_j \in r(\mathcal{A})$, $i \le N$, $j \le M$, the sets A_i are pairwise disjoint, the B_j are pairwise disjoint, $\|f-\phi\| < \varepsilon/3$, and $\|g-\psi\| < \varepsilon/3$. Taking appropriate intersections we can assume that for each $j \le M$ there exists

$i \leq N$ such that $B_j \subseteq A_i$. Let $B_j \subseteq A_i$. Then we define h on B_j in the following way:

$$h|B_j = \begin{cases} \psi|B_j & \text{if } |d_j - c_i| \geq 2\varepsilon/3\,, \\ \psi|B_j + 2\varepsilon & \text{if } |d_j - c_i| < 2\varepsilon/3\,. \ \blacksquare \end{cases}$$

LEMMA 3.12. *If $\mathcal{A}$ is a σ-class of subsets of Z then for every countable family $\mathcal{G} \subseteq \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A})$ there exists $g \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A})$ such that $\inf|f-g| > 0$ for every $f \in \mathcal{G}$.*

P r o o f. By Lemma 3.11 for any $f \in \mathcal{G}$ the family $\{h \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}) : \inf|f-h| > 0\}$ is residual in $\mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A})$. As the latter space is complete, the lemma follows by the Baire category theorem. $\blacksquare$

THEOREM 3.13. *Let $\mathcal{A}$ be a hereditary σ-class on $\mathcal{T}$. Let $\mathcal{A}(Z)$, for some uncountable $Z \in \mathcal{T}$, have a universal set in $\mathcal{A}(\mathcal{C} \times Z)$. Then there exists a function $f \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z))$ for which there is no countable partition $Z = \bigcup\{Z_m : m \in \mathbb{N}\}$ such that $f|Z_m \in \underline{\mathbf{M}}\mathcal{A}(Z)|Z_m + \overline{\mathbf{M}}\mathcal{A}(Z)|Z_m$ for each $m \in \mathbb{N}$. In other words*,

$$\mathrm{dec}(\mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z)), R(\underline{\mathbf{M}}\mathcal{A}(Z) + \overline{\mathbf{M}}\mathcal{A}(Z))) > \aleph_0\,.$$

P r o o f. Let $C \subseteq Z$ be homeomorphic to $\mathcal{C}$. By Lemma 3.8 for each $n \in \mathbb{N}$ there exists $G_n \in \underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(C \times Z) + \overline{\mathbf{M}}_{[-n,n]}\mathcal{A}(C \times Z)$ which is a universal function for $\underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z) + \overline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)$. Let $\pi = (\pi_1, \pi_2, \ldots) : C \to {}^{\mathbb{N}}C$ be a fixed homeomorphism. Let $g_n(t) = G_n(\pi_n(t), t)$ for every $t \in C$. It is easy to see that $g_n \in \underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z) + \overline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)$. By Lemma 3.12 there exists $f \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z))$ such that $f(t) \neq g_n(t)$ for each $t \in Z$ and for each $n \in \mathbb{N}$.

Assume there is a partition $Z = \bigcup\{Z_m : m \in \mathbb{N}\}$ such that $f|Z_m \in \underline{\mathbf{M}}(\mathcal{A}(Z)|Z_m) + \overline{\mathbf{M}}(\mathcal{A}(Z)|Z_m)$ for each $m \in \mathbb{N}$. Let $f|Z_m = l_m + u_m$ where $l_m \in \underline{\mathbf{M}}\mathcal{A}(Z)|Z_m$ and $u_m \in \overline{\mathbf{M}}\mathcal{A}(Z)|Z_m$. Let $Z_{m,n} = \{x \in Z_m : |l_m(x)| \leq n \text{ and } |u_m(x)| \leq n\}$. Of course $\bigcup\{Z_{m,n} : n \in \mathbb{N}\} = Z_m$. Let $l_{m,n} \in \underline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)$ and $u_{m,n} \in \overline{\mathbf{M}}_{[-n,n]}\mathcal{A}(Z)$ be extensions of $l_m|Z_{m,n}$ and $u_m|Z_{m,n}$, respectively (see Theorem 2.D). There exists $c \in C$ such that for each pair $m, n \in \mathbb{N}$ there exists $i(m,n) \in \mathbb{N}$ such that $l_{m,n}(t) + u_{m,n}(t) = G_{i(m,n)}(\pi_{i(m,n)}(c), t)$ for each $t \in \mathbb{N}$. Let $c \in Z_{m,n}$ for some $m, n \in \mathbb{N}$. Then $f(c) = l_{m,n}(c) + u_{m,n}(c) = g_{i(m,n)}(c)$, which is a contradiction. $\blacksquare$

COROLLARY 3.14. *If Z is an uncountable Polish space*, *then for any $\alpha < \omega_1$*

$$\mathrm{dec}(\mathrm{cl}(\mathrm{b}\mathbf{L}_\alpha(Z) + \mathrm{b}\mathbf{U}_\alpha(Z)), R(\mathbf{L}_\alpha(Z) + \mathbf{U}_\alpha(Z))) > \aleph_0\,. \ \blacksquare$$

COROLLARY 3.15. *If Z is an uncountable Polish space*, *then for any $n \in \mathbb{N}$*

$$\mathrm{dec}(\mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\Sigma^1_n(Z) + \mathrm{b}\overline{\mathbf{M}}\Sigma^1_n(Z)), R(\underline{\mathbf{M}}\Sigma^1_n(Z) + \overline{\mathbf{M}}\Sigma^1_n(Z))) > \aleph_0. \ \blacksquare$$

LEMMA 3.16. *Let $\mathcal{A}$ be a σ-class of subsets of some set Z. If $f \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A})$, then for given $n \in \mathbb{N}$ and $\delta > 0$ there exists $v \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}$ such that $\|f - v\| < 2^{-n+1}$, $\mathrm{Rg}\, v \subseteq \{i \cdot 2^{-n} : i \in \mathbb{Z}\}$ and $v(x) < v(y)$ implies $f(x) < f(y) + \delta$ for all $x, y \in Z$.*

Proof. Let $m \in \mathbb{N}$, $m > n + 1$ and $2^{-m+2} < \delta$. Let $g \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}$ and $\|f - g\| < 2^{-m}$. By Lemma 3.10 there exists

$$w = \sum_{i=-M}^{M} i \cdot 2^{-m-1} \chi_{A_i}\,,$$

where $M \in \mathbb{N}$, $\|g - w\| < 2^{-m}$, the sets A_i are pairwise disjoint and, for each i, $A_i \in r(\mathcal{A})$ and thus $\chi_{A_i} \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}$. Then $\|w - f\| < 2^{-m+1}$. Let $v(x) = [2^n w(x)] \cdot 2^{-n}$. Obviously $\mathrm{Rg}\, v \subseteq \{i \cdot 2^{-n} : i \in \mathbb{Z}\}$ and $\|f - v\| < 2^{-n+1}$. If $v(x) < v(y)$ then $w(x) < w(y)$, whence $f(x) < f(y) + 2^{-m+2} < f(y) + \delta$. ∎

THEOREM 3.17. *Let $\mathcal{A}$ be a σ-class of subsets of some set Z. Then each $f \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A})$ can be expressed as the superposition $f = g \circ h$ where $h \in \mathrm{b}\underline{\mathbf{M}}_{[-1,1]}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}_{[-1,1]}\mathcal{A}$ and $g \in C([-2,2]) \cap VB([-2,2])$.*

Proof. Let $f \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A})$. Let $\|f\| < N \in \mathbb{N}$. The function f is the uniform limit $f = \lim_{n\to\infty} v_n$ of functions $v_n \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A} + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}$. By Lemma 3.16 we can assume that $\mathrm{Rg}\, v_n \subseteq \{i/2^n : i \in \mathbb{Z}\}$, $\|f - v_n\| < 2^{-n+1}$ and that

$$(*) \qquad v_n(x) < v_n(y) \quad \text{implies} \quad f(x) < f(y) + \delta_n\,,$$

where $\delta_n = (N+1)^{-1} \cdot 2^{-2n-2}$. Let $v_n = l_n + u_n$, where $l_n \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A}$ and $u_n \in \mathrm{b}\overline{\mathbf{M}}\mathcal{A}$. Let $\mathbf{v}(x) = (v_1(x), v_2(x), \ldots)$. Let s be the function defined on $\mathrm{Rg}\,\mathbf{v}$ as $s(v_1(x), v_2(x), \ldots) = \lim_{n\to\infty} v_n(x)$. Of course $f = s \circ \mathbf{v}$.

Now, let α_n, $n \in \mathbb{N}$, satisfy the following conditions:

1° $\alpha_n > 0$, $n \in \mathbb{N}$;
2° $\alpha_n \sup |l_n| < 2^{-n}$ and $\alpha_n \sup |u_n| < 2^{-n}$;
3° $\alpha_n 2^{-n} > 2 \sum_{i>n} \alpha_i (\sup |l_i| + \sup |u_i|)$.

Let

$$\phi(\mathbf{v}(x)) = \sum_{n=1}^{\infty} \alpha_n v_n(x) = \sum_{n=1}^{\infty} \alpha_n l_n(x) + \sum_{n=1}^{\infty} \alpha_n u_n(x)\,.$$

The convergence of the series follows from 2°. From 3° it follows that ϕ is 1-1. We have $f = s \circ \mathbf{v} = (s \circ \phi^{-1}) \circ (\phi \circ \mathbf{v})$. Of course $\phi \circ \mathbf{v} \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A} + \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}$ and we put $h = \phi \circ \mathbf{v}$.

Now we show that $\varphi = s \circ \phi^{-1}$ can be extended to a function $g \in VB([-2,2]) \cap C([-2,2])$. To this end it is enough to show that φ can be extended to a continuous function $\widetilde{\varphi}$ on $\mathrm{cl}(\mathrm{dom}\,\varphi)$ and that φ is of bounded variation on its domain.

Let $s_k = \sum_{n=1}^{\infty} \alpha_n v_n(x_k)$ and $t_k = \sum_{n=1}^{\infty} \alpha_n v_n(x'_k)$ and $s_k \nearrow q \swarrow t_k$. We shall show that the sequences $\varphi(s_k)$ and $\varphi(t_k)$ converge to the same limit $\widetilde{\varphi}(q)$. For each $n \in \mathbb{N}$ there is some $k(n)$ such that $v_n(x_k) = v_n(x'_k)$ for $k > k(n)$. Indeed, as ϕ preserves the lexicographic order on $\mathbf{v}(Z)$ the sequence $v_1(x_k)$ is nondecreasing and, as $|\mathrm{Rg}\, v_1| < \aleph_0$, it is constant for $k \geq n_1$, for some $n_1 \in \mathbb{N}$. Then for $k \geq n_1$ the sequence $v_2(x_k)$ is nondecreasing and is constant for $k \geq n_2$ for some $n_2 \geq n_1$. Inductively we prove that for each $m \in \mathbb{N}$ the sequence $v_m(x_k)$ is constant for $k \geq n_m$ for some $n_m \geq n_{m-1}$. Similarly, putting $n'_0 = 0$, by induction we show that for each $m \in \mathbb{N}$ the sequence $v_m(x'_k)$ is nonincreasing for $k \geq n'_{m-1}$ and constant for $k \geq n'_m$ for some $n'_m \geq n'_{m-1}$.

If $v_1(x_{n_1}) < v_1(x'_{n'_1})$ then for all $k \geq \max(n_1, n'_1)$ we would have $v_1(x'_k) = v_1(x'_{n'_1}) > v_1(x_{n_1}) = v_1(x_k)$ and by 3°, $t_k - s_k > \varepsilon$ for some fixed $\varepsilon > 0$, which is a contradiction. Inductively $v_m(x_{n_m}) = v_m(x'_{n'_m})$ for each $m \in \mathbb{N}$. As the sequence v_m is uniformly convergent to f and for any $m \in \mathbb{N}$ we have $v_m(x_k) = v_m(x'_k)$ for $k \geq \max(n_m, n'_m)$, the sequences $f(x_k) = \varphi(s_k)$ and $f(x'_k) = \varphi(t_k)$ are convergent and $f(x_k) - f(x'_k) \to 0$. Thus φ can be extended to a continuous function $\widetilde{\varphi}$ defined on $\mathrm{cl}(\mathrm{dom}\, \varphi)$.

Now we show that φ is of bounded variation on its domain. Let $t_1 < \ldots < t_m$, where $t_i = \sum_{n=1}^{\infty} \alpha_n v_n(x_i) = \phi \circ \mathbf{v}(x_i)$, $i \leq m$. We shall estimate the sum $\sum_{i \in \mathcal{I}} (\varphi(t_i) - \varphi(t_{i+1}))$, where $\mathcal{I} \subseteq \{1, \ldots, m-1\}$ is the set of all i for which $\varphi(t_i) - \varphi(t_{i+1}) > 0$. Let $A_n = \{i \in \mathcal{I} : \min\{j : v_j(x_i) < v_j(x_{i+1})\} = n\}$. We have

$$\sum_{i \in \mathcal{I}} (\varphi(t_i) - \varphi(t_{i+1})) = \sum_{n=1}^{\infty} \sum_{i \in A_n \cap \mathcal{I}} (\varphi(t_i) - \varphi(t_{i+1}))\,.$$

For $i \in A_n$ we have, by $(*)$, $\varphi(t_i) - \varphi(t_{i+1}) = f(x_i) - f(x_{i+1}) < \delta_n$, whence

$$\sum_{i \in \mathcal{I}} (\varphi(t_i) - \varphi(t_{i+1})) < \sum_{n=1}^{\infty} 2(N+1) \cdot 2^n \delta_n < 1\,.$$

Thus φ is of bounded variation. ∎

By [Ma, Th. 2] we have the following converse theorem:

Theorem 3.18. *If $\mathcal{R}$ is any algebra of functions such that $\mathrm{cl}\,\mathcal{R} = \mathcal{R}$ then for any $f \in \mathrm{b}\mathcal{R}$ and any function g continuous on a closed interval containing $\mathrm{Rg}\, f$ we have $g \circ f \in \mathcal{R}$.* ∎

Remark. Corollary 3.14 and Theorem 3.17 show that Theorem 14 in [L] is false. That there was a mistake in its proof in [L] was already noticed by A. Lindenbaum himself in [L, corr.].

Lemma 3.19. *Let $\mathcal{A}$ be a σ-class of subsets of some set Z. Let $X \subseteq Z$. Then* *every* $f \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}(\mathcal{A}(Z)|X) + \mathrm{b}\overline{\mathbf{M}}(\mathcal{A}(Z)|X))$ *can be extended to a function* $f^* \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z))$.

Proof. By Theorem 3.17, $f = h \circ g$ where $h \in C(\mathbb{R})$ and $g \in \mathrm{b}\underline{\mathbf{M}}(\mathcal{A}(Z)|X) + \mathrm{b}\overline{\mathbf{M}}(\mathcal{A}(Z)|X)$. The function g can be extended to some $g^* \in \mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z)$. Then $f^* = h \circ g^* \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z))$ by Theorem 3.18. ∎

THEOREM 3.20. *Let* $\mathcal{A}$ *be a hereditary* σ*-class on* $\mathcal{T}$ *and suppose* $\Sigma^0_1(X) \subseteq \mathcal{A}(X)$ *for every* $X \in \mathcal{T}$. *Let* $Z \in \mathcal{T}$ *and suppose* $\mathcal{A}(Z)$ *has a universal set in* $\mathcal{A}(\mathcal{C} \times Z)$. *Then there is an* $F \in \mathbf{M}\mathcal{A}'(\mathcal{N} \times Z)$ *which is a universal function for* $\mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z))$.

Proof. By Lemma 3.8 there is a function $H \in \underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(\mathcal{C} \times Z) + \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(\mathcal{C} \times Z)$ universal for $\underline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z) + \overline{\mathbf{M}}_{[-1,1]}\mathcal{A}(Z)$. Let $\phi : \mathcal{N} \to \mathcal{C}$ be a continuous surjection ([Ku, 37, I, Th. 1]). Let $G(w,x) = H(\phi(w),x)$ for $w \in \mathcal{N}$ and $x \in Z$. Then $G \in \mathbf{M}\mathcal{A}'(\mathcal{N} \times Z)$ because $H \in \mathbf{M}\mathcal{A}'(\mathcal{C} \times Z)$ and, as is easy to see, $\mathcal{A}'$ is closed under continuous substitutions. Let $\psi : \mathcal{N} \to C([-2,2])$ be a continuous surjection ([Ku, 37, I, Th. 1]). We write $\psi_w(\cdot)$ for $\psi(w)$ in the sequel. Let $\xi = (\xi_1, \xi_2)$ be any homeomorphism from $\mathcal{N}$ onto $\mathcal{N}^2$. Let $F(w,x) = \psi_{\xi_1(w)}(G(\xi_2(w),x))$. By Theorems 3.17 and 3.18, F is universal for $\mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z))$.

We show that $F \in \mathbf{M}\mathcal{A}'(\mathcal{N} \times Z)$. Let $\Psi(w,s) = \psi_{\xi_1(w)}(s)$, $w \in \mathcal{N}$, $s \in [-2,2]$, $\widetilde{G}(w,x) = G(\xi_2(w),x)$ and $\Phi(w,x) = (w, \widetilde{G}(w,x))$. We have $F(w,x) = \Psi(\Phi(w,x))$. Then $\widetilde{G} \in \mathbf{M}\mathcal{A}'(\mathcal{N} \times Z)$ because $\mathcal{A}'$ is closed under continuous substitutions. An easy argument shows that Ψ is continuous. Finally, $\Phi^{-1}(U) \in \mathcal{A}'(\mathcal{N} \times Z)$ for any open set $U \subseteq \mathcal{N} \times \mathbb{R}$: indeed, as $U = \bigcup_{i=1}^{\infty} V_i \times W_i$, where the V_i are open in $\mathcal{N}$ and W_i are open in $\mathbb{R}$, we have

$$\Phi^{-1}(U) = \bigcup_{i=1}^{\infty} (V_i \times Z) \cap \widetilde{G}^{-1}(W_i) \in \mathcal{A}'(\mathcal{N} \times Z)\,. \ \blacksquare$$

In the next theorem we add new assumptions on the hereditary σ-class $\mathcal{A}$ and the family $\mathcal{T}$. Namely, we assume that $\mathcal{T}$ satisfies the following stronger form of (i):

(i*) if $X \subseteq Z \in \mathcal{T}$ and X, as a subspace of Z, is completely metrizable by some metric ρ, then $(X,\rho) \in \mathcal{T}$.

We then assume that $\mathcal{A}$ satisfies for any $Z \in \mathcal{T}$ and $X \subseteq Z$:

(II*) $\mathcal{A}(Z)|X = \mathcal{A}(X)$ where X is considered with any metric ρ such that $(X,\rho) \in \mathcal{T}$ is topologically a subspace of Z.

Assume also $\Sigma^0_1 \subseteq \mathcal{A}$.

However, the conditions imposed on $\mathcal{A}$ are not very restrictive as the classes Σ^0_α, Σ^0_n still satisfy them.

THEOREM 3.21. *Let $\mathcal{A}$ be a hereditary σ-class satisfying* (II*). *Let $Z \in \mathcal{T}$ be uncountable and suppose $\mathcal{A}(Z)$ has a universal set in $\mathcal{A}(\mathcal{C} \times Z)$. Then there exists $g \in \mathbf{M}\mathcal{A}'(Z)$ for which there is no countable partition $Z = \bigcup\{Z_n : n \in \mathbb{N}\}$ such that $g|Z_n \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}(\mathcal{A}(Z)|Z_n) + \mathrm{b}\overline{\mathbf{M}}(\mathcal{A}(Z)|Z_n))$ for each $n \in \mathbb{N}$. In other words*,

$$\mathrm{dec}(\mathbf{M}\mathcal{A}'(Z), R(\mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z))) > \aleph_0\,.$$

Proof. Let $\mathcal{N}'$ be any subset of Z homeomorphic to $\mathcal{N}$ ([Ku, 36, IV, Cor. 2]). Let $\varphi = (\varphi_1, \varphi_2, \ldots) : \mathcal{N}' \to {}^{\mathbb{N}}\mathcal{N}$ be any homeomorphism. By Theorem 3.20 there exists a universal function $F \in \mathbf{M}\mathcal{A}'(\mathcal{N} \times Z)$. Let $F_n(s,x) = F(\varphi_n(s), x)$ for $s \in \mathcal{N}'$ and $x \in Z$. Then $F_n \in \mathbf{M}\mathcal{A}'(\mathcal{N}' \times Z)$ and thus $f_n : \mathcal{N}' \to \mathbb{R}$ defined as $f_n(s) = F_n(s,s)$ belongs to $\mathbf{M}\mathcal{A}'(\mathcal{N}')$ because, by our assumption on $\mathcal{A}$, $\mathcal{A}'(\mathcal{N}' \times \mathcal{N}') = \mathcal{A}'(\mathcal{N}' \times Z)|(\mathcal{N}' \times \mathcal{N}')$. By Corollary 2.F and the fact that $\mathcal{A}'(\mathcal{N}') = \mathcal{A}'(Z)|\mathcal{N}'$ and $\mathcal{N}' \in (\mathcal{A}(Z))_\delta$, f_n can be extended to a function $f^*_n \in \mathbf{M}\mathcal{A}'(Z)$. By Theorem 2.A there exists $g \in \mathbf{M}\mathcal{A}'(Z)$ such that $g(x) \neq f^*_n(x)$ for each $x \in Z$ and $n \in \mathbb{N}$.

Now assume that $g = \bigcup\{g_n : n \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$, $g_n \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}(\mathcal{A}(Z)|\mathrm{dom}\, g_n) + \mathrm{b}\overline{\mathbf{M}}(\mathcal{A}(Z)|\mathrm{dom}\, g_n))$. By Lemma 3.19 each g_n has an extension $g^*_n \in \mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\mathcal{A}(Z) + \mathrm{b}\overline{\mathbf{M}}\mathcal{A}(Z))$ for all $n \in \mathbb{N}$. There is an $s \in \mathcal{N}'$ such that $F_n(s,x) = g^*_n(x)$ for each $n \in \mathbb{N}$. But then $f_n(s) = g_n(s)$ for each $n \in \mathbb{N}$ and, as $g(s) \in \{g_n(s) : n \in \mathbb{N}\}$, we obtain $g(s) = f_{n_0}(s)$ for some $n_0 \in \mathbb{N}$, which is a contradiction. ∎

For any uncountable Polish space Z we derive from Theorem 3.21 the following immediate corollaries.

COROLLARY 3.22.

$$\mathrm{dec}(\mathbf{B}_{\alpha+1}(Z), R(\mathrm{cl}(\mathrm{b}\mathbf{L}_\alpha(Z) + \mathrm{b}\mathbf{U}_\alpha(Z))) > \aleph_0\,. \ \blacksquare$$

COROLLARY 3.23.

$$\mathrm{dec}(\mathbf{M}\Sigma^1_{n+1}(Z), R(\mathrm{cl}(\mathrm{b}\underline{\mathbf{M}}\Sigma^1_n(Z) + \mathrm{b}\overline{\mathbf{M}}\Sigma^1_n(Z))) > \aleph_0\,. \ \blacksquare$$

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