STUDIA MATHEMATICA 103 (3) (1992)

Almost everywhere summability of Laguerre series. II

by

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Abstract. Using methods from [9] we prove the almost everywhere convergence of the Cesàro means of Laguerre series associated with the system of Laguerre functions $\mathcal{L}_n^a(x) = (n!/\Gamma(n+a+1))^{1/2}e^{-x/2}x^{a/2}L_n^a(x)$, $n = 0,1,2,\ldots$, $a \geq 0$. The novel ingredient we add to our previous technique is the A_p weights theory. We also take the opportunity to comment and slightly improve on our results from [9].

1. Introduction. This paper is a natural continuation of [9] where we investigated almost everywhere summability of expansions with respect to the system of functions

$$\ell_n^a(x) = (n!/\Gamma(n+a+1))^{1/2}e^{-x/2}L_n^a(x)\,, \tag{1.1}$$

orthonormal in $L^2(\mathbb{R}_+, x^a dx)$, $a \geq 0$. There are several ways of studying Laguerre expansions. A first, and say, the most principal one is by dealing with Laguerre polynomials

$$L_n^a(x) = (n!)^{-1}e^x x^{-a}(d/dx)^n(e^{-x}x^{n+a})\,,$$

which form a complete orthogonal system in $L^2(\mathbb{R}_+, x^a e^{-x}dx)$. Investigating Poisson integrals for this type of expansion Muckenhoupt [6] was the first to prove an a.e. convergence result for Laguerre expansions. In fact, he pointed out that polynomial expansions possess some unexpected features by showing that for any $1 \leq p < 2$ there is a function in $L^p(x^a e^{-x}dx)$ whose Abel–Poisson means diverge everywhere.

A much more pleasant type of expansion occurs to be that with the functions $\ell_n^a(x)$ as an orthonormal basis in $L^2(\mathbb{R}_+, x^a\,dx)$, $a \geq 0$. We proved in [9] that the Cesàro means of order $\delta > a+2/3$ converge to f a.e. (in §4 we show how to lower the bound on the parameter δ). Incidentally, as pointed out in [9], this result also gives a.e. convergence of the Cesàro means of polynomial

1991 *Mathematics Subject Classification*: Primary 42C15; Secondary 42C10, 43A55.
Key words and phrases: Laguerre expansions, Cesàro means, almost everywhere convergence.

expansions of functions in $L^2(\mathbb{R}_+, e^{-x}x^a\,dx)$ and thus in $L^p(\mathbb{R}_+, e^{-x}x^a\,dx)$, $p > 2$.

A third type of expansion takes place when we consider the more famous (which in our opinion does not mean more natural) system of Laguerre functions

$$\mathcal{L}_n^a(x) = (n!/\Gamma(n+a+1))^{1/2} e^{-x/2} x^{a/2} L_n^a(x)$$

forming an orthonormal basis in $L^2(\mathbb{R}_+, dx)$. The convergence for this type of expansion has been investigated in the literature at least since Askey and Wainger proved their celebrated mean convergence result. In 1979 Markett [4] proved that for any $a \geq 0$ the Cesàro means

$$C_n^\delta f(x) = (A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \langle f, \mathcal{L}_k^a\rangle_{L^2(dx)} \mathcal{L}_k^a(x) \tag{1.2}$$

of order $\delta > 1/2$ converge in L^p norm to f for any $f \in L^p(\mathbb{R}_+, dx)$, $1 \leq p < \infty$, and, moreover, $\delta = 1/2$ is the critical index here. Długosz' paper [3] was the first to bring interesting a.e. convergence results.

The main objective of this paper is to prove the following result concerning the a.e. summability of the Cesàro means for expansions with respect to the system of Laguerre functions $\mathcal{L}_n^a(x)$.

THEOREM 1.1. *Let $a \geq 0$ and $\delta > a + 2/3$. Then for $C_n^\delta f$ given by* (1.2)

$$C_n^\delta f(x) \to f(x)$$

almost everywhere as $n \to \infty$ for every $f \in L^p(\mathbb{R}_+, dx)$, $1 \leq p < \infty$.

The purely real-variable proof of this theorem, given in §3, relies on our technique developed in [9] and applies well-known facts from Muckenhoupt's A_p weights theory. We also prove a similar result for still other, probably the most exotic, Laguerre expansions. This time we use the system of functions

$$\psi_n^a(x) = \mathcal{L}_n^a(x^2)(2x)^{1/2},$$

$a \geq 0$, orthonormal in $L^2(\mathbb{R}_+, dx)$. These expansions were considered by Markett [5] and Thangavelu [10]. Thangavelu proved that for $a \geq 1/2$ the Cesàro means of order $\delta > 1/6$ converge to f a.e. and in L^p norm for any $f \in L^p(\mathbb{R}_+, dx)$, $1 \leq p < \infty$, and $1/6$ is the critical index here.

We prove the following.

THEOREM 1.2. *Let $a \geq 0$ and $\delta > a + 2/3$. Then for every $f \in L^p(\mathbb{R}_+, dx)$, $1 \leq p < \infty$,*

$$(A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \langle f, \psi_k^a\rangle_{L^2(dx)} \psi_k^a(x) \to f(x)$$

almost everywhere as $n \to \infty$.

Even if the Cesàro index $\delta_0 = a + 2/3$ far exceeds the critical index $1/6$ we decided to include the proof for its simplicity. Moreover, the theorem says something about the case $0 \leq a < 1/2$ which is not covered by [10]. Getting in [10] the critical $\delta = 1/6$ required much more effort.

The paper is organized as follows. In §2 we prove a simple lemma on A_p weights. §3 is devoted to the proofs of the main theorems. In §4 we gather improvements and comments on our previous results from [9]. For all unexplained notions and symbols we refer to [9].

2. Preliminaries. Given a fixed $\gamma > -1$ we consider the half-axis $\mathbb{R}_+$ equipped with the measure $d\mu(x) = x^\gamma\,dx$ and the Euclidean distance $\varrho(x,y) = |x-y|$. Then $d\mu$ satisfies the doubling condition

$$\mu(B_{2\varepsilon}(x)) \leq C\mu(B_\varepsilon(x)),$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}_+ : |x-y| < \varepsilon\}$. Thus $(\mathbb{R}_+, d\mu, \varrho)$ is a homogeneous space and the maximal operator

$$Mf(x) = \sup_{\varepsilon>0} \mu(B_\varepsilon(x))^{-1} \int_{B_\varepsilon(x)} |f(y)|\,d\mu(y) \tag{2.1}$$

is of weak type $(1,1)$ and strong type (p,p), $1 < p \leq \infty$.

The Muckenhoupt theory of A_p weights has originally been developed for the Euclidean spaces and then extended to spaces of homogeneous type. In particular, in the context of the homogeneous space which is considered here we say that a nonnegative weight $w(y)$ on $\mathbb{R}_+$ satisfies the $A_1 = A_1\,(d\mu)$ condition provided

$$\frac{1}{\mu(I)} \int_I w(y)\,d\mu(y) \leq C \operatorname*{ess\,inf}_I w$$

for any interval $I \subset \mathbb{R}_+$. Similarly, $w(y)$ satisfies the A_p condition, $1 < p < \infty$, if

$$\Big(\int_I w(y)\,d\mu(y)\Big)\Big(\int_I w(y)^{-1/(p-1)}\,d\mu(y)\Big)^{p-1} \leq C\mu(I)^p.$$

The importance of these classes of functions is explained by the fact that for $w \in A_1$ the operator M given by (2.1) maps $L^1(wd\mu)$ continuously into weak $L^1(wd\mu)$, that is,

$$\int_{\{Mf>\lambda\}} w\,d\mu \leq \frac{C}{\lambda} \int_0^\infty |f| w\,d\mu,$$

and, if $w \in A_p$, $1 < p < \infty$, then M is bounded on $L^p(w d\mu)$, i.e.

$$\int_0^\infty |Mf|^p w\, d\mu \le C \int_0^\infty |f|^p w\, d\mu .$$

We will need the following simple $A_p\,(d\mu)$ characterization of power functions.

LEMMA 2.1. *Let $\gamma > -1$, $1 \le p < \infty$ and $d\mu(x) = x^\gamma\, dx$. Then $x^b \in A_1\,(d\mu)$ if and only if $-(\gamma+1) < b \le 0$. Similarly, $x^b \in A_p(d\mu)$, $1 < p < \infty$, if and only if $-(\gamma+1) < b < (\gamma+1)(p-1)$.*

Proof. The necessity is obvious. For the sufficiency, if $p = 1$ and b satisfies $-(\gamma+1) < b \le 0$ then all we need to check is the inequality

$$t^{b+\gamma+1} - u^{b+\gamma+1} \le Ct^b(t^{\gamma+1} - u^{\gamma+1})$$

with arbitrary $0 \le u < t < \infty$. By homogeneity, this is equivalent to

$$1 - z^{b+\gamma+1} \le C(1 - z^{\gamma+1}),$$

$0 \le z < 1$, which is clearly satisfied with $C = 1$. If $1 < p < \infty$ we have to show

$$(t^{b+\gamma+1} - u^{b+\gamma+1})(t^{\gamma-\frac{b}{p-1}+1} - u^{\gamma-\frac{b}{p-1}+1})^{p-1} \le C(t^{\gamma+1} - u^{\gamma+1})^p$$

with arbitrary $0 \le u < t < \infty$, or, by homogeneity,

$$(1 - z^{b+\gamma+1})(1 - z^{\gamma-\frac{b}{p-1}+1})^{p-1} \le C(1 - z^{b+\gamma+1})^p$$

with arbitrary $0 \le z < 1$. An elementary argument shows that there is a constant $C > 0$ such that the last inequality is satisfied.

3. Proofs. Proving Theorems 1.1 and 1.2 we work with $\alpha \ge 1$ and then translate the results for the parameter $a = \alpha - 1$. Until the end of this section we use the notation $d\mu(x) = x^{2\alpha-1}\, dx$. As in [9] we consider the system of functions

$$\varphi_n^*(x) = (2n!/\Gamma(n+\alpha))^{1/2} e^{-x^2/2} L_n^{\alpha-1}(x^2),$$

orthonormal in $L^2(\mathbb{R}_+, d\mu)$. The main result of [9], Theorem 5.5, says that for $\delta > \alpha - 1/3$ the Cesàro means

$$C_n^\delta f(x) = (A_n^\delta)^{-1} \sum_{k=0}^n A_{n-k}^\delta \langle f, \varphi_k^*\rangle_{L^2(d\mu)} \varphi_k^*(x)$$

converge to $f(x)$ as $n \to \infty$ almost everywhere for any $f \in L^p(d\mu)$, $1 \le p < \infty$. This result is achieved by establishing the crucial estimate

$$C_*^\delta f(x) \le CMf(x) \tag{3.1}$$

for the maximal operator $C_*^\delta f(x) = \sup_n |C_n^\delta f(x)|$, where f is any locally integrable (with respect to $d\mu$) function on $\mathbb{R}_+$. Here Mf denotes the maximal function defined in §2 with $\gamma = 2\alpha - 1$. We now observe that all we need to show for the proof of Theorem 1.1 is the almost everywhere convergence

$$C_n^\delta f(x) \to f(x) \tag{3.2}$$

with $\delta > a + 2/3$ for functions from the weighted space $L^p(w_{p,\alpha}\, d\mu)$ where

$$w_{p,\alpha}(x) = x^{(\alpha-1)(p-2)}.$$

This is so because for fixed $1 \le p < \infty$ the mapping

$$Df(x) = 2^{-1/2} f(x^{1/2}) x^{(\alpha-1)/2}$$

is a bijection from $L^p(w_{p,\alpha}\, d\mu)$ onto $L^p(dx)$ satisfying

(a) $D\varphi_n^* = \mathcal{L}_n^{\alpha-1}$,
(b) $\|Df\|_{L^p(dx)} = 2^{1/p-1/2}\|f\|_{L^p(w_{p,\alpha} d\mu)}$,
(c) $\langle Df, Dg\rangle_{L^2(dx)} = \langle f, g\rangle_{L^2(d\mu)}$.

Thus, replacing x by $x^{1/2}$ in (3.2) and then multiplying both sides of (3.2) by $2^{-1/2}x^{(\alpha-1)/2}$ gives $C_n^\delta f(x) \to f(x)$ for every $f \in L^p(\mathbb{R}_+, dx)$ with $C_n^\delta f$ given by (1.2). Therefore, by (3.1) we are done provided we show that

$$\|Mf\|_{L^p(w_{p,\alpha} d\mu)} \le C\|f\|_{L^p(w_{p,\alpha} d\mu)} \tag{3.3}$$

for $1 < p < \infty$ and

$$\int_{\{Mf > \lambda\}} w_{1,\alpha}\, d\mu \le \frac{C}{\lambda}\|f\|_{L^1(w_{1,\alpha}\, d\mu)}. \tag{3.4}$$

But both inequalities follow from Lemma 2.1 with $\gamma = 2\alpha - 1$ and $b = (\alpha-1)(p-2)$.

Similarly, to prove Theorem 2.2 we observe that the mapping $Hf(x) = f(x)x^{\alpha-1/2}$ is, for fixed $1 \le p < \infty$, a bijection from $L^p(v_{p,\alpha} d\mu)$ onto $L^p(dx)$ where

$$v_{p,\alpha}(x) = x^{(\alpha-1/2)(p-2)}.$$

Moreover, H satisfies

(a) $H\varphi_n^* = \psi_n^{\alpha-1}$,
(b) $\|Hf\|_{L^p(dx)} = \|f\|_{L^p(v_{p,\alpha}\, d\mu)}$,
(c) $\langle Hf, Hg\rangle_{L^2(dx)} = \langle f, g\rangle_{L^2(d\mu)}$.

The proof is concluded by observing that (3.3) and (3.4) are satisfied with $w_{p,\alpha}$ replaced by $v_{p,\alpha}$. Once more we use Lemma 2.1 with $\gamma = 2\alpha - 1$ and $b = (\alpha - 1/2)(p-2)$.

4. Improvements and comments. In this section we first improve on our results concerning the summability of the Cesàro means

$$C_n^\delta f(x) = (A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \langle f, \ell_k^a \rangle_{L^2(d\mu)} \ell_k^a(x) \tag{4.1}$$

of expansions with respect to the system of functions $\ell_k^a(x)$ given by (1.1). By improving we mean lowering the Cesàro index δ for which the mean or a.e. convergence holds.

In Theorem 1.3 of [9] we proved that if $\delta > a+1/2$, $a \ge 0$, then $C_n^\delta f \to f$ in L^p norm for every $f \in L^p(x^a\,dx)$ and $1 \le p < \infty$. It occurs that the lower bound $a + 1/2$ is best possible for $p = 1$. For other values of p, $1 < p < \infty$, we apply Muckenhoupt's result from [7] and then interpolate to get the following.

PROPOSITION 4.1. *Let $a \ge 0$, $1 \le p < \infty$ and*

$$\delta(a,p) = \begin{cases} 0 & \text{for } \dfrac{4(a+1)}{2a+3} < p < \dfrac{4(a+1)}{2a+1}, \\ 2(a+1)\left|\dfrac{1}{p} - \dfrac{1}{2}\right| - \dfrac{1}{2} & \text{otherwise.} \end{cases}$$

If $\delta > \delta(a,p)$ then

$$\|C_n^\delta f - f\|_{L^p(x^a dx)} \to 0 \tag{4.2}$$

as $n \to \infty$ for every $f \in L^p(x^a\,dx)$. The lower bound $\delta(a,p)$ is best possible in the sense that if $0 \le \delta < \delta(a,p)$ then there exists an $f \in L^p(x^a dx)$ such that (4.2) does not hold.

Proof. By the Banach–Steinhaus theorem the mean convergence (4.2) for every $f \in L^p(x^a\,dx)$ is equivalent (up to the existence of a dense class of functions for which (4.2) holds) to the uniform boundedness

$$\|C_n^\delta\|_{p,p} \le C \tag{4.3}$$

of the operator norms of C_n^δ on $L^p(x^a\,dx)$. It follows from [7] (cf. Theorem 7 of §5 with $A = B = a = b$ equal to $a(1/p - 1/2)$) that the partial sums $S_n = C_n^0$ are uniformly bounded on $L^p(x^a\,dx)$ if and only if

$$\frac{4(a+1)}{2a+3} < p < \frac{4(a+1)}{2a+1}.$$

Using Stein's complex interpolation theorem we then prove that for any fixed p, $1 < p \le 4(a+1)/(2a+3)$ or $4(a+1)/(2a+1) \le p < \infty$, C_n^δ are uniformly bounded on $L^p(x^a\,dx)$ provided $\delta > 2(a+1)|1/p - 1/2| - 1/2$.

To show that the index $\delta(a,p)$ is critical suppose $1 \le p < 4(a+1)/(2a+3)$ and assume (4.3) is valid for a δ, $0 \le \delta < \delta(a,p)$. Then the formula

$$C_n^0 = \sum_{k=0}^{n} A_k^\delta A_{n-k}^{-\delta-1} C_k^\delta$$

gives the bound

$$\|S_n\|_{p,p} \le Cn^\delta \tag{4.4}$$

on the partial sum operators $S_n = C_n^0$ (cf. [1] for details). The mapping $f \to \widehat{f}(n) = \int_0^\infty f(x)\ell_n^a(x)x^a\,dx$, for any fixed $n = 0, 1, \ldots$, gives rise to a bounded functional on $L^p(x^a\,dx)$ with norm equal to $\|\ell_n^a\|_q$, $1/p + 1/q = 1$. Thus we can take a function $f \in L^p(x^a\,dx)$, $\|f\|_p = 1$, such that $\frac{1}{2}\|\ell_n^a\|_q \le |\widehat{f}(n)|$. Then, by (4.4), $|\widehat{f}(n)|\|\ell_n^a\|_p \le Cn^\delta$ so

$$\|\ell_n^a\|_p \|\ell_n^a\|_q \le Cn^\delta\,.$$

But this is only possible when $\delta \ge \delta(a,p)$ since by the estimates from [5] we have ($a_n \sim b_n$ stands for $a_n = O(b_n)$ and $b_n = O(a_n)$ as $n \to \infty$)

$$\|\ell_n^a\|_p \sim \begin{cases} n^{(a+1)(1/p-1/2)}, & 1 \le p < \dfrac{4(a+1)}{2a+1}, \\ n^{-1/4}(\log n)^{1/p}, & p = 4, \\ n^{(a+1)(1/2-1/p)-1/2}, & \dfrac{4(a+1)}{2a+1} < p \le \infty, \end{cases} \tag{4.5}$$

so $\|\ell_n^a\|_p\|\ell_n^a\|_q \sim n^{2(a+1)(1/p-1/2)-1/2}$. This finishes the proof of Proposition 4.1.

As far as the a.e. convergence is concerned we proved in [9] that the Cesàro means (4.1) converge to $f(x)$ a.e. for every $f \in L^p(x^a dx)$, $1 \le p < \infty$, provided $\delta > 2/3$. Keeping in mind the mean convergence result it is tempting to lower this exponent by 1/6. Even if we still cannot do this for the whole range of p's we will show here how to do this for all p's from the interval $(4/3, \infty)$. The argument presented below was communicated to the author in a letter by S. Thangavelu to whom we are greatly indebted.

From now on we strictly follow the notation from the proofs of Lemma 5.4 and Theorem 5.5 of [9]. As in §3 of the present paper we work with the parameter $\alpha \ge 1$ and then take $a = \alpha - 1$. To prove in [9] the basic estimate

$$C_*^\delta f(x) = \sup_n |C_n^\delta f(x)| \le Cf^*(x)$$

we first noted that $C_n^\delta f(x) = \Phi_{n,\delta} \times f$ and then showed the inequality

$$|\Phi_{n,\delta}(x)| \le C\omega_{\sqrt{n}}(x) \tag{4.6}$$

with $\delta > \alpha - 1/3$, $\omega(x) = (1+x^2)^{-(\alpha+\varepsilon)}$ and $\varepsilon > 0$ chosen in such a way that $\delta > \alpha - 1/3 + 2\varepsilon$. In fact, as one can check just by following the proof of Lemma 5.4, $\delta > \alpha - 1/2$ is enough as long as we are outside the interval

$(\sqrt{\nu/2}, \sqrt{3\nu/2})$, $\nu = 4n + 2(\alpha + \delta) + 2$. Therefore, let $\Phi_{n,\delta} = \Phi^1_{n,\delta} + \Phi^2_{n,\delta}$, where

$$\Phi^1_{n,\delta} = \chi_{\{\nu/2 < x^2 \le 3\nu/2\}} \Phi_{n,\delta}\,,$$

and, for $i = 1, 2$, let

$$C^\delta_{*,i} f(x) = \sup_n |\Phi^i_{n,\delta} \times f(x)|\,.$$

We will show that if $r > 4/3$ and $\delta > \alpha - 1/2$ then

$$C^\delta_{*,1} f(x) \le C((|f|^r)^*(x))^{1/r}\,. \tag{4.7}$$

We have

$$|\Phi^1_{n,\delta} \times f(x)| \le \int\limits_{\sqrt{\nu/2}}^{\sqrt{3\nu/2}} |\Phi^1_{n,\delta}(y)| T^x_E(|f|)(y)\, d\mu(y) = \int\limits_{\sqrt{\nu/2}}^{\sqrt{\nu}} + \int\limits_{\sqrt{\nu}}^{\sqrt{3\nu/2}} = I_1 + I_2\,.$$

Only the first integral will be considered. The second is estimated in a similar way. By the definition of $\Phi_{n,\delta}$

$$|\Phi_{n,\delta}(x)| \le Cn^{-\delta} e^{-x^2/2} |L^{\alpha+\delta}_n(x^2)| \le Cn^{(\alpha-\delta)/2} x^{-(\alpha+\delta)} |\mathcal{L}^{\alpha+\delta}_n(x^2)|\,.$$

Therefore, for $\nu/2 < x^2 < 3\nu/2$, using the fundamental estimate on the Laguerre functions (cf. [6], p. 235), we get

$$\begin{aligned} |\Phi^1_{n,\delta}(x)| &\le Cn^{(\alpha-\delta)/2} \nu^{-(\alpha+\delta)/2} \nu^{-1/4} (\nu^{1/3} + |x^2 - \nu|)^{-1/4} \\ &\le Cn^{-1/4-\delta} (\nu^{1/3} + |x^2 - \nu|)^{-1/4}\,. \end{aligned}$$

Thus, with $1/r + 1/q = 1$, we estimate I_1 as follows:

$$\begin{aligned} I_1 &\le Cn^{-1/4-\delta} \int\limits_{\sqrt{\nu/2}}^{\sqrt{\nu}} (\nu^{1/3} + \nu - y^2)^{-1/4} T^x_E(|f|)(y)\, d\mu(y) \\ &\le Cn^{-1/4-\delta} \Big(\int\limits_{\sqrt{\nu/2}}^{\sqrt{\nu}} (\nu^{1/3} + \nu - y^2)^{-q/4}\, d\mu(y) \Big)^{1/q} \\ &\quad \times \Big(\int\limits_{\sqrt{\nu/2}}^{\sqrt{\nu}} T^x_E(|f|^r)(y)\, d\mu(y) \Big)^{1/r} \\ &\le Cn^{-1/4-\delta} \nu^{\alpha/r} \Big(\int\limits_{\nu/2}^{\nu} (\nu^{1/3} + \nu - t)^{-q/4} t^{\alpha-1}\, dt \Big)^{1/q} ((|f|^r)^*(x))^{1/r} \\ &\le Cn^{-1/4-\delta+\alpha/r+(\alpha-1)/q} \Big(\int\limits_{\nu/2}^{\nu} (\nu^{1/3} + \nu - t)^{-q/4}\, dt \Big)^{1/q} ((|f|^r)^*(x))^{1/r} \\ &\le C((|f|^r)^*(x))^{1/r}\,. \end{aligned}$$

The last inequality comes from the assumption that $\delta > \alpha - 1/2$ and $q < 4$.

As we have already mentioned the condition $\delta > \alpha - 1/2$ is sufficient to show that $C^\delta_{*,2} f(x) \le Cf^*(x)$, which implies

$$\|C^\delta_{*,2} f\|_p \le C\|f\|_p\,,$$

since the maximal operator $f \to f^*$ is of strong type (p,p), $1 < p < \infty$. Similarly, if $p > 4/3$, by taking $4/3 < r < p$ and using (4.7)

$$\|C^\delta_{*,1} f\|_p \le C\|((|f|^r)^*)^{1/r}\|_p \le C\|f\|_p\,.$$

Now, since $C^\delta_* f \le C^\delta_{*,1} f + C^\delta_{*,2} f$, for $\delta > \alpha - 1/2$ and $p > 4/3$ we eventually get

$$\|C^\delta_* f\|_p \le C\|f\|_p\,, \tag{4.8}$$

which is sufficient to prove the a.e. convergence of the δ-Cesàro means (4.1), $\delta > a + 1/2$, to $f(x)$ for any $f \in L^p(x^a dx)$, $p > 4/3$.

To lower further the index δ for which (4.8) holds we use the interpolation method from [8] (§5 of Chapter 7). First we quote a result valid for general orthonormal systems.

LEMMA 4.2 ([2], p. 238). *Suppose that* $\|C^{\delta_0}_* f\|_2 \le C\|f\|_2$ *for a* $\delta_0 > 0$. *Then* $\|C^\delta_* f\|_2 \le C\|f\|_2$ *for every* $\delta > 0$.

Thus, interpolating between $p = 2$ with arbitrarily small δ and $p = \infty$ or $p = 1$ with $\delta > \alpha - 1/2$ or $\delta > \alpha - 1/3$ respectively, gives $\|C^\delta_* f\|_p \le C\|f\|_p$ with $\delta > (2\alpha - 1)(1/2 - 1/p)$ for $2 \le p \le \infty$ and $\delta > (2\alpha - 2/3)(1/p - 1/2)$ for $1 \le p \le 2$. Consequently, we obtain

PROPOSITION 4.3. *Let* $a \ge 0$, $1 \le p < \infty$ *and*

$$\delta(a,p) = \begin{cases} 2(a + 2/3)(1/p - 1/2), & 1 \le p \le 2, \\ 2(a + 1/2)(1/2 - 1/p), & 2 \le p \le \infty. \end{cases}$$

Then for every $f \in L^p(x^a\, dx)$ *and* $\delta > \delta(p,a)$

$$(A^\delta_n)^{-1} \sum_{k=0}^{n} A^\delta_{n-k} \langle f, \ell^a_k \rangle_{L^2(d\mu)} \ell^a_k(x) \to f(x)$$

almost everywhere as $n \to \infty$.

In addition we strengthen Corollary 1.2 of [9].

COROLLARY 4.4. *Let* $a \ge 0$ *and* $\delta > 0$. *Then for every* $f \in L^2(x^a e^{-x}\, dx)$ *with the expansion* $f \sim \sum_{k=0}^\infty d_k L^a_k(x)$

$$(A^\delta_n)^{-1} \sum_{k=0}^{n} A^\delta_{n-k} d_k L^a_k(x) \to f(x)$$

almost everywhere as $n \to \infty$.

We are now going to prove the following localization principle for Laguerre expansions.

THEOREM 4.5. *Let $a \ge 0$, $\delta > a + 2/3$, $1 \le p < \infty$ and let $f \in L^p(x^a dx)$ have the expansion $f \sim \sum_{k=0}^{\infty} b_k \ell_k^a$. If f vanishes in a neighborhood of a point $y \in (0, \infty)$ then*

$$(A_n^\delta)^{-1} \sum_{k=0}^{n} A_{n-k}^\delta \ell_k^a(y) \to 0 \quad \text{as } n \to \infty. \tag{4.9}$$

The theorem will follow from the more general result.

PROPOSITION 4.6. *Let $\alpha \ge 1$ and $k \in L^1(x^{2\alpha-1}\,dx)$ be such that $|k(x)| \le C(1+x^2)^{-(\alpha+\varepsilon)}$, $\varepsilon > 0$. If $f \in L^p(x^{2\alpha-1}dx)$, $1 \le p < \infty$, vanishes in a neighborhood of a point $y \in (0,\infty)$ then $\lim_{t\to 0} k_t * f(y) = 0$.*

Proof. We consider the case $p = 1$ only; for $p > 1$ the argument is similar. Let $\eta > 0$. By the assumption on k there is a constant $C > 0$ such that for $x > \eta$ and $0 < t < 1$ we have $|k_t(x)| \le C$. Suppose now that g vanishes on $(y-\eta, y+\eta)$. Then also $T_E^y g(x) = 0$ for $x < \eta$ and

$$|k_t * g(y)| \le \int_\eta^\infty |T_E^y g(x)||k_t(x)|\,d\mu(x) \le C\|g\|_1 .$$

Given a function f vanishing on $(y-\eta, y+\eta)$ we now choose a compactly supported continuous function h vanishing on $(y-\eta, y+\eta)$ and such that $\|f-h\|_1$ is small. Since $\lim_{t\to 0} k_t * h(y) = 0$ and

$$|k_t * f(y)| \le |k_t * h(y)| + C\|f-h\|_1$$

we get the result.

We now return to the proof of Theorem 4.5. By the remarks from [9], pp. 132, 133, all we need to prove is an analogue of (4.9) with a, $\delta > a + 2/3$, f and $\ell_k^a(x)$ replaced by $\alpha \ge 1$, $\delta > \alpha - 1/3$, $f \in L^p(x^{2\alpha-1}dx)$ and $\varphi_k^*(x)$. Since we have (cf. [9], p. 145)

$$|\mathcal{C}_n^\delta f(x)| = |\Phi_{n,\delta} \times f(x)| \le (|\Phi_{n,\delta}| * |f|)(x) \le C\omega_{\sqrt{n}} * |f|(x)$$

where $\omega(x) = (1+x^2)^{-(\alpha+\varepsilon)}$ and $\varepsilon > 0$ is chosen in such a way that $\delta > \alpha - 1/3 + 2\varepsilon$, we are done.

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Received June 2, 1992 (2950)