ACTA ARITHMETICA
LXIV.1 (1993)

The *abc*-inequality and the generalized Fermat equation in function fields

by

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1. Introduction. Let $K = k(t)$ be a rational function field of one variable with constant field k algebraically closed of characteristic 0. It is a classical result that the Fermat equation

$$z_1^r + z_2^r = z_3^r, \qquad r \geq 3\,,$$

has no solution in non-constant polynomials in $k(t)$ with no common factors. Newman and Slater [N-S] showed that this result also holds for the Euler–Fermat equation

$$z_1^r + \ldots + z_n^r = z_{n+1}^r, \qquad r \geq 8n^2\,.$$

Let $K^* = K - \{0\}$ and let $c_1, \ldots, c_n$ be elements in K^*. Bounds for the heights and for the number of solutions of the generalized Fermat equation

$$c_1 z_1^r + \ldots + c_n z_n^r = 0\,, \qquad c_i \in K^*\,, \tag{1.1}$$

which depend on r and on the degrees of the coefficients c_i have been obtained by Silverman [S] and Voloch [V]. They showed that (1.1) has no non-trivial solutions when the degrees of the c_i's are small relative to r. Recently, a result uniform with respect to the coefficients c_i was obtained by Bombieri and Mueller [B-M]. They showed that if $r > n!(n! - 2)$ and $n \geq 3$, then solutions to (1.1) fall into at most $n!^{n!}$ families, each with explicitly given simple structure. In the case $n = 3$ and $r > 30$ the author [M] has shown that

$$c_1 z_1^r + c_2 z_2^r = c_3, \qquad c_i \in K^*, \qquad 1 \leq i \leq 3\,,$$

has at most two distinct solutions in $K^* \times K^*$, provided either $c_1/c_3 \notin (K^*)^r$ or $c_2/c_3 \notin (K^*)^r$.

The main objective of this paper is to show that the bound $n!^{n!}$ in [B-M] can be replaced by $2(n!)^{2n-1}$. This result is stated in Theorem 1. The strategy of our proof, which relies on the *abc*-inequality, follows the

Supported in part by NSF grant DMS-900-2876.

lines in [B-M], but we introduce a new idea which allows us a much more efficient counting of the number of classes of solutions. In Theorems 2 and 3, we have singled out some special cases of our Theorem 1 which are of independent interest.

I would like to thank Professor Enrico Bombieri for his valuable advice concerning the final form of this paper, I also wish to thank the referee for his helpful suggestions.

2. Statement of results. Denote by

$$F(\boldsymbol{x}) = c_1x_1 + \ldots + c_nx_n = 0\,, \qquad c_i \in K^*\,, \tag{2.1}$$

the generalized Fermat equation. By a solution of (2.1) we mean a solution $\boldsymbol{x}$ with every coordinate $x_i \in (K^*)^r$. Let $\boldsymbol{X}$ be the set of such solutions of (2.1). Let $I = \{1, \ldots, n\}$ and let $\pi : I = \bigcup R$ be a partition of I. We say $\boldsymbol{x} \in \boldsymbol{X}$ is *associated with* π if

$$F(\boldsymbol{x}) = \sum_R \sum_{j\in R} c_jx_j = 0$$

where for each $R \in \pi$,

$$F_R(\boldsymbol{x}_R) = \sum_{j\in R} c_jx_j = 0\,. \tag{2.2}$$

Define

$$\boldsymbol{X}(\pi) = \{\boldsymbol{x} \in \boldsymbol{X} \mid \boldsymbol{x} \text{ is associated with } \pi\}$$

and

$$\boldsymbol{X}_R(\pi) = \{\boldsymbol{x}_R \mid \boldsymbol{x}_R = (x_j)_{j\in R} \text{ is a solution of (2.2)}\}\,.$$

Then it is easily seen that

$$\boldsymbol{X}(\pi) = \bigcap_{R\in\pi} \boldsymbol{X}_R(\pi) \quad \text{and} \quad \boldsymbol{X} = \bigcup_{\pi} \boldsymbol{X}(\pi)\,. \tag{2.3}$$

Definition. Let $\boldsymbol{e}_R$ be a vector with each $e_j \in (K^*)^r$, $j \in R$, and let $\boldsymbol{x}_R \in \boldsymbol{X}_R(\pi)$. We say $\boldsymbol{x}_R$ is *compatible with* $\boldsymbol{e}_R$ if there is $w \in (K^*)^r$ and $v_j \in k$ such that

$$x_j = e_jv_jw, \qquad \forall j \in R\,.$$

Let us write

$$\boldsymbol{X}_R(\pi, \boldsymbol{e}_R) = \{\boldsymbol{x}_R \in \boldsymbol{X}_R(\pi) \mid \boldsymbol{x}_R \text{ is compatible with } \boldsymbol{e}_R\}$$

and

$$\boldsymbol{X}(\pi, \boldsymbol{e}) = \bigcap_{R\in\pi} \boldsymbol{X}_R(\pi, \boldsymbol{e}_R)\,.$$

We say $\boldsymbol{X}_R(\pi, \boldsymbol{e}_R)$ is a *compatible class* of solutions of (2.2), and $\boldsymbol{X}(\pi, \boldsymbol{e})$ is a compatible class of solutions of (2.1).

Our main result is the following

THEOREM 1. *Suppose $r > n!(n!-2)$. Then there are partitions π of I and vectors $\boldsymbol{e} \in (K^{*r})^n$ such that $\boldsymbol{X}$ is the union of at most $2(n!)^{2n-1}$ compatible classes $\boldsymbol{X}(\pi, \boldsymbol{e})$.*

The next theorem is a version of Theorem 1 in a special case where the coefficients c_i in (2.1) are restricted to be sums of rth power elements in K^*.

THEOREM 2. *Let $l_i, 1 \le i \le n$, be positive integers and let the coefficients c_i in (2.1) be given by*

$$c_i = \sum_{j=1}^{l_i} a_{ij}, \qquad i = 1, \ldots, n, \tag{2.4}$$

where $a_{ij} \in (K^*)^r$, $1 \le i \le n$, $1 \le j \le l_i$. Suppose $r > l(l-2)$ where

$$l = \sum_{i=1}^{n} l_i\,. \tag{2.5}$$

Then $\boldsymbol{X}$ is the union of at most $2(n!)l^{2n-2}$ compatible classes $\boldsymbol{X}(\pi, \boldsymbol{e})$.

Our final result is the following theorem which improves the condition $r > 30$ in the Main Theorem in [M] to $r > 24$.

THEOREM 3. *Let $c_i \in K^*$, $1 \le i \le 3$, such that either $c_1/c_3 \notin (K^*)^r$ or $c_2/c_3 \notin (K^*)^r$, and let $r > 24$. Then the binomial equation*

$$c_1x_1 + c_2x_2 = c_3$$

has at most two distinct solutions in $(K^*)^r \times (K^*)^r$.

3. Proof of Theorem 1.

The main tool of our method is the *abc*-inequality (Theorem B of [Br-Ma]). In Lemma 1 we state a version of that inequality which works especially well for homogeneous equations and which follows from the proof of Lemma 2 of [B-M].

LEMMA 1. *Let k and K be as before. Suppose*

$$p_1^r + \ldots + p_d^r = 0\,, \qquad p_i \in K^*\,, \tag{3.1}$$

and no proper subsum of (3.1) vanishes. If $r > d(d-2)$, then $p_i/p_j \in k$.

For a proof of Lemma 1, see the proof of Lemma 2 of [B-M].

Our first step towards proving Theorem 1 is to construct a system of "rth power" equations. That is, equations whose monomials are rth power elements in K^*. We start by ordering the elements in $\boldsymbol{X}$ so that the first m elements $\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(m)}$ in $\boldsymbol{X}$ are linearly independent over K where $m < n$

is the rank of the matrix $(\boldsymbol{x}^{(i)})_{i=1,2,\ldots}$. Let J be any subset of I of Card $J = m+1$, and let S be the set of all such subsets of I. If $\boldsymbol{x} \in \boldsymbol{X}$, then

$$\operatorname{rank}\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{x}^{(i)}\end{pmatrix}_{i=1,\ldots,m} = m\,.$$

Hence for every $J \in S$,

$$\det\begin{pmatrix}\boldsymbol{x}_J\\ \boldsymbol{x}_J^{(i)}\end{pmatrix}_{i=1,\ldots,m} = \sum \varepsilon(\sigma) x^{(1)}_{\sigma_1}\ldots x^{(m)}_{\sigma_m} x_{\sigma_{m+1}} = 0 \tag{3.2}$$

where the sum is over the set $\mathcal{M}(J)$ of permutations σ of J and $\varepsilon(\sigma) = \pm 1$ according to the parity of σ. Write

$$m_\sigma(\boldsymbol{x}_J) = \varepsilon(\sigma) x^{(1)}_{\sigma_1}\ldots x^{(m)}_{\sigma_m} x_{\sigma_{m+1}}\,, \tag{3.3}$$

and

$$L_J(\boldsymbol{x}_J) = \det\begin{pmatrix}\boldsymbol{x}_J\\ \boldsymbol{x}_J^{(i)}\end{pmatrix}_{i=1,\ldots,m\,.}$$

Then (3.3) gives a system of linear forms in $x_1, \ldots, x_n$,

$$L_J(\boldsymbol{x}_J) = \sum_{\sigma\in\mathcal{M}(J)} m_\sigma(\boldsymbol{x}_J) = 0, \qquad \forall J \in S\,. \tag{3.4}$$

Let $\boldsymbol{x} \in \boldsymbol{X}$. We say $L_J(\boldsymbol{x}_J)$ *decomposes into components* $\mathcal{N}$ if

$$L_J(\boldsymbol{x}_J) = \sum_{\mathcal{N}} \sum_{\sigma\in\mathcal{N}} m_\sigma(\boldsymbol{x}_J) = 0 \tag{3.5}$$

where $\mathcal{M}(J) = \bigcup \mathcal{N}$ is a partition and where

$$\sum_{\sigma\in\mathcal{N}} m_\sigma(\boldsymbol{x}_J) = 0 \tag{3.6}$$

is a vanishing subsum for every component $\mathcal{N}$ of $\mathcal{M}(J)$, but no proper subsum of it vanishes.

DEFINITION. Let $\boldsymbol{x}, \boldsymbol{x}' \in \boldsymbol{X}$. We say $\boldsymbol{x}$ and $\boldsymbol{x}'$ are *proportional* (i.e. $\boldsymbol{x} \sim \boldsymbol{x}'$) if $x_j/x'_j \in k$ for each j.

DEFINITION. Let $\boldsymbol{x} \in \boldsymbol{X}$. We say $\boldsymbol{x}$ is a *singular solution* if for every $J \in S$ and for every decomposition of $L_J(\boldsymbol{x}_J) = 0$, each component $\mathcal{N}$ of $\mathcal{M}(J)$ has the property that if $\sigma, \sigma' \in \mathcal{N}$, then $m_\sigma/m_{\sigma'} \in k$, where m_σ and $m_{\sigma'}$ are given by (3.3). We say m_σ and $m_{\sigma'}$ are *proportional*, and we write $m_\sigma \sim m_{\sigma'}$.

LEMMA 2. *Suppose* $r > n!(n!-2)$. *Then every* $\boldsymbol{x} \in \boldsymbol{X}$ *is a singular solution.*

Proof. Suppose $\boldsymbol{x} \in \boldsymbol{X}$ is not a singular solution. Then there is a $J \in S$, a decomposition of $L_J(\boldsymbol{x}_J) = 0$ and a component $\mathcal{N}$ of $\mathcal{M}(J)$ such that for some $\sigma, \sigma' \in \mathcal{N}$, $m_\sigma/m_{\sigma'} \notin k$.

From Lemma 1 and the fact that $\mathrm{Card}\,\mathcal{N} \le (m+1)! \le n!$, we obtain $r \le n!(n!-2)$. This proves Lemma 2.

Our immediate object is to show that each $\boldsymbol{x} \in \boldsymbol{X}$ is associated with a partition π_0 which arises in a natural way. First, we define a projection map p from $\mathbb{Z}^{m+1}$ to $\mathbb{Z}$ by

$$p(\sigma) = p(\sigma_1, \ldots, \sigma_{m+1}) = \sigma_{m+1}, \qquad \forall \sigma \in \mathcal{M}^*,$$

where

$$\mathcal{M}^* = \bigcup_J \mathcal{M}(J) = \bigcup_J \bigcup \mathcal{N}.$$

Next, for each $J \in S$, let G_J be the incidence graph of the sets $p(\mathcal{N}) \subset \mathbb{Z}$. Thus the vertices of G_J are the sets $p(\mathcal{N})$ with $\mathcal{N}$ running over all the components of the decomposition (3.5), and there is an edge connecting $p(\mathcal{N})$ to $p(\mathcal{N}')$ precisely if $p(\mathcal{N}) \cap p(\mathcal{N}') \ne \emptyset$. The graph G_J splits into the disjoint union of connected components $G_{J,\nu}$ and we define

$$\pi_0 \ : \quad I = \bigcup_{\nu=1}^{s} R_\nu \tag{3.7}$$

where $R_\nu = \bigcup p(\mathcal{N})$ with $p(\mathcal{N}) \in G_{J,\nu}$. Since $\boldsymbol{x} \in \boldsymbol{X}$, we have

$$F(\boldsymbol{x}) = \sum_{j \in I} c_j x_j = \sum_{R_\nu \in I} \sum_{j \in R_\nu} c_j x_j = 0.$$

We claim that for $\nu = 1, \ldots, s$,

$$F_{R_\nu}(\boldsymbol{x}_{R_\nu}) = \sum_{j \in R_\nu} c_j x_j = 0. \tag{3.8}$$

To see this, we remark that in Lemma 1 of [B-M] it has been shown that $F(\boldsymbol{x})$ is a linear combination of the $L_J(\boldsymbol{x})$; that is, there exist $\lambda_J \in k$ such that

$$F(\boldsymbol{x}) = \sum_J \lambda_J L_J(\boldsymbol{x}_J) = \sum_J \lambda_J \sum_{\mathcal{N}} \sum_{\sigma \in \mathcal{N}} m_\sigma(\boldsymbol{x}_J).$$

Therefore

$$c_j = \sum_J \lambda_J \sum_{\substack{\mathcal{N} \subset \mathcal{M}(J) \\ j \in p(\mathcal{N})}} \sum_{\substack{\sigma \in \mathcal{N} \\ p(\sigma) = j}} \varepsilon(\sigma) x^{(1)}_{\sigma_1} \ldots x^{(m)}_{\sigma_m}.$$

Since the $p(\mathcal{N})$'s involved in the middle sum all belong to the same component, say $G_{J,\nu}$, we then have

$$F_{R_\nu}(\boldsymbol{x}_{R_\nu}) = \sum_J \lambda_J \sum_{\substack{\mathcal{N} \subset \mathcal{M}(J) \\ p(\mathcal{N}) \in G_{J,\nu}}} \sum_{\sigma \in \mathcal{N}} m_J(\boldsymbol{x}_J). \tag{3.9}$$

Now (3.9) in conjunction with (3.6) yield (3.8) and our claim is proved. We remark that $\boldsymbol{x}$ is a singular solution if and only if $\boldsymbol{x}_{R_\nu}$ is a singular solution for each ν.

A crucial idea in the proof of Theorem 1 is to show that any singular solution is compatible with a finite number of vectors which are determined by the monomials m_σ, $\sigma \in \mathcal{M}^*$. Define, for each $\sigma \in \mathcal{M}^*$,

$$\tau(\sigma) = (\sigma_1, \ldots, \sigma_m), \qquad a_{\tau(\sigma)} = x_{\sigma_1}^{(1)} \ldots x_{\sigma_m}^{(m)} ,$$

and

$$E = \{a_{\tau(\sigma)}/a_{\tau(\sigma')} \mid \sigma, \sigma' \in \mathcal{M}^*\} . \tag{3.10}$$

Then $m_\sigma = a_{\tau(\sigma)} x_{p(\sigma)}$, and the set E is what we want. One sees easily that

$$\operatorname{Card} E \le \left(m! \binom{n}{m} \right)^2 \le n!^2 .$$

Proof of Theorem 1. Let π_0 and R_ν be given by (3.7) and let $b_\nu = \operatorname{Card} R_\nu$. Our first object is to show that $\boldsymbol{X}_{R_\nu}(\pi_0)$ is the union of at most

$$(b_\nu - 1)!(b_\nu!)^{2(b_\nu - 1)} \tag{3.11}$$

compatible classes $\boldsymbol{X}_{R_\nu}(\pi_0, \boldsymbol{e}_{R_\nu})$.

For simplicity we shall set $R_\nu = I$ and adjust our notations accordingly in what follows. For example, we write $I = \bigcup p(\mathcal{N}_\alpha)$, where the $p(\mathcal{N}_\alpha)$'s are given by (3.7), and we let $\boldsymbol{X}(\pi_0)$ stand for $\boldsymbol{X}_{R_\nu}(\pi_0)$ and $\boldsymbol{X}(\pi_0, \boldsymbol{e})$ stand for $\boldsymbol{X}_{R_\nu}(\pi_0, \boldsymbol{e}_{R_\nu})$.

We remark that one may pick $\boldsymbol{x} \in \boldsymbol{X}(\pi_0)$ such that $x_1 = 1$. To see this, we write $\boldsymbol{x} = x_1 x_1^{-1} \boldsymbol{x} = x_1(1, \ldots, x_1^{-1} x_n)$, $x_1 \in (K^*)^r$. Then $\boldsymbol{x}$ is compatible with $(1, \ldots, x_1^{-1} x_n)$. We will now construct a sequence of subsets $T_\alpha \subset p(\mathcal{N}_\alpha)$ with the properties

(i) every T_α is connected with some $T_{\alpha'}$, $\alpha' < \alpha$,

(ii) suppose T_α and $T_{\alpha'}$, $\alpha' < \alpha$, are connected; then they have exactly one element in common,

(iii) $I = \bigcup T_\alpha$.

We start by setting $T_1 = p(\mathcal{N}_1)$ where $\mathcal{N}_1$ may be chosen so that $\delta(1) = 1 \in p(\mathcal{N}_1)$. To define T_2, we pick $p(\mathcal{N}_2) \not\subset T_1$ and $p(\mathcal{N}_1) \cap p(\mathcal{N}_2) \neq \emptyset$. Let $\delta(2)$ be the least integer in the set $\{\beta \mid \beta \in p(\mathcal{N}_2) \cap p(\mathcal{N}_1) \neq \emptyset\}$, and let

$$T_2 = \{\delta(2)\} \cup \{\beta \in p(\mathcal{N}_2) \mid \beta \notin T_1\} .$$

Now suppose for some integer $q \ge 3$, $T_1, \ldots, T_{q-1}$ have been defined by this procedure, where $T_{\alpha+1} \not\subset T_\alpha$ and $\bigcup_{\alpha=1}^{q-1} T_\alpha \subsetneq I$. To construct T_q, we first pick $p(\mathcal{N}_q) \not\subset \bigcup_{\alpha=1}^{q-1} T_\alpha$ such that

$$p(\mathcal{N}_q) \cap \bigcup_{\alpha=1}^{q-1} T_\alpha \neq \emptyset \tag{3.12}$$

and let $\delta(q)$ be the least integer in the non-empty set (3.12). Set

$$T_q = \{\delta(q)\} \cup \Big\{\beta \in p(\mathcal{N}_q) \mid \beta \notin \bigcup_{\alpha=1}^{q-1} T_\alpha \Big\}.$$

Then there is a positive integer k such that

$$I = \bigcup_{\alpha=1}^{k} T_\alpha \,. \tag{3.13}$$

Clearly the sets T_α in (3.13) satisfy the above stated properties (i)–(iii).

Next, let $1 \le j \le n$ be such that $j \in T_1$. Since $1 \in T_1$, there are permutations $\sigma^{(1,j)}$ and $\sigma^{(\delta(1))}$ in $\mathcal{N}_1$ such that

$$p(\sigma^{(1,j)}) = j \quad \text{and} \quad p(\sigma^{(\delta(1))}) = 1\,.$$

Moreover, since $\boldsymbol{x}$ is singular, we have

$$m_{\sigma^{(1,j)}} \sim m_{\sigma^{(\delta(1))}}\,.$$

Let $\tau(\cdot)$ stand for $\tau(\sigma^{(\cdot)})$, and let

$$m_{\sigma^{(1,j)}} = a_{\tau(1,j)} x_j \quad \text{and} \quad m_{\sigma^{(\delta(1))}} = a_{\tau(\delta(1))} x_1\,.$$

Then we get, since $x_1 = 1$,

$$x_j \sim e(1,j) x_1 = e(1,j)\,, \tag{3.14}$$

where

$$e(1,j) = a_{\tau(\delta(1))}/a_{\tau(1,j)} \in E \quad \text{and} \quad j \in T_1\,.$$

From (3.14) and the fact that the cardinality of E is at most $n!^2$, we see that the number of proportional classes of x_j, $j \in T_1$, is at most $n!^2$. Next, suppose that the proportional classes of x_j, $j \in T_\alpha$, $1 \le \alpha \le k-1$, have been determined and suppose $j \in T_k$. Then since $\delta(k) \in T_k$, there are permutations $\sigma^{(k,j)}$ and $\sigma^{(\delta(k))}$ in $\mathcal{N}_k$ such that

$$p(\sigma^{(k,j)}) = j, \quad p(\sigma^{(\delta(k))}) = \delta(k), \quad j \in T_k\,.$$

Writing

$$m_{\sigma^{(k,j)}} = a_{\tau(k,j)} x_j \quad \text{and} \quad m_{\sigma^{(\delta(k))}} = a_{\tau(\delta(k))} x_{\delta(k)}\,,$$

from $m_{\sigma^{(k,j)}} \sim m_{\sigma^{(\delta(k))}}$ we get

$$x_j \sim e(\delta(k),j) x_{\delta(k)}\,, \tag{3.15}$$

where

$$e(\delta(k),j) = a_{\tau(\delta(k))}/a_{\tau(k,j)} \in E, \quad j \in T_k,$$

and where $\delta(k) \in T_k \cap T_j$ for some $1 \le j \le k-1$. Since by the hypothesis, the proportional classes of $x_{\delta(k)}$ have already been determined, each $e(\delta(k), j)$ in (3.15) then determines a proportional class of x_j, $j \in T_k$. The number of such proportional classes is at most $n!^2$. It follows that there is a permutation σ of $\{1, \ldots, n\}$ such that $\sigma(1) = 1$ and such that $x^{(\sigma)} = (1, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ may fall into not more than $(n!)^{2(n-1)}$ proportional classes determined by vectors $\boldsymbol{e}$ with each coordinate $e_i \in E$. Since the number of permutations σ is at most $(n-1)!$, we get at most $(n-1)!n!^{2(n-1)}$ proportional classes for any $\boldsymbol{x} \in \boldsymbol{X}(\pi_0)$ such that $x_1 = 1$. Hence we have shown that $\boldsymbol{X}(\pi_0)$ is the union of at most $(n-1)!n!^{2(n-1)}$ compatible classes $\boldsymbol{X}(\pi_0, \boldsymbol{e})$. Since we have set $R_\nu = I$ in the above arguments, it is now clear that we have proved (3.11).

Finally, from (2.3) and (3.11) we deduce that $\boldsymbol{X}$ is a union of at most μ compatible classes, where

$$
\begin{aligned}
(3.16)\qquad \mu &\le \sum_{s=1}^{n} \sum_{\substack{b_1+\ldots+b_s=n \\ b_\nu \ge 1}} \Big(\prod_{\nu=1}^{s} n!^{2b_\nu - 2}(b_\nu - 1)! \Big) \frac{n!}{\prod_{\nu=1}^{s} b_\nu!} \\
&< \sum_{s=1}^{n} \sum_{\substack{b_1+\ldots+b_s=n \\ b_\nu \ge 1}} (n!)^{2n-2s+1} < (n!)^{2n-1} \sum_{s=0}^{n-1} \binom{n-1}{s-1} (n!)^{-2s} \\
&< (n!)^{2n-1} \Big[1 + \sum_{s=1}^{n-1} n^{s-1} (n!)^{-2s} \Big] \\
&< 2(n!)^{2n-1}, \quad n \ge 3\,.
\end{aligned}
$$

This completes the proof of Theorem 1.

4. Proof of Theorem 2. Let a_{ij} and l_i be given by (2.4) and (2.5), and let

$$\mathcal{M}_0 = \{(i,j) \mid 1 \le j \le l_i\}\,.$$

Define $p : \mathbb{Z}^2 \to \mathbb{Z}$ by $p(i,j) = i$, and

$$m_{(i,j)} = a_{ij} x_i\,, \qquad \forall (i,j) \in \mathcal{M}_0\,.$$

Then (2.1) is a rth power equation

$$(4.1)\qquad F(\boldsymbol{x}) = \sum_{i=1}^{n} \Big(\sum_{j=1}^{l_i} a_{ij} \Big) x_i = \sum_{\mathcal{M}_0} m_{(i,j)} = \sum_{\mathcal{N} \subset \mathcal{M}_0} \sum_{(i,j) \in \mathcal{N}} m_{(i,j)} = 0\,,$$

where $\mathcal{M}_0 = \bigcup \mathcal{N}$. Since Theorem 2 is a version of Theorem 1 in a special case we shall use the results in Section 3 to prove Theorem 2 with minor changes. First, we replace E in (3.10) by

$$E_0 = \{a_{ij}/a_{i'j'} \mid a_{ij}, a_{i'j'} \text{ are given by (2.4)}\}\,. \tag{4.2}$$

Then it is easily seen that the cardinality of E_0 is at most l^2. Next, we replace the bound in (3.16) by

$$\sum_{s=1}^{n} \sum_{\substack{b_1+\ldots+b_s=n \\ b_\nu \ge 1}} \Big(\prod_{\nu=1}^{s} l^{2b_\nu-2}(b_\nu-1)!\Big) \frac{n!}{\prod_{\nu=1}^{s} b_\nu!}\,. \tag{4.3}$$

But (4.3) is

$$\begin{aligned} &< (n!)l^{2n-2} \sum_{s=0}^{n-1} \binom{n-1}{s-1} (l!)^{-2s} \\ &< (n!)l^{2n-2}\Big[1 + \sum_{s=1}^{n-1} n^{s-1}(n!)^{-2s}\Big] \le 2(n!)l^{2n-2}\,, \end{aligned}$$

where $l \ge n \ge 3$. This proves Theorem 2.

We remark that for the Euler–Fermat equation

$$x_1 + \ldots + x_n = 0 \tag{4.4}$$

where $r > n(n-2)$, the number of compatible classes of solutions of (4.4) is bounded by the number of partitions of I and the latter is at most n^n.

5. Proof of Theorem 3. Theorem 3 is included here mainly as an example of how the technique of proof of Theorem 1 can be used in practice. Although our proof of Theorem 3 follows the general lines in [M], the arguments have been simplified a great deal. In fact, it is easily seen that Theorem 3 is an immediate consequence of the following lemmas.

LEMMA 3. *Suppose $c_i \in K^*$, $1 \le i \le 3$, such that either $c_1/c_3 \notin (K^*)^r$ or $c_2/c_3 \notin (K^*)^r$. Let $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ be two distinct solutions of*

$$c_1x + c_2y + c_3 = 0\,. \tag{5.1}$$

Then $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are non-proportional (*i.e. either $x_1/x_2 \notin k$ or $y_1/y_2 \notin k$*).

LEMMA 4. *Suppose $r > 24$. Then any three distinct solutions of* (5.1) *are mutually proportional.*

It follows from Lemmas 3 and 4 that (5.1) cannot have three distinct solutions. Therefore Theorem 3 is proved.

Proof of Lemma 3. Suppose $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are distinct solutions of (5.1) such that $\boldsymbol{x}^{(1)}$ and $\boldsymbol{x}^{(2)}$ are proportional (i.e. $\boldsymbol{x}^{(1)} \sim \boldsymbol{x}^{(2)}$). Writing

$\boldsymbol{x}^{(i)} = (x_i, y_i, 1)$ with x_i, $y_i \in (K^*)^r$, there are constants α and β in k such that

$$x_2 = \alpha x_1 \quad \text{and} \quad y_2 = \beta y_1 . \tag{5.2}$$

From (5.1) and (5.2) we get $c_1(1-\alpha)x_1 + c_2(1-\beta)y_1 = 0$, which gives

$$\frac{c_1}{c_2} = \frac{\beta - 1}{1-\alpha}\frac{y_1}{x_1} \in (K^*)^r .$$

Also from (5.1) and (5.2) we get

$$c_1\left(1 - \frac{\alpha}{\beta}\right)\frac{x_1}{y_1} + c_3\left(1 - \frac{1}{\beta}\right)\frac{1}{y_1} = 0 ,$$

which gives

$$\frac{c_1}{c_3} = \left(\frac{\beta^{-1} - 1}{1 - \alpha\beta^{-1}}\right)\frac{1}{x_1} \in (K^*)^r .$$

But this contradicts the hypothesis of Lemma 3. Thus Lemma 3 is proved.

Proof of Lemma 4. We remark first that the hypothesis $r > 24$ implies that every solution of (5.1) is singular (see Lemma 2). Let $\boldsymbol{x}^{(1)}$, $\boldsymbol{x}^{(2)}$ and $\boldsymbol{x}^{(3)}$ be three distinct solutions of (5.1). Since the rank of the matrix $(\boldsymbol{x}^{(i)})_{i=1,2,3}$ is at most 2, we have

$$\det(\boldsymbol{x}^{(i)})_{i=1,2,3} = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 .$$

By expanding the determinant in full, we get

$$L = x_1y_2 + x_2y_3 + x_3y_1 - x_1y_3 - x_2y_1 - x_3y_2 = 0 .$$

In what follows, we will proceed to show that each of the following cases is either impossible or it leads to three mutually proportional solutions.

Case (i): L has no proper subsum that vanishes. Then, since every solution of (5.1) is singular, the monomials in L are mutually proportional. From the following proportional monomials:

$$x_1y_2 \sim x_3y_2,\ x_2y_3 \sim x_1y_3,\ x_2y_3 \sim x_2y_1 \quad \text{and} \quad x_3y_2 \sim x_3y_1 ,$$

we get

$$\boldsymbol{x}^{(1)} \sim \boldsymbol{x}^{(2)} \sim \boldsymbol{x}^{(3)} . \tag{5.3}$$

Case (ii): L decomposes into three components c_i, $1 \le i \le 3$, of two monomials each. Writing $c_i = u + v = 0$, we claim that u and v must be monomials with the same sign. Suppose $u = x_1y_2$ and $v = -x_1y_3$ or $v = -x_3y_2$. Then from $u + v = 0$ we get $y_2 = y_3$ or $x_1 = x_3$, which together with (5.1) gives two equal solutions. If $v = -x_2y_1$, then $x_1/y_1 = x_2/y_2$, which together with (5.1) again yields two equal solutions. Thus our claim is proved. But this implies that the number of positive and negative monomials in (5.1) must be even, which is not the case. Therefore, case (ii) is impossible.

Case (iii): L decomposes into two components c_1 and c_2 of four and two monomials respectively. Then one sees from case (ii) that the two monomials in c_2 must have the same sign. Up to sign we may represent c_1 and c_2 by

$$\begin{aligned} c_1 &= x_i y_j + x_j y_k + x_k y_i - x_i y_k = 0, \\ c_2 &= -x_j y_i - x_k y_j = 0\,, \end{aligned} \tag{5.4}$$

where (i, j, k) is a permutation of $(1, 2, 3)$. From both equations in (5.4) we get

$$y_j \sim y_k, \quad x_j \sim x_i, \quad \frac{x_j}{x_k} \sim \frac{y_i}{y_k} \quad \text{and} \quad \frac{x_j}{x_k} = -\frac{y_j}{y_i}\,,$$

which yields $x_i \sim x_j$, $y_i^2 \sim y_j^2$ and hence

$$x_i \sim x_j \qquad \text{and} \qquad y_i \sim y_j\,. \tag{5.5}$$

The proportionality relation $y_i \sim y_j$ is obtained from $y_i^2 \sim y_j^2$ and the fact that the constant field k is algebraically closed. Combining (5.4) and (5.5) we get $x_i \sim x_j \sim x_k$ and $y_i \sim y_j \sim y_k$, which again gives (5.3).

Case (iv): L decomposes into two components c_1 and c_2 of three monomials each. This is the last and also the most complex of the four cases. Since each monomial x_i or y_i may appear at most twice in a component, it suffices for us to consider components such that in one of them, say c_1, one of the following four cases holds: (a) both x_i and y_j appear twice, (b1) x_i appears twice but no y_i appears twice, (b2) y_i appears twice but no x_i appears twice, (c) both x_i and y_i appear exactly once. To be more explicit, we have:

$$\begin{aligned} c_1 &= x_i y_j - x_i y_k - x_k y_j = 0 \\ c_2 &= x_j y_k - x_j y_i + x_k y_i = 0 \end{aligned} \qquad \text{(case (a))} \tag{5.6}$$

$$\begin{aligned} c_1 &= x_i y_j - x_i y_k + x_k y_i = 0 \\ c_2 &= x_j y_k - x_j y_i - x_k y_j = 0 \end{aligned} \qquad \text{(case (b1))} \tag{5.7}$$

$$\begin{aligned} c_1 &= x_i y_j - x_j y_i - x_k y_i = 0 \\ c_2 &= -x_i y_k + x_j y_k + x_k y_i = 0 \end{aligned} \qquad \text{(case (b2))} \tag{5.8}$$

$$\begin{aligned} c_1 &= x_i y_j + x_j y_k + x_k y_i = 0 \\ c_2 &= -x_i y_k - x_j y_i - x_k y_j = 0 \end{aligned} \qquad \text{(case (c))} \tag{5.9}$$

From both the first and the second equations in (5.6) we get $y_j \sim y_k$, $x_i \sim x_k$, $y_k \sim y_i$, and $x_j \sim x_k$, which clearly yields (5.3). From both the first and the second equations in (5.7) we get $y_j \sim y_k \sim y_i$, which then gives

$x_i \sim x_k \sim x_j$ and hence (5.3). Similarly, we may obtain (5.3) from (5.8). Finally, from the first equation in (5.9) we get

$$y_k \sim \frac{x_i y_j}{x_j} \quad \text{and} \quad \frac{y_i}{y_j} \sim \frac{x_i}{x_k}\,, \tag{5.10}$$

and from the second equation in (5.9) we get

$$y_k \sim \frac{x_j y_i}{x_i} \quad \text{and} \quad \frac{y_i}{y_j} \sim \frac{x_k}{x_j}\,. \tag{5.11}$$

It is easily seen that (5.10) and (5.11) together give

$$\frac{y_i}{y_j} \sim \frac{x_i^2}{x_j^2} \quad \text{and} \quad \frac{y_i^2}{y_j^2} \sim \frac{x_i}{x_j}\,,$$

which yields $x_i^3 \sim x_j^3$ and $y_i^3 \sim y_j^3$. Since k is algebraically closed, we get

$$x_i \sim x_j \quad \text{and} \quad y_i \sim y_j\,. \tag{5.12}$$

Now, (5.3) follows from (5.9) and (5.12).

This completes the proof of Lemma 4. Thus Theorem 3 is proved.

The idea of the proof of Lemma 4 was inspired by the article [E-G-S-T].

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Received on 13.9.1991
and in revised form on 25.9.1992 (2173)