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Power moments of the error term in the approximate functional equation for $\zeta^2(s)$

by

Aleksandar Ivić (Beograd)

Let as usual $s=\sigma+it$ be a complex variable, $d(n)$ the number of divisors of n, and $\zeta(s)$ the Riemann zeta-function. One may consider (see e.g. Th. 4.2 of [3])

$$R\left(s;\frac{t}{2\pi}\right):=\zeta^2(s)-\sum_{n\le t/(2\pi)}{}^{'}\, d(n)n^{-s}-\chi^2(s)\sum_{n\le t/(2\pi)}{}^{'}\, d(n)n^{s-1} \qquad (0\le\sigma\le 1)$$

as the error term in the approximate functional equation for $\zeta^2(s)$, where

$$\chi(s)=\zeta(s)/\zeta(1-s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\,.$$

In his important works [10], [11] Y. Motohashi established a very precise formula for the function $R(s;t/(2\pi))$, which connects it with

$$\Delta(x):=\sum_{n\le x}{}^{'}\, d(n)-x(\log x+2\gamma-1)-1/4\,,$$

the error term in the classical divisor problem. Here γ is Euler's constant, and in general $\sum'_{n\le y}$ denotes that the last term in the sum is to be halved if y is an integer. In particular, Motohashi has shown that

$$\chi(1-s)R\left(s;\frac{t}{2\pi}\right)=-\sqrt{2}\left(\frac{t}{2\pi}\right)^{-1/2}\Delta\left(\frac{t}{2\pi}\right)+O(t^{-1/4})\,. \tag{1}$$

By using (1) and the author's bounds (see [2] or Ch. 13 of [3])

$$\int_1^T|\Delta(x)|^A\,dx\ll\begin{cases}T^{1+A/4+\varepsilon}, & 0\le A\le 35/4,\\ T^{19/54+35A/108+\varepsilon}, & A\ge 35/4,\end{cases} \tag{2}$$

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I. Kiuchi [7] obtained the bounds

$$\int_1^T \left|R\left(\frac{1}{2}+it;\frac{t}{2\pi}\right)\right|^A dt \ll \begin{cases} T^{1-A/4+\varepsilon}, & 0\le A\le 4,\\ 1, & A\ge 4.\end{cases} \tag{3}$$

Here, as usual, both $f(x)=O(g(x))$ and $f(x)\ll g(x)$ mean that $|f(x)|\le Cg(x)$ for $x\ge x_0$, $g(x)>0$ and some $C>0$. In the special case $A=2$ a precise result may be obtained. Kiuchi and Matsumoto [8] obtained

$$\int_1^T \left|R\left(\frac{1}{2}+it;\frac{t}{2\pi}\right)\right|^2 dt = \sqrt{2\pi}\Big\{\sum_{n=1}^\infty d^2(n)h^2(n)n^{-1/2}\Big\}T^{1/2}+F(T) \tag{4}$$

with $F(T)=O(T^{1/4}\log T)$, and I. Kiuchi improved this in [6] to $F(T)=O(\log^5 T)$. In (4) the function $h(n)$ is defined as

$$h(n)=\left(\frac{2}{\pi}\right)^{1/2}\int_0^\infty (y+n\pi)^{-1/2}\cos(y+\pi/4)\,dy\,. \tag{5}$$

Two integrations by parts give

$$h(n)=\left(\frac{2}{\pi}\right)^{1/2}\Big\{-(2\pi n)^{-1/2}+(2\pi n)^{-3/2} -\frac{3}{4}\int_0^\infty (y+n\pi)^{-5/2}\cos(y+\pi/4)\,dy\Big\},$$

which easily yields

$$h(n)=-\frac{n^{-1/2}}{\pi}+\frac{n^{-3/2}}{2\pi^2}+O(n^{-5/2}),\qquad h(n)<0\qquad (n\in\mathbb{N}), \tag{6}$$

so that the series in (4) converges, since $d(n)\ll n^\varepsilon$ for any $\varepsilon>0$.

The object of this note is to improve (3), and at the same time to indicate how a simple proof of (4) with $F(T)=O(\log^5 T)$ may be obtained. The results are contained in the following

THEOREM. *Let $A\ge 0$ be a given constant. For $0\le A<4$ there exists a positive constant $C(A)$ such that*

$$\int_1^T \left|R\left(\frac{1}{2}+it;\frac{t}{2\pi}\right)\right|^A dt \sim C(A)T^{1-A/4}\qquad (T\to\infty). \tag{7}$$

Moreover, there exist effectively computable constants $B>0$ and D such that, for any $\varepsilon>0$,

$$\int_1^T \left|R\left(\frac{1}{2}+it;\frac{t}{2\pi}\right)\right|^4 dt = B\log T+D+O(T^{\varepsilon-1/23}), \tag{8}$$

and for $A > 4$

$$\int\limits_1^T \left|R\left(\frac{1}{2}+it;\frac{t}{2\pi}\right)\right|^A dt = D(A) + O(E(T,A))\,, \tag{9}$$

where

$$D(A) = \int\limits_1^\infty \left|R\left(\frac{1}{2}+it;\frac{t}{2\pi}\right)\right|^A dt \tag{10}$$

is finite and positive, *and*

$$E(T,A) = \begin{cases} T^{1-A/4}, & 4 < A < 28/3, \\ T^{(4-2A)/11+\varepsilon}, & A \ge 28/3. \end{cases} \tag{11}$$

P r o o f. We begin with the case $A > 4$, which is not difficult to settle. Instead of (2) we may use the bounds

$$\int\limits_1^T |\Delta(x)|^A\,dx \ll \begin{cases} T^{1+A/4+\varepsilon}, & 0 \le A \le 28/3, \\ T^{1+7(A-2)/22+\varepsilon}, & A \ge 28/3. \end{cases}$$

This result is obtained in the same way as (2) was obtained, only instead of $\Delta(x) \ll x^{35/108+\varepsilon}$ one uses the sharper estimate $\Delta(x) \ll x^{7/22+\varepsilon}$ of Iwaniec and Mozzochi [5], e.g. in (13.71) of [3] and in the estimate that follows it. Moreover, from the proof of D. R. Heath-Brown [1] one obtains then

$$\int\limits_1^T |\Delta(x)|^A\,dx \ll \begin{cases} T^{1+A/4}, & 0 \le A < 28/3, \\ T^{1+7(A-2)/22+\varepsilon}, & A \ge 28/3, \end{cases} \tag{12}$$

and in the bound for $A \ge 28/3$ one could actually replace T^ε by a suitable power of the logarithm. Since $|\chi(1/2 \pm it)| = 1$, it follows from (1) and (12) that

$$\int\limits_Y^{2Y} \left|R\left(\frac{1}{2}+it;\frac{t}{2\pi}\right)\right|^A dt \ll Y^{1-A/4} + Y^{-A/2}\int\limits_Y^{2Y} |\Delta(y)|^A\,dy \ll E(Y,A)\,.$$

This easily yields (9), since both exponents of T in the definition (11) of $E(T,A)$ are negative for $A > 4$.

To obtain the remaining results of the Theorem it is necessary to use the classical Voronoï formula for $\Delta(x)$ (see Ch. 3 of [3]), namely

$$\Delta(x) = (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n=1}^\infty d(n)n^{-3/4}\cos(4\pi\sqrt{xn}-\pi/4) + O(x^{-1/4})\,, \tag{13}$$

which in truncated form may be written as

$$\begin{aligned} \Delta(x) = (\pi\sqrt{2})^{-1}x^{1/4}\sum_{n\le N} d(n)n^{-3/4}\cos(4\pi\sqrt{xn}-\pi/4) \\ + O(x^\varepsilon + x^{1/2+\varepsilon}N^{-1/2}) \end{aligned} \tag{14}$$

for any given $\varepsilon > 0$, and $1 \le N \le x^C$, where $C > 0$ is any fixed number. The key idea, suggested by (1), is to make the connection between the functions $R(\cdot)$ and $\Delta(\cdot)$ in such a way that the appropriate analogues of (13) and (14) may by obtained for $R(\cdot)$. The relation (1) is too weak for this purpose, and we shall use the following formula which follows from Motohashi's work (e.g. pp. 74–75 of [11]):

$$\begin{aligned}(15)\quad \chi\Big(\frac12 - it\Big) R\Big(\frac12 + it; \frac{t}{2\pi}\Big) &= -\sqrt{2}\Big(\frac{t}{2\pi}\Big)^{-1/2}\Delta\Big(\frac{t}{2\pi}\Big) \\ &\quad + (\pi\sqrt{2})^{-1}\Big(\frac{t}{2\pi}\Big)^{-1/2}\Big(\frac16\log\Big(\frac{t}{2\pi}\Big) + \frac{\gamma}{3} + 1\Big) \\ &\quad + (2\pi)^{-1/2}\Big(\frac{t}{2\pi}\Big)^{-1/4}\sum_{n=1}^{\infty} d(n) n^{-1/4} h_1(n)\cos(2\sqrt{2\pi t n} - \pi/4) \\ &\quad + O(t^{-3/4}),\end{aligned}$$

where

$$h_1(n) := \int_0^{\infty} (y + n\pi)^{-3/2}\cos(y - \pi/4)\,dy \ll n^{-3/2}.$$

Now we define

$$(16)\qquad g(t) := t^{1/2}\chi\Big(\frac12 - it\Big) R\Big(\frac12 + it; \frac{t}{2\pi}\Big),$$

so that $g(t)$ is real for t real, and

$$(17)\qquad |g(t)| = t^{1/2}\Big|R\Big(\frac12 + it; \frac{t}{2\pi}\Big)\Big|.$$

Noting that an integration by parts gives

$$h_1(n) = \Big(\frac{2}{\pi n}\Big)^{1/2} + (2\pi)^{1/2} h(n),$$

where $h(n)$ is given by (5), we deduce from (13) and (15) that

$$\begin{aligned}(18)\quad g(t) - (6\sqrt{\pi})^{-1}\Big(\log\frac{t}{2\pi} + 2\gamma + 6\Big) &= (2\pi t)^{1/4}\sum_{n=1}^{\infty} d(n) h(n) n^{-1/4}\cos(2\sqrt{2\pi t n} - \pi/4) + O(t^{-1/4}).\end{aligned}$$

On the other hand, by using (14) and the fact that

$$\sum_{n>N} d(n)n^{-1/4}h_1(n)\cos(2\sqrt{2\pi tn}-\pi/4) \ll N^{-3/4}\log N\,,$$

we obtain from (15), for $1 \le N \le t^C$,

$$\begin{aligned} g(t) - (6\sqrt{\pi})^{-1}\left(\log\frac{t}{2\pi} + 2\gamma + 6\right) &= (2\pi t)^{1/4}\sum_{n\le N} d(n)h(n)n^{-1/4}\cos(2\sqrt{2\pi tn}-\pi/4) \\ &\quad + O(t^{\varepsilon} + t^{1/2+\varepsilon}N^{-1/2})\,. \end{aligned} \tag{19}$$

If we now set

$$G(t) := g(t) - (6\sqrt{\pi})^{-1}\left(\log\frac{t}{2\pi} + 2\gamma + 6\right), \tag{20}$$

then the analogy between $\Delta(x)$ and $G(t)$ is indeed striking: (13) corresponds to (18) and (14) to (19), only the scaling factors are different and $n^{-3/4}$ is replaced by

$$n^{-1/4}h(n) \sim -\pi^{-1}n^{-3/4}\,.$$

Thus essentially *the results on $\Delta(x)$ based only on the use of* (13) *and* (14) *have their counterparts for $G(t)$*, and through the use of (16) and (20) one can then obtain the corresponding results for $R(1/2+it;t/(2\pi))$. To stress our point, note that the result

$$\Delta(x) - \Delta(y) \ll (x+y)^{\varepsilon}(|x-y|+1) \qquad (x,y\ge 1)\,,$$

which follows trivially from $d(n) \ll n^{\varepsilon}$ and the definition of $\Delta(x)$, does not seem obtainable by (13) or (14). Thus we cannot infer automatically the corresponding bound

$$G(x) - G(y) \ll (x+y)^{\varepsilon}(|x-y|+1) \qquad (x,y\ge 1)$$

for $G(t)$ (or $g(t)$). Indeed, it is not obvious how the last bound can be proved.

After the above discussion it is easy to see why (4) holds with $F(T) = O(\log^5 T)$. Namely T. Meurman [9] proved

$$\int_1^X \Delta^2(x)\,dx = \frac{\zeta^4(3/2)}{6\pi^2\zeta(3)}X^{3/2} + R(X) \tag{21}$$

with $R(X) = O(X\log^5 X)$. This was obtained much earlier by K.-C. Tong [13], but Meurman's method is substantially simpler than Tong's. E. Preissmann [12] indicated how at one place in the proof a variant of Hilbert's inequality may be used to save a further log-power, so that now even $R(X) = O(X\log^4 X)$ is known. Since the works of Meurman and Preissmann use only (13) and (14), it follows that the analogue of (21) may be obtained for $G(t)$, and this will be

$$\int_1^T G^2(t)\,dt = A_1 T^{3/2} + R_1(T)\,, \qquad R_1(T) = O(T\log^4 T)\,, \tag{22}$$

with the value

$$A_1 = \frac{\sqrt{2\pi}}{3}\sum_{n=1}^\infty d^2(n)h^2(n)n^{-1/2}\,.$$

From (20) and (22) one obtains

$$\begin{aligned}\int_1^T g^2(t)\,dt = &\int_1^T G^2(t)\,dt + (3\sqrt{\pi})^{-1}\int_1^T G(t)\Big(\log\frac{t}{2\pi} + 2\gamma + 6\Big)\,dt\\ &+ (36\pi)^{-1}\int_1^T \Big(\log\frac{t}{2\pi} + 2\gamma + 6\Big)^2\,dt\\ &= A_1 T^{3/2} + R_2(T)\,,\end{aligned}$$

say, where

$$R_2(T) = R_1(T) + O(T^{3/4}\log T) + (36\pi)^{-1}\int_1^T \Big(\log\frac{t}{2\pi} + 2\gamma + 6\Big)^2\,dt\,, \tag{23}$$

since by the first derivative test (Lemma 2.1 of [3]) one easily finds that

$$H(T) := \int_1^T G(t)\Big(\log\frac{t}{2\pi} + 2\gamma + 6\Big)\,dt \ll T^{3/4}\log T\,. \tag{24}$$

Thus we have $R_2(T) = O(T\log^4 T)$, so that (17) and integration by parts give

$$\int_1^T \Big|R\Big(\frac12 + it; \frac{t}{2\pi}\Big)\Big|^2\,dt = 3A_1T^{1/2} + F(T)$$

with

$$F(T) = R_2(T)T^{-1} - R_2(1) - 3A_1 + \int_1^T R_2(t)t^{-2}\,dt\,. \tag{25}$$

Hence the bound $R_2(T) = O(T\log^4 T)$ gives immediately (4) with $F(T) = O(\log^5 T)$, obtained by I. Kiuchi [6], whose proof is much more involved, as it reproduces the details of the method of Meurman and Preissmann. Note also that if

$$R_1(T) = o(T\log^2 T) \qquad (T\to\infty)$$

could be proved, then from (23) and (25) it would follow that

$$F(T) = \left(\frac{1}{108\pi} + o(1)\right)\log^3 T \qquad (T \to \infty)\,.$$

This would mean the appearance of a new main term in (4), and a similar observation was made by Kiuchi [6]. It may also be remarked that by the method of [4] it follows that there exist constants $B_1, B_2 > 0$ such that for $T \ge T_0$ every interval $[T, T + B_1T^{1/2}]$ contains points t_1, t_2 such that

$$H(t_1) > B_2 t_1^{3/4}\log t_1\,, \qquad H(t_2) < -B_2 t_2^{3/4}\log t_2\,,$$

where $H(t)$ is defined by (24), and a sharp mean square formula for $H(t)$ may be also derived. This observation coupled with the bound in (24) prompts one to state the optimistic conjecture that

$$\int_1^T \left|R\left(\frac{1}{2} + it; \frac{t}{2\pi}\right)\right|^2 dt = 3A_1T^{1/2} + a_0\log^3 T + a_1\log^2 T + a_2\log T + a_3 + O(T^{\varepsilon - 1/4}) \tag{26}$$

holds with any $\varepsilon > 0$, and effectively computable constants a_0, a_1, a_2 and a_3.

We return now to the proof of the Theorem. K.-M. Tsang [14] recently proved

$$\int_1^X \Delta^4(x)\,dx = 3c_2(64\pi^4)^{-1}X^2 + O(X^{45/23+\varepsilon}) \tag{27}$$

with

$$c_2 = \sum_{\substack{k,l,m,n=1\\ k^{1/2}+l^{1/2}=m^{1/2}+n^{1/2}}}^{\infty} (klmn)^{-3/4}d(k)d(l)d(m)d(n)\,, \tag{28}$$

which he showed to be a convergent series. Tsang's proof is entirely based on (14), hence we may follow it to derive the corresponding result for $g(t)$, which will be

$$\int_1^T g^4(t)\,dt = \frac{3\pi}{8}c_3T^2 + U(T)\,, \qquad U(T) = O(T^{45/23+\varepsilon})\,, \tag{29}$$

where

$$c_3 := \sum_{\substack{k,l,m,n=1\\ k^{1/2}+l^{1/2}=m^{1/2}+n^{1/2}}}^{\infty} (klmn)^{-1/4}h(k)h(l)h(m)h(n)d(k)d(l)d(m)d(n)\,. \tag{30}$$

Since $h(n) < 0$ and $h(n) \ll n^{-1/2}$, one shows that c_3 is finite and positive in the same way as Tsang did for c_2 in (28). Using (17) and integrating (29) by parts we easily obtain (8) with

$$B = \frac{3\pi c_3}{4} > 0\,, \qquad D = 2\int_1^\infty \frac{U(t)}{t^3}\,dt - U(1)\,.$$

Let now $0 \le A < 4$. From (4), (29) and Hölder's inequality for integrals it follows that

$$T^{1-A/4} \ll \int_1^T \left|R\left(\frac{1}{2} + it; \frac{t}{2\pi}\right)\right|^A dt \ll T^{1-A/4} \qquad (0 \le A < 4)\,. \tag{31}$$

D. R. Heath-Brown [1] proved the existence of

$$\lim_{X\to\infty} X^{-1-k/4} \int_1^X |\Delta(x)|^k\,dx$$

for $0 \le k \le 9$ by a general method. In view of (17) and (31) this method gives, when applied to $g(t)$, the existence of

$$\lim_{T\to\infty} T^{-1-k/4} \int_1^T |g(t)|^k\,dt$$

for $0 \le k < 4$. From (17) and integration by parts we deduce that

$$C(A) = \lim_{T\to\infty} T^{A/4-1} \int_1^T \left|R\left(\frac{1}{2} + it; \frac{t}{2\pi}\right)\right|^A dt$$

exists for $0 \le A < 4$. Since (31) holds we obtain $C(A) > 0$, hence (7) is proved.

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KATEDRA MATEMATIKE RGF-A
UNIVERSITETA U BEOGRADU
DJUŠINA 7
11000 BEOGRAD, SERBIA (YUGOSLAVIA)

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