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Some division theorems for vector fields

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Abstract. This paper is concerned with the problem of divisibility of vector fields with respect to the Lie bracket $[X, Y]$. We deal with the local divisibility. The methods used are based on various estimates, in particular those concerning prolongations of dynamical systems. A generalization to polynomials of the adjoint operator $\mathrm{ad}(X)$ is given.

0. Introduction. The Lie bracket of differentiable vector fields on a smooth manifold is one of the fundamental operations not only in differential geometry. We deal with the following problem of division:

Given vector fields X, Z, does there exist a vector field Y such that $[X, Y] = Z$?

The problem has been considered only for local vector fields and the full and positive answer is known whenever X has a nonvanishing germ. In this case X has local representation $\partial/\partial x_1$ and the "quotient" Y can be taken to be

$$Y(x_1, \ldots, x_n) = \int_{-\delta}^{x_1} Z(t, x_2, \ldots, x_n)\, dt$$

for $\|x\| = \sup |x_i| \leq \delta$. This fact has been broadly exploited in papers concerning the well-known Pursell–Shanks theorem and its generalizations.

Since our problem will also be of local character it can be assumed that X and Z are vector fields defined in a neighbourhood of the origin 0 of $\mathbb{R}^n$ and the equality $[X, Y] = Z$ is meant in the sense of germs, that is, there exists a neighbourhood U of 0 in which it holds.

Thus X, Y, Z will be elements of the Lie algebra $\mathfrak{X}(\mathbb{R}^n)$ of local C^∞ vector fields defined near the origin of $\mathbb{R}^n$. In view of the above, the question remains open only for homogeneous vector fields X, Z, that is, with $X(0) = Z(0) = 0$. From now on the notation $\mathfrak{X}(\mathbb{R}^n)$ will be used for the subalgebra of homogeneous elements.

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In order to justify what we deal with in the section that follows, let us see how the flow Ψ_t of a given vector field X can be involved in the problem of divisibility by X.

For any field Z the transfer of Z along the trajectories of X is defined by

$$(\Psi_t)_*Z = (D\Psi_{-t} \circ \Psi_t)Z \circ \Psi_t\,,$$

i.e.

$$(\Psi_t)_*Z(x) = D\Psi_{-t}(\Psi_t(x))Z(\Psi_t(x))\,.$$

The Lie bracket $[Z, X]$ is just the infinitesimal version of that and we have

$$[Z, X] = \frac{d}{dt}\bigg|_{t=0} (\Psi_t)_*Z\,.$$

More generally,

$$\frac{d}{dt}(\Psi_t)_*Z = [(\Psi_t)_*Z, X]\,.$$

Setting $Y_t = (\Psi_t)_*Z$ we can write

$$Y_t' = [Y_t, X] \qquad \left(Y' = \frac{d}{dt}Y\right).$$

This gives

$$Z = -\Big[\int\limits_0^t Y_s\,ds, X\Big] + Y_t \tag{0.1}$$

since $Y_0 = Z$. Without loss of generality we can assume that X is complete so the range of t is $(-\infty, \infty)$. If necessary we can replace X by fX where f is a C^∞ function which is 1 in a neighbourhood of 0 and has a compact support in the set where X is defined. Suppose that

1° $Y_t \to 0$ as $t \to \infty$,

2° the integral

$$Y(x) = \int\limits_0^\infty Y_t(x)\,dt \tag{0.2}$$

is convergent and Y is C^∞ in a neighbourhood of 0. Then $Z = [X, Y]$ so Y is a solution to the problem.

Since $X(0) = 0$ we have $\Psi_t(0) = 0$ for all t. If $x = 0$ is asymptotically stable then $\Psi_t(x) \to 0$ as $t \to \infty$ for small $\|x\|$. As also $Z(0) = 0$, it follows that $Z(\Psi_t(x)) \to 0$, but what we need is that $D\Psi_{-t}(\Psi_t(x))Z(\Psi_t(x))$ and all its x-derivatives converge to 0 as $t \to \infty$, uniformly in x. The study of this question will be the subject of the next section.

1. Some bounds to flows. Consider the system of differential equations

$$x' = X(x) \qquad (x' = dx/dt) \tag{1.1}$$

where x, $X(x)$ are n-vectors, X is C^∞ in a neighbourhood of $x = 0$ and $X(0) = 0$. Thus X can be written

$$X(x) = Ax + h(x) \tag*{(1.1)$'$}$$

with $A = DX(0)$ and $\|h(x)\| \le L\|x\|^2$. We may assume that the Lipschitz constant L is global, so that solutions to (1.1) are defined globally.

Assume that all the eigenvalues λ_i of A satisfy $\operatorname{Re}\lambda_i < 0$ for $i = 1, \ldots, n$ (for short: $\operatorname{Re}\lambda < 0$). It is known that under this condition there exist positive constants K and c such that

$$\|e^{tA}\| \le Ke^{-ct} \qquad \text{for } t > 0\,,$$

and $\delta > 0$ such that

$$\|\Psi_t(x)\| \le Ke^{-ct/2}\|x\| \qquad \text{for } \|x\| \le \delta\,. \tag{1.2}$$

Here $\Psi_t(x)$ is the solution of (1.1) passing through x at $t = 0$ (the flow of X). For the constant c we may take any number $< \min(|\operatorname{Re}\lambda|)$ (this is easily seen by writing A in Jordan canonical form).

Lemma 1.1. *If there is a bound*

$$\|\Psi_t(x)\| \le Ke^{c(t)}\|x\| \qquad \textit{for } \|x\| \le \delta\,,\ t \ge 0\,,$$

with K a constant and $c(t)$ depending only on the eigenvalues of A and not on their multiplicities (*as in the above case*), *then the derivatives $D^k\Psi_t(x)$, $k = 1, 2, \ldots,$ also have similar bounds with the same δ and $c(t)$ and different constants K_k.*

Proof. Consider the following variational equation (kth prolongation of (1.1) with respect to x):

$$\begin{cases} x' = X(x)\,, \\ \xi_1' = DX(x)\xi_1\,, \\ \xi_2' = D^2X\xi_1\xi_1 + DX\xi_2\,, \\ \ldots \\ \xi_k' = \displaystyle\sum_{s=1}^{k} D^sX \sum_{\substack{\alpha_1+\ldots+\alpha_s=k \\ \alpha_i>0}} \xi_{\alpha_1}\ldots\xi_{\alpha_s}\,, \end{cases} \tag{1.3}$$

with $\xi_\alpha \in \mathbb{R}^n$ for $\alpha = 1, \ldots, k$. With brief notation $(x, \xi_1', \ldots, \xi_k') = F(x, \xi_1, \ldots, \xi_k)$ the Hessian of this equation, i.e. $DF(0)$, is of the form

$$\begin{pmatrix} A & & & \\ & A & & \\ & & \ddots & \\ & & & A \end{pmatrix} \qquad \text{(of dimension } (k+1)n)\,.$$

Thus $DF(0)$ has the same eigenvalues as A.

For any constant vector $v \in \mathbb{R}^n$ the system

$$(\Psi_t(x), D\Psi_t(x)v, \ldots, D^k\Psi_t(x)v^k)$$

is a solution to (1.3) passing through $(x, v, 0, \ldots, 0) \in \mathbb{R}^{(k+1)n}$. In fact,

$$(D^k\Psi_t v^k)' = (D^k\Psi_t)'v^k = D^k\Psi_t' v^k = D^k(X \circ \Psi_t)v^k$$
$$= \Big(\sum_{s=1}^{k} D^s(X)\Psi_t \sum_{\substack{\alpha_1+\ldots+\alpha_s=k \\ \alpha_i>0}} D^{\alpha_1}\Psi_t \ldots D^{\alpha_s}\Psi_t\Big)v^k$$

and $(D^{\alpha_1}\Psi_t \ldots D^{\alpha_s}\Psi_t)v^k = (D^{\alpha_1}\Psi_t v^{\alpha_1}) \ldots (D^{\alpha_s}\Psi_t v^{\alpha_s})$. Therefore, if a bound $\|\Psi_t(x)\| \le ke^{c(t)}\|x\|$ holds for $\|x\| \le \delta$ and $t \ge 0$, then

$$\|D^l\Psi_t(x)v^l\| \le K_l e^{c(t)}\|(x, v, 0, \ldots, 0)\|, \qquad l = 1, \ldots, k,$$

for all $\|x\| \le \delta$ and any $\|v\| \le 1$. This gives $\|D^l\Psi_t\| \le K_l' e^{c(t)}$.

LEMMA 1.2. *We have*

$$|\det D\Psi_t(x)| \ge Me^{(\operatorname{tr} A)t} \quad \textit{for } \|x\| \le \delta, \tag{1.4}$$

with a positive constant M.

Proof. Set $\Delta_t(x) = \det D\Psi_t(x)$. Then $\Delta_{t+s}(x) = \Delta_t(\Psi_s(x))$. Hence

$$\Delta_s'(x) = \Delta_0'(\Psi_0(x))\Delta_s(x).$$

A routine computation leads to $\Delta_0'(\xi) = \operatorname{tr} DX(\xi)$ and finally

$$\Delta_s(x) = \exp \int_0^t \operatorname{tr} DX(\Psi_s(x))\, ds, \tag{1.4}$$

since $\Delta_0(x) = 1$. By applying $(1.1)'$ this can be written as

$$\Delta_s(x) = e^{(\operatorname{tr} A)t} \exp \int_0^t \operatorname{tr} Dk(\Psi_s(x))\, ds.$$

Since

$$|\operatorname{tr} Dk(\Psi_s(x))| \le C\|\Psi_s(x)\|^2 \le C\delta Ke^{-ct},$$

the integral $\int_0^t \operatorname{tr} Dk(\Psi_s(x))\, ds$ is bounded from below by $-C\delta K/c$. Thus we can take $M = \exp(-C\delta K/c)$.

LEMMA 1.3. *There are constants K_k and L_k such that*

$$\|D^k\Psi_t(x)\| \le K_k e^{-ct/2} \tag{1.5}$$

$$\|D^k[D\Psi_{-t}(\Psi_t(x))]\| \le L_k e^{(k+1)at}, \tag{1.6}$$

where $a = -\operatorname{tr} A - (n-1)c$ and $\|x\| \le \delta$.

Proof. The bounds (1.5) follow immediately from Lemma 1.1 with reference to (1.2).

For (1.6), from the identity $\Psi_{-t}(\Psi_t(x)) = x$ it follows that $D\Psi_{-t}(\Psi_t(x))$ is equal to the inverse matrix to $D\Psi_t(x)$. In view of (1.5) and Lemma 1.2 the elements of $(D\Psi_t(x))^{-1}$ are majorized in absolute value by

$$e^{(-\operatorname{tr} A-(n-1)c)t} \ (= e^{at})$$

up to a constant multiplicative factor.

Now from $(D\Psi_t)^{-1} \circ D\Psi_t = I$ we get

$$D(D\Psi_t)^{-1} D\Psi_{-t} + (D\Psi_t)^{-1} D^2\Psi_t = 0\,,$$

which gives

$$\|D(D\Psi_t)^{-1}\| \le L_1 e^{2at}\,,$$

and (1.6) follows by induction.

We denote by $\mathfrak{X}_m(\mathbb{R}^n)$ the space of local vector fields, m-flat at 0.

THEOREM 1.4. *Suppose X is as above and $Z \in \mathfrak{X}_m(\mathbb{R}^n)$. If $(k+1)a - mc/2 < 0$ then there exists a C^k vector field Y in a neighbourhood of 0 such that $[X,Y] = Z$.*

Proof. Z being m-flat satisfies $\|D^k Z(x)\| \le M_k \|x\|^{m-k}$ for $0 \le k \le m-1$ and it is bounded for $k \ge m$ when $\|x\| \le \delta$. We have

$$\|D^k(D\Psi_{-t}(\Psi_t(x))Z(\Psi_t(x)))\| \le \sum_{r+s=k} \|D^r(D\Psi_{-t}\circ\Psi_t)\|\,\|D^s(Z\circ\Psi_t)\|\,. \tag{1.7}$$

By (1.2) and (1.5)

$$\|(D^u Z)\circ\Psi_t\| \mathrel{\dot{\le}} e^{-(m-u)ct/2} \qquad \text{for } u \le m-1\,,$$

and the left hand side is bounded for $u \ge m$. Here and below, $\dot{\le}$ indicates that the bound holds up to a multiplicative constant.

As the other term in (1.7) is bounded by $e^{-uct/2}$ we get

$$\|D^s(Z\circ\Psi_t)\| \mathrel{\dot{\le}} e^{-mct/2} \tag{1.8}$$

for all s since $u \ge m$. In view of (1.6)–(1.8) we have

$$\|D^k(D\Psi_{-t}\circ\Psi_t)Z\circ\Psi_t|\,\| \mathrel{\dot{\le}} e^{((k+1)a-mc/2)t}\,.$$

Suppose $(k+1)a - mc/2 < 0$ for a positive integer k. Then the integral

$$F(X,Z) = \int_0^\infty (\Psi_t)_* Z\,dt \qquad \Big(= \int_0^\infty (\exp tX)_* Z\,dt\Big) \tag{1.9}$$

is a vector field of class C^k in the ball $\|x\| \le \delta$.

Since clearly $\|(\Psi_t)_* Z\| \to 0$ as $t \to \infty$, we get by (0.1)

$$[X, F(X,Z)] = Z\,, \tag{1.10}$$

which was to be proved.

2. Divisibility by linear vector fields. Suppose $X = Ax$. Then $\Psi_t(x) = e^{tA}x$. As previously we assume that A satisfies $\operatorname{Re}\lambda < 0$.

Let c be any constant $< \min(|\operatorname{Re}\lambda|)$ and b any constant $> \max(|\operatorname{Re}\lambda|)$. Then

$$\|e^{ta}\| \le Ke^{-ct}\,, \qquad \|e^{-tA}\| \le Le^{bt}\,, \qquad t \ge 0\,. \tag{2.1}$$

We call

$$d(X) = \frac{\max(|\operatorname{Re}\lambda|)}{\min(|\operatorname{Re}\lambda|)}$$

the *dispersion* of X. Obviously $b/c > d(X)$.

THEOREM 2.1. *Suppose that Z is m-flat at $x = 0$ and $m \ge d(X) + 1$. Then Z is divisible by X with a quotient $F(X, Z)$ defined by* (1.9).

Proof. Now

$$(\Psi_t)_*Z(x) = e^{-tA}Z(e^{tA})$$

and

$$D^k((\Psi_t)_*Z(x) = e^{-tA}D^kZ(e^{tA})e^{ktA}\,, \qquad k \ge 0\,.$$

Exploiting the m-flatness of Z as in the proof of Theorem 1.4 we come to the following estimate:

$$\|D^k((\Psi_t)_*Z(x))\| \mathrel{\dot{\le}} e^{(b-mc)t}\,, \qquad k = 0, 1, \ldots$$

The constants b and c may be taken such that $b/c < d(X)+1$. It follows that $b - mc < 0$ for $m \ge d(X) + 1$. Consequently, the integral (1.9) converges uniformly together with all its derivatives. Thus $F(X, Z)$ is a C^∞ vector field in any ball contained in the domain of Z.

In particular, if $X = x$ then $d(X) = 1$ and taking $m = 2$ we conclude from Theorem 2.1 that each Z from $\mathfrak{X}_2(\mathbb{R}^n)$ is divisible by X.

3. Divisibility by means of linearization. Consider again the general case $X = Ax + h(x)$ as in (1.1). The field $X_0 = Ax$ is called the *linearization* of X at 0. From now on the vector field X will be thought of locally as the germ at 0 of a smooth map $X : \mathbb{R}^n \to \mathbb{R}^n$.

Suppose that X is C^∞-equivalent to its linearization X_0, that is, there exists a C^∞-diffeomorphism f of $\mathbb{R}^n$, with $f(0) = 0$, such that $f_*X = X_0$ in a neighbourhood of 0.

For a given Z set $Z_0 = f_*Z$ and assume that there is a Y_0 such that $Z_0 = [Y_0, X_0]$. This means

$$f_*Z = [Y_0, f_*X] = f_*[(f^{-1})_*Y_0, X]\,.$$

Hence $Z = [Y, X]$ with $Y = (f^{-1})_*Y_0$, and we obtain

LEMMA 3.1. *If f_*Z is divisible by the linearization of X then Z is divisible by X.*

Note that the transformation f_* does not change the order of flatness of Z.

Which (germs of) vector fields are linearizable? The answer is: almost all. This can be concluded from the following theorems of Sternberg:

Either of the conditions below implies that a vector field X with $X(0) = 0$ is C^∞-equivalent to its linearization $DX(0)x$.

(i) *Each eigenvalue λ of $DX(0)$ satisfies* $\operatorname{Re}\lambda < 0$ *and*

$$X(x) = DX(0)x + o(x^\infty)\,. \tag{3.1}$$

(ii) *Each eigenvalue λ_i $(i = 1, \ldots, n)$ satisfies*

$$\lambda_i \neq m_1\lambda_1 + \ldots + m_n\lambda_n \tag{3.2}$$

whenever the m_j are non-negative integers with $m_1 + \ldots + m_n \geq 2$ ([1], [2]).

Combining these facts with our results of previous sections, via Lemma 3.1, we come to the following conclusion.

Theorem 3.2. *Suppose that X is a C^∞ vector field and $DX(0)$ has all eigenvalues with negative real parts. If X satisfies either* (3.1) *or* (3.2) *then every vector field Z, m-flat with $m \geq d(X) + 1$, is C^∞-divisible by X.*

Sternberg's algebraic condition (3.2) is also directly involved in the problem of divisibility of vector fields. Namely, let

$$\sum a^i_\alpha x^\alpha\,, \quad \sum b^i_\alpha x^\alpha\,, \quad \sum c^i_\alpha x^\alpha$$

be the Taylor series at $x = 0$ for X, Y, Z respectively. The equality $[X, Y] = Z$ gives

$$\sum_{\substack{\alpha+\beta=\gamma \\ j=1,\ldots,n}} b^i_{\alpha+1_j} a^j_\beta - a^i_{\alpha+1_j} b^j_\beta = c^i_\gamma \tag{3.3}$$

with 1_j standing for the multiindex $(0, \ldots, 0, 1, 0, \ldots, 0)$, where 1 is in the jth place. For given coefficients a and c there is a purely algebraic problem of solvability of this equation with respect to the unknown coefficients b.

Let us take $X = \sum_{i=1}^n \lambda_i x_i \partial/\partial x_i$. Then the λ_i are the eigenvalues of $DX(0)$.

Let $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_1 + \ldots + \alpha_n = |\alpha|$. In this case all a^i_α in formula (3.3) vanish for $|\alpha| \geq 2$. Hence (3.3) is now

$$\sum_j \Big(\sum_k b^i_{\alpha+1_j-1_k} a^j_k - a^i_j b^j_\alpha \Big) = c^i_\alpha\,.$$

Since $a^i_j = \lambda_i \delta^i_j$ and the number of the indices j is α_j we get

$$\Big(\sum_j \alpha_j \lambda_j - \lambda_i \Big) b^i_\alpha = c^i_\alpha\,.$$

Suppose that Z is m-flat and $c^i_\alpha \neq 0$ for $|\alpha| \geq m$; then for the existence of Y such that $[X,Y] = Z$ it is necessary to have

$$\lambda_i \neq \sum_j \alpha_j \lambda_j \tag{3.4}$$

for any non-negative integers $\alpha_1, \ldots, \alpha_n$ satisfying $|\alpha| \geq m$. This is exactly Sternberg's condition for $m = 2$ (in the regularity class $k = \infty$).

If $\lambda_i = \sum \alpha_j \lambda_j$ and $\operatorname{Re}\lambda < 0$ (or $\operatorname{Re}\lambda > 0$) then

$$|\operatorname{Re}\lambda_i| = \sum \alpha_j |\operatorname{Re}\lambda_j| \geq |\alpha| \min(|\operatorname{Re}\lambda_j|)\,.$$

This implies $|\alpha| \leq \max(|\operatorname{Re}\lambda_j|)/\min(|\operatorname{Re}\lambda_j|) = d(X)$. Thus for $m \geq d(X) + 1$ we have $|\alpha| \leq m-1$ and the condition (3.4) is satisfied (as it should be in view of Theorem 3.2). This also shows that the lower bound $d(X)+1$ for m in Theorem 3.2 is sharp.

On the other hand, if there are both negative and positive numbers in $\operatorname{Re}\lambda$ then the equality $\lambda_i = \sum \alpha_j \lambda_j$ may occur for all $|\alpha|$.

4. Generalization to polynomials. For some applications to actions of infinite Lie groups it is useful to know when polynomials of the adjoint mapping $\operatorname{ad}(X)$ act surjectively in the space of infinitely flat vector fields. An answer to this question is given in the following:

THEOREM 4.1. *Let $P(\xi) = a_0 + a_1 + \ldots + a_r\xi^r$ be a polynomial of degree $r > 0$. Suppose that X satisfies $\operatorname{Re}\lambda < 0$. For any vector field Z vanishing up to infinite order at $x = 0$ there exists a vector field Y such that $Z = P(\operatorname{ad}(X))Y$. The Y can be defined by*

$$Y(x) = -\int_0^\infty f(t)(\Psi_t)_* Z(x)\,dt \tag{4.1}$$

where $f(t)$ is the solution of the differential equation

$$a_0\xi - a_1\xi' + \sum_{k=2}^{r}(k-1)a_k\xi^{(k)} = 0\,, \tag{4.2}$$

with initial conditions $\xi(0) = \ldots = \xi^{(r-2)}(0) = 0$, $\xi^{(r-1)}(0) = 1/((r-1)a_r)$ for $r \geq 2$ and $\xi(0) = -1/a_1$ for $r = 1$.

P r o o f. Equation (4.2) being with constant coefficients, there are positive constants α, β such that

$$|f(t)| \leq \alpha e^{\beta t} \qquad \text{for } t \geq 0\,. \tag{4.3}$$

As in Section 1, we have the following bounds:

$$\|f(t)D^k(\Psi_t)_* Z(x)\| \leq \alpha M^k_m e^{(\beta+\gamma_k - mc)t}\,, \qquad c > 0\,, \tag{4.4}$$

for $t \geq 0$ and $\|x\| \leq \delta$. With k fixed we can choose m great enough so that $\beta+\gamma_k-mc<0$. This makes the integral (4.1) uniformly convergent in $B(\delta)$ together with all derivatives. Thus Y is C^∞ in $B(\delta)$.

Set $Y_t = (\Psi_t)_* Z$. In the introduction we saw that $Y_t' = \mathrm{ad}(X)Y_t$. Hence

$$Y_t^{(k)} = [\mathrm{ad}(X)]^k Y_t\,, \qquad k \geq 1\,.$$

Therefore

$$P(\mathrm{ad}(X))fY_t = a_0 fY_t + a_1 fY_t' + \ldots + a_r fY_t^{(r)}\,. \tag{4.5}$$

From

$$(fY_t)^{(k)} = fY_t^{(k)} + k(f'Y_t)^{(k-1)} + (1-k)f^{(k)}Y_t$$

for $k \geq 1$, we get

$$fY_t^{(k)} = (fY_t)^{(k)} - k(f'Y_t)^{(k-1)} + (k-1)f^{(k)}Y_t\,.$$

On inserting this into (4.5) one gets for $r \geq 2$

$$\begin{aligned} P(\mathrm{ad}(X))fY_t &= \Big(a_0 f - a_1 f' + \sum_{k=2}^{r} a_k (k-1) f^{(k)}\Big)Y_t + a_1(fY_t)' \\ &\quad + \sum_{k=2}^{r} a_k[(fY_t)^{(k)} - k(f'Y_t)^{(k-1)}] \\ &= a_1(fY_t)' + \sum_{k=2}^{r} a_k[(fY_t)^{(k)} - k(f'Y_t)^{(k-1)}] \end{aligned}$$

according to our assumption on f. Now, by integrating either side with respect to t over the interval $(0,\infty)$ and using notation (4.1) we obtain

$$P(\mathrm{ad}(X))Y = a_1 fY_t|_0^\infty + \Big\{\sum_{k=1}^{r} a_k[(fY_t)^{(k-1)} - k(f'Y_t)^{(k-2)}]\Big\}_0^\infty\,, \tag{4.6}$$

and f satisfies $f(0) = f'(0) = \ldots = f^{(r-2)}(0) = 0$, $f^{(r-1)}(0) = 1/((r-1)a_r)$. So, in view of (4.4) for $k=0$, we have

$$fY_t|_0^\infty = -f(0)Y_0 = -f(0)Z = 0\,.$$

As the bound (4.3) can be extended to all derivatives of f and the operator $\mathrm{ad}(X)$ is bounded in $B(\delta)$, there is a constant M such that

$$\|f^{(p)}Y_t^{(q)}\| \leq \alpha e^{\beta t}\|\mathrm{ad}(X)^q\|\,\|Y_t\| \leq Me^{(\beta+\gamma_0-mc)t}\,, \qquad p,q \geq 0\,,$$

with $\beta+\gamma_0-mc<0$. Therefore

$$I_k = [(fY_t)^{(k-1)} - k(f'Y_t)^{(k-2)}]_0^\infty = 0$$

for $2 \leq k \leq r-1$. For $k=r$

$$I_r = -f^{(r-1)}(0)Y_0 + rf^{(r-1)}(0)Y_0 = (r-1)f^{(r-1)}(0)Z\,.$$

Coming back to (4.6) we finally get

$$P(\mathrm{ad}(X))Y = a_r(r-1)f^{(r-1)}(0)Z = Z\,,$$

as required.

For $r = 1$, we take $f(0) = -1/a_1$. Then

$$P(\mathrm{ad}(X))Y = a_1 f Y_t|_0^\infty = -a_1 f(0) Z = Z\,.$$

In particular:

(i) If $P(u) = a + u$, then $f(t) = -e^{at}$ and

$$Y = \int_0^\infty e^{at}(\Psi_t)_* Z\, dt\,.$$

(ii) If $P(u) = u^r$, $r \geq 2$, then

$$Y = \frac{1}{(r-1)!(r-1)} \int_0^\infty t^{r-1}(\Psi_t)_* Z\, dt\,.$$

This Y satisfies

$$Z = [X, \ldots [X, [X, Y]]] \qquad (r \text{ commutators})\,.$$

As we see from the proof one can expect existence of a solution to the equation $P(\mathrm{ad}(X))Y = Z$ also in the case where Z vanishes at $x = 0$ up to a finite order m. This would depend on the polynomial P and the required regularity class of Y which is to be defined by formula (4.1).

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