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Isolated intersection multiplicity and regular separation of analytic sets

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Abstract. An isolated point of intersection of two analytic sets is considered. We give a sharp estimate of their regular separation exponent in terms of intersection multiplicity and local degrees.

1. Separation. Let M be an m-dimensional normed complex vector space. Following ([4], IV.7) we say that a pair of closed sets X, Y in an open subset G of M satisfies the *condition* (S) at a point $a \in G$ if either $a \notin X \cap Y$, or $a \in X \cap Y$ and

$$\varrho(z, X) + \varrho(z, Y) \geq c\varrho(z, X \cap Y)^p$$

for z in a neighbourhood of a, for some $c, p > 0$ ($\varrho(\cdot, Z)$ denotes the distance function to the set $Z \subset M$).

In the sequel we will consider only isolated points of the intersection of X and Y.

We say that X and Y are *p-separated at $a \in G$* if a is an isolated point of $X \cap Y$ and the pair X, Y satisfies the condition (S) at a, with exponent p and some constant $c > 0$.

As a simple consequence of properties of (S) (see [4], IV.7.1) we get the following lemma.

Lemma 1.1. *Let $H_1 \subset G$ and H_2 be open subsets of normed, finite-dimensional complex vector spaces and let $f : H_1 \to H_2$ be a biholomorphism. Then closed subsets X and Y of G are p-separated at a point $a \in H_1$ if and only if $f(X \cap H_1)$ and $f(Y \cap H_1)$ are p-separated at $f(a)$.*

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By the above lemma our condition can be carried over—in a classical manner—to the case of manifolds. (In this paper all manifolds are assumed to be second-countable.)

Namely, we say that closed subsets X, Y of an m-dimensional complex manifold M are *p-separated at $a \in M$* if for some (and hence for every) chart $\varphi : \Omega \to G \subset \mathbb{C}^m$ such that $a \in \Omega$, the sets $\varphi(X \cap \Omega)$, $\varphi(Y \cap \Omega)$, closed in G, are p-separated at $\varphi(a)$.

It is clear that if X and Y are p-separated at $a \in M$ and $X \cap Y = \{a\}$, then the pair X, Y satisfies the "condition of regular separation" (see [4], IV.7.1).

Now, suppose that X and Y are analytic subsets of M and $a \in M$ is an isolated point of $X \cap Y$. The principal topic of our research is a detailed study of the set

$$P = \{p > 0 : X \text{ and } Y \text{ are } p\text{-separated at } a\},$$

and of the *best exponent*

$$p_0 = p_0(X, Y; a) = \inf P.$$

If $\dim M = m \geq 1$, then a standard calculation yields $p_0 \geq 1$. Obviously, $p_0 = 0$ for $m = 0$.

LEMMA 1.2. *Let M be an open subset of a normed, finite-dimensional complex vector space. Suppose that a is an accumulation point of X. Then X and Y are p-separated at a if and only if there exists a neighbourhood U of a and $c > 0$ such that*

$$\varrho(x, Y) \geq c|x - a|^p \quad \text{for } x \in X \cap U.$$

Proof. It suffices to show that the above condition implies that X and Y are p-separated at a. Without loss of generality we can assume that $c \in (0, 1)$ and U is contained in the ball $B(a, 1)$. Since a is an accumulation point of X, we see that $p \geq 1$.

Fix $r > 0$ such that $B(a, 2r) \subset U$. If $z \in B(a, r)$ then there exist $x \in X \cap B(a, 2r)$ and $y \in Y \cap B(a, 2r)$ such that $\varrho(z, X) = |z - x|$ and $\varrho(z, Y) = |z - y|$. An easy computation shows that

$$l = \varrho(z, X) + \varrho(z, Y) \geq |x - y| \geq \varrho(x, Y) \geq c|x - a|^p.$$

Moreover,

$$l \geq \varrho(z, X) = |z - x| \geq c|z - x|^p.$$

Combining these inequalities we deduce that

$$l \geq \frac{c}{2}(|x - a|^p + |z - x|^p) \geq \frac{c}{2^p}|z - a|^p \quad \text{for } z \in B(a, r),$$

and the proof is complete.

We now state a result which we shall frequently use.

LEMMA 1.3. *Let M be a complex manifold. If $a \in M$ and $p > 0$ then the following conditions are equivalent*:

(1) *X and Y are p-separated at a*,
(2) *$X \times Y$ and Δ_M are p-separated at (a, a)*,

where $\Delta_M = \{(x, x) \in M^2 : x \in M\}$ is the diagonal in M^2.

Proof. Without loss of generality we can assume that M is an open subset of a normed complex vector space N with $\dim N \geq 1$.

Consider N^2 with the norm $|(x, y)| = |x| + |y|$. Observe that, for $z \in M$,

$$\varrho((z, z), X \times Y) = \varrho(z, X) + \varrho(z, Y), \qquad |(z, z) - (a, a)| = 2|z - a|\,.$$

Lemma 1.2 now shows that condition (2) is satisfied if and only if

$$\varrho(z, X) + \varrho(z, Y) \geq c|z - a|^p\,,$$

in a neighbourhood of a, for some $c > 0$. This completes the proof.

2. Multiplicity of isolated intersection. For the convenience of the reader we repeat, from [1], basic definitions and facts on isolated intersections of analytic sets.

Let Z be a pure k-dimensional locally analytic subset of a complex manifold M of dimension m. Let N be a submanifold of M of dimension n such that N intersects Z at an isolated point $a \in M$. We denote by $\mathcal{F}_a(Z, N)$ the set of all locally analytic subsets V of M satisfying:

(1) V has pure dimension $m - k$,
(2) $N_a \subset V_a$,
(3) a is an isolated point of $V \cap Z$,

where N_a, V_a denote the germs of N and V at a.

Observe that for $V \in \mathcal{F}_a(Z, N)$ the intersection of Z and V is proper at a and we can consider the classical intersection multiplicity $\mathrm{i}(Z \cdot V; a)$ in the sense of Draper [2] (cf. [9]). We define

$$\widetilde{\mathrm{i}}(Z \cdot N; a) = \min\{\mathrm{i}(Z \cdot V; a) : V \in \mathcal{F}_a(Z, N)\},$$
$$\mathcal{P}_a(Z, N) = \{V \in \mathcal{F}_a(Z, N) : \mathrm{i}(Z \cdot V; a) = \widetilde{\mathrm{i}}(Z \cdot N; a)\}\,.$$

Note that ([1], Th. 4.4) gives the full characterization of the family $\mathcal{P}_a(Z, N)$.

Having disposed of this preliminary step we can now turn to the general case. Let X, Y be pure dimensional locally analytic subsets of a complex manifold M such that a is an isolated point of $X \cap Y$. The positive integer

$$\mathrm{i}(X \cdot Y; a) = \widetilde{\mathrm{i}}((X \times Y) \cdot \Delta_M; (a, a))$$

is defined to be the *multiplicity of intersection* of X and Y at a.

If Y is a submanifold the definition of $\mathrm{i}(X \cdot Y; a)$ presented above coincides with that of $\widetilde{\mathrm{i}}(X \cdot Y; a)$ introduced earlier.

Finally, observe that in the case $Y = \{a\}$ we get

$$\mathfrak{i}\,(X \cdot Y; a) = \widetilde{\mathfrak{i}}\,(X \cdot Y; a) = \deg_a X\,,$$

where $\deg_a X$ is the classical degree (the Lelong number) of X at a (see e.g. [1], [2]).

3. Main results. In this part we apply the "diagonal construction" to separation of analytic sets. Let us begin with the following theorem motivated by [7].

THEOREM 3.1. *Let Z be a pure dimensional analytic subset and let N be a closed submanifold of a complex manifold M of dimension $m \geq 1$. Suppose that $a \in M$ is an isolated point of $Z \cap N$ and set*

$$P = \{p > 0 : Z \text{ and } N \text{ are } p\text{-separated at } a\}\,.$$

Then

1) $p_0 = \inf P \in P \cap \mathbb{Q}$,
2) $1 \leq p_0 \leq \mathfrak{i}\,(Z \cdot N; a) - \deg_a Z + 1$.

Proof. Let $V \in \mathcal{P}_a(Z; N)$ (see Section 2). We know that $\mathfrak{i}\,(Z \cdot N; a) = \mathfrak{i}\,(V \cdot N; a)$, and ([1], Th. 4.4) implies that V_a is a germ of a manifold. Suppose that $\dim Z = k$, $\dim N = n$.

We can assume, by using Lemma 1.1 if necessary, that:

- $M = B \times D \times \mathbb{C}^n$, where B and D are the unit balls in $\mathbb{C}^k$, $\mathbb{C}^{m-n-k}$ respectively,
- $N = \{0\} \times \mathbb{C}^n$, $0 \in \mathbb{C}^{m-n}$,
- $V = \{0\} \times D \times \mathbb{C}^n$, $0 \in \mathbb{C}^k$,
- $Z \cap V = \{0\}$,
- $\pi|Z : Z \to B \times D$ is proper, where $\pi : M \to B \times D$ is the natural projection.

In this situation, by ([1], Th. 4.4, Lemma 2.4), we obtain $C_0(\pi(Z)) \cap (\{0\} \times D) = \{0\}$, where $C_0(\pi(Z))$ is the tangent cone of the set $\pi(Z)$ at $0 \in \mathbb{C}^{m-n}$. An easy computation and ([7], Th. (1.2)) show that there exists an open neighbourhood $W \subset B \times D$ of $0 \in \mathbb{C}^{m-n}$ and a constant $A > 0$ such that

$$(*) \qquad (x, y) \in \pi(Z) \cap W \;\Rightarrow\; |y| \leq A|x|\,.$$

After these preparations let us define

$$Q = \{q > 0 : \exists \widetilde{c} > 0 \;:\; |z| + |y| \leq \widetilde{c}\,|x|^q \text{ for } (x, y, z) \in Z \text{ in some neighbourhood of } 0\}\,.$$

By ([7], Th. (1.2)) we get:

1′) $q_0 = \sup Q \in (Q \cap \mathbb{Q}) \cup \{+\infty\}$, 2′) $d^{-1} \in Q$,

where $d = \mathfrak{i}(Z \cdot N; 0) - \deg_0 Z + 1$.

Now, observe that Lemma 1.2 implies that Z and N are p-separated at $0 \in \mathbb{C}^m$ if there exists $c > 0$ such that

$$|x| + |y| \geq c(|x| + |y| + |z|)^p \quad \text{for } (x, y, z) \in Z$$

in some neighbourhood of $0 \in \mathbb{C}^m$.

We prove that

$$(**) \qquad P = \{1/q : q \in Q,\ q \leq 1\}.$$

First, suppose that $q \in Q$, $q \leq 1$. Then $p = 1/q \geq 1$ and $|x| \geq c_1(|z|+|y|)^p$ for $(x, y, z) \in Z$ in some neighbourhood of 0 and for some constant $c_1 \in (0, 1)$. This implies $|x| \geq (c_1/2^p)(|x|+|y|+|z|)^p$ and finally, there exists $c_2 > 0$ such that $|x| + |y| \geq c_2(|x| + |y| + |z|)^p$ for $(x, y, z) \in Z$ in some neighbourhood of 0. Hence $p = 1/q \in P$.

Now, let $p \in P$. Then $p \geq 1$ and there exists $c > 0$ such that

$$|x| + |y| \geq c(|x| + |y| + |z|)^p \quad \text{for } (x, y, z) \in Z$$

in some neighbourhood of 0. By property $(*)$ we get

$$|x| \geq c_3(|y| + |z|)^p,$$

and finally there exists $c_4 > 0$ such that

$$|y| + |z| \leq c_4|x|^q, \quad \text{where } q = 1/p,$$

for $(x, y, z) \in Z$ in some neighbourhood of 0. Therefore $p = 1/q$ where $q \in Q$ and $q \leq 1$, which proves $(**)$. Since $d \geq 1$, condition 2′) implies $d \in P$.

It is easily seen that $p_0 = \max\{1, 1/q_0\} \leq d$. From 1′) we conclude that $p_0 \in P \cap \mathbb{Q}$, and the proof is complete.

In the remainder of this paper we assume that X and Y are analytic subsets of an m-dimensional ($m \geq 1$) complex manifold M, and that a is an isolated point of $X \cap Y$.

Define

$$P = \{p > 0 : X \text{ and } Y \text{ are } p\text{-separated at } a\}.$$

We can now state our main result.

Theorem 3.2. *If X and Y are pure dimensional, then*

1) $p_0 = \inf P \in P \cap \mathbb{Q}$,
2) $1 \leq p_0 \leq \mathfrak{i}(X \cdot Y; a) - \deg_a X \cdot \deg_a Y + 1$.

Proof. Define

$$Z = X \times Y \subset M^2, \quad N = \Delta_M \subset M^2,$$
$$\widetilde{P} = \{p > 0 : Z \text{ and } N \text{ are } p\text{-separated at } (a, a)\}.$$

By Lemma 1.3, $P = \widetilde{P}$. It is obvious that $\mathfrak{i}(X \cdot Y; a) = \mathfrak{i}(Z \cdot \Delta_M; (a,a))$ and $\deg_{(a,a)} Z = \deg_a X \cdot \deg_a Y$. Now, Theorem 3.1 completes the proof.

In the last two theorems we have been working under the assumption that X, Y are pure dimensional. To study the general case suppose that $X_1, \ldots, X_r$ and $Y_1, \ldots, Y_s$ are all components of X and Y, respectively, passing through a. We can extend our definitions from the pure dimensional case (cf. [1]) by the following natural formulas:

$$\mathfrak{i}(X \cdot Y; a) = \sum_{k=1}^{r} \sum_{l=1}^{s} \mathfrak{i}(X_k \cdot Y_l; a),$$

$$\deg_a X = \sum_{k=1}^{r} \deg_a X_k, \qquad \deg_a Y = \sum_{l=1}^{s} \deg_a Y_l .$$

We can now state the analogue of the last theorem.

Corollary 3.3. *Under the above definitions*:

1) $p_0 = \inf P \in P \cap \mathbb{Q}$,
2) $1 \le p_0 \le \mathfrak{i}(X \cdot Y; a) - \deg_a X \cdot \deg_a Y + 1$.

Proof. It is clear that $p_0 = \max\{p_0(X_k, Y_l; a) : k = 1, \ldots, r,\ l = 1, \ldots, s\}$ (see Section 1), which implies 1), by Theorem 3.2. Let $p_0 = p_0(X_k, Y_l; a)$ for some fixed k, l. Observe that Theorem 3.2 gives

$$1 \le p_0 = p_0(X_k, Y_l; a) \le \mathfrak{i}(X_k \cdot Y_l; a) - \deg_a X_k \cdot \deg_a Y_l + 1 .$$

An easy computation shows that

$$\mathfrak{i}(X_k \cdot Y_l; a) - \deg_a X_k \cdot \deg Y_l \le \mathfrak{i}(X \cdot Y; a) - \deg_a X \cdot \deg_a Y ,$$

and the proof is complete.

The following corollary yields information about "1-separation" in terms of tangent cones of sets.

Corollary 3.4. *The following conditions are equivalent*:

1) X *and* Y *are* 1*-separated at* a,
2) $C_a(X) \cap C_a(Y) = \{0\}$.

Proof. Without loss of generality we can assume that M is an open subset of $\mathbb{C}^m$ and that $a = 0$.

First, suppose that X and Y are 1-separated at 0 and, by contradiction, that $v \in C_0(X) \cap C_0(Y)$, $v \neq 0$. This implies $(v, v) \in C_0(X \times Y) \cap \Delta_{\mathbb{C}^m}$ and so, by definition, there exist sequences $(x_\nu, y_\nu) \in X \times Y$ and $\lambda_\nu \in \mathbb{C}$ such that

$$x_\nu \to 0,\ y_\nu \to 0,\ \lambda_\nu(x_\nu, y_\nu) \to (v, v) \quad \text{as } \nu \to \infty .$$

Since X and Y are 1-separated, $|x_\nu - y_\nu| \geq C|x_\nu|$ for some $c > 0$ and sufficiently large ν. Then $|\lambda_\nu x_\nu - \lambda_\nu y_\nu| \geq C|\lambda x_\nu|$, which is impossible.

Next, if $C_0(X) \cap C_0(Y) = \{0\}$ then ([1], Th. 5.6) implies $\mathrm{i}\,(X \cdot Y; 0) = \deg_0 X \cdot \deg_0 Y$. By Corollary 3.3 we get $p_0(X, Y; 0) = 1$, which completes the proof.

We shall now construct an example showing that the estimate of p_0 presented in our basic Theorem 3.1 is optimal.

Example 3.5. Let $s \geq d \geq 1$ be integers. Define $M = \mathbb{C}^2$, $a = 0$ and

$$Z = \{(x, y) \in \mathbb{C}^2 : y^s + xy^{d-1} + x^d = 0\}, \qquad N = \{(x, y) \in \mathbb{C}^2 : x = 0\}.$$

Straightforward calculation yields that $\deg_0 Z = d$, $\mathrm{i}\,(Z \cdot N; 0) = s$ and

$$p_0 = p_0(Z, N; 0) = s - d + 1.$$

References

[1] R. Achilles, P. Tworzewski and T. Winiarski, *On improper isolated intersection in complex analytic geometry*, Ann. Polon. Math. 51 (1990), 21–36.

[2] R. Draper, *Intersection theory in analytic geometry*, Math. Ann. 180 (1969), 175–204.

[3] S. Łojasiewicz, *Ensembles semi-analytiques*, I.H.E.S., Bures-sur-Yvette, 1965.

[4] —, *Introduction to Complex Analytic Geometry*, Birkhäuser, Basel 1991.

[5] —, *Sur la séparation régulière*, Univ. Studi Bologna, Sem. Geom. 1985, 119–121.

[6] A. Płoski, *Multiplicity and the Łojasiewicz exponent*, preprint 359, Polish Academy of Sciences, Warszawa 1986.

[7] —, *Une évaluation pour les sous-ensembles analytiques complexes*, Bull. Polish Acad. Sci. Math. 31 (1983), 259–262.

[8] P. Tworzewski and T. Winiarski, *Analytic sets with proper projections*, J. Reine Angew. Math. 337 (1982), 68–76.

[9] T. Winiarski, *Continuity of total number of intersection*, Ann. Polon. Math. 47 (1986), 155–178.

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