COLLOQUIUM MATHEMATICUM

VOL. LXIV 1993 FASC. 1

A NOTE ON LACUNARY APPROXIMATION ON $[-1,1]$

BY

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1. Introduction. Denote by $C^N_{[-1,1]}$ the class of functions which have N continuous derivatives on the interval $[-1,1]$. Let Π_n be the set of algebraic polynomials of degree $\le n$, and

$$\Pi_n^k = \Big\{ f(x) = \sum_{j=0}^n a_j x^j : a_k = 0 \Big\} ,$$

where, here and throughout the paper, k is always a natural number. For $f \in C_{[-1,1]} := C^0_{[-1,1]}$, define

$$E_n(f) = \inf\{\|f-p\| : p \in \Pi_n\} = \inf\{ \max_{-1\le x\le 1} |f(x)-p(x)| : p \in \Pi_n\} ,$$

$$E_{n,k}(f) = \inf\{\|f-q\| : q \in \Pi_n^k\} = \|f(\cdot) - p_n^k(f,\cdot)\| .$$

Throughout the paper, we use $C(x)$ to indicate a positive constant depending upon x only, and C a positive absolute constant, which may be different in different relations.

The study of the approximation to continuous functions by lacunary general polynomials in $[a,1]$ for $a \ge 0$ started from the work of Müntz [7] in 1914, and great advance has been made in the field since then. There are many works concerning the Jackson type theorems for Müntz approximation.

On the other hand, several references [2]–[4], [6], [8]–[10] investigated the approximation of continuous functions on $[0,1]$ and $[-1,1]$ by elements from Π_n^k (actually, lacunary approximation on $[-1,1]$ is not a special case of the usual Müntz approximation). For instance, Hasson [2] proved that

$$E_{n,k}(x^k) \approx n^{-k} . \tag{1}$$

By applying (1) Hasson [2] established that if $f \in C^k_{[-1,1]}$, $f^{(k)}(0) \ne 0$, then

$$\lim_{n\to\infty} E_{n,k}(f)/E_n(f) = \infty .$$

(Lorentz [6] proved this result in a different way.)

1991 *Mathematics Subject Classification*: 41A10, 41A30.

Supported in part by an NSERC Postdoctoral Fellowship.

In estimating $E_{n,k}(f)/E_n(f)$ by smoothness of the function $f(x)$, Hasson conjectured that

Conjecture A. *If $f \in C_{[-1,1]}$ and f' does not exist at some interior point of $[-1,1]$, then*

$$\limsup_{n\to\infty} E_{n,k}(f)/E_n(f) < \infty\,.$$

Xu [8], Yang [9] and Zhou [10] gave negative answers to the above conjecture in different ways. Zhou proved

Theorem B. *There exist continuous and nowhere differentiable functions f on $[-1,1]$ such that*

$$\limsup_{n\to\infty} E_{n,k}(f)/E_n(f) = \infty\,.$$

Theorem C. *There exists an infinitely differentiable function f on $[-1,1]$ such that*

$$\limsup_{n\to\infty} E_{n,k}(f)/E_n(f) < \infty\,.$$

The above results thus indicate that the boundedness of the ratios $E_{n,k}(f)/E_n(f)$ is indeed irrelevant to smoothness of functions.

If we consider the relation between smoothness and lacunary approximation, a natural question arises if there are any Jackson type estimates for lacunary approximation. Note that all known results such as that of Hasson cited above (see also [4], [8]–[10]) require the condition $f^{(k)}(0) \neq 0$ (which makes things easier to deal with) and thus another natural question is what happens if we drop this condition.

The present paper will investigate those two questions.

Let $\omega_k^{\phi}(f,t)$ be the Ditzian–Totik modulus of smoothness of order k:

$$\omega_k^{\phi}(f,t) = \sup_{0<h\le t} \|\Delta_{h\phi}^k f(x)\|\,,$$

where $\phi(x) = \sqrt{1-x^2}$,

$$\Delta_{h\phi}^k f(x) = \begin{cases} \sum_{j=0}^{k}(-1)^j\binom{k}{j} f(x+(k/2-j)h\phi(x))\,, & \\ & x \pm kh\phi(x)/2 \in [-1,1], \\ 0, & \text{otherwise}, \end{cases}$$

and as usual, we denote by $\omega_k(f,t)$ the ordinary modulus of smoothness of order k:

$$\omega_k(f,t) = \sup\left\{\left|\Delta_h^k f(x) := \sum_{j=0}^{k}(-1)^j\binom{k}{j} f(x+jh)\right| : \right.$$
$$\left. 0 < h \le t,\ x \in [-1,1-kh]\right\}.$$

The main results of this note are the following:

THEOREM 1. *Let* $\phi(x)=\sqrt{1-x^2}$. *Then*

$$E_{n,k}(f)\le C(k)\omega_k^{\phi}(f,n^{-1})\,.$$

COROLLARY 1. *We have*

$$E_{n,k}(f)\le C(k)\omega_k(f,n^{-1})\,.$$

THEOREM 2. *Let* $0<\varrho<1$. *Then the set*

$$A:=\left\{f\in C_{[-1,1]}:\liminf_{h\to 0+}h^{-k}\Delta_h^k f(0)=0\,,\ \limsup_{n\to\infty}\frac{E_{n,k}(f)}{n^{\varrho}\omega_{k+1}(f,n^{-1})}=\infty\right.$$
$$\left.\text{and } \omega_{k+1}(f,t)>0 \text{ for } t>0\right\}$$

is residual in $C_{[-1,1]}$.

COROLLARY 2. *Let* $0<\varrho<1$. *Then the set*

$$\left\{f\in C_{[-1,1]}:\liminf_{h\to 0+}h^{-k}\Delta_h^k f(0)=0\,,\right.$$
$$\left.\limsup_{n\to\infty}\frac{E_{n,k}(f)}{n^{\varrho}\omega_{k+1}^{\phi}(f,n^{-1})}=\infty \text{ and } \omega_{k+1}(f,t)>0 \text{ for } t>0\right\}$$

is residual in $C_{[-1,1]}$.

We adopt the familiar categorical vocabulary as in Borwein [1]. A set is *nowhere dense* if the interior of its closure is empty. A set is *category* 1 if it is a countable union of nowhere dense sets. A set is *residual* if it is the complement of a category 1 set. So a residual set contains almost all functions from the Baire category point of view.

The following Corollary 3 improves Theorem B.

COROLLARY 3. *There exist continuous and nowhere differentiable functions* f *on* $[-1,1]$ *such that*

$$\liminf_{h\to 0+}h^{-k}\Delta_h^k f(0)=0 \quad\text{and}\quad \limsup_{n\to\infty}\frac{E_{n,k}(f)}{\omega_{k+1}(f,n^{-1})}=\infty\,.$$

Proof. This follows since the class of all continuous but nowhere differentiable functions is also residual in $C_{[-1,1]}$ (cf. [10]). ∎

2. Proof of Theorems 1 and 2

Proof of Theorem 1. We can prove Theorem 1 by following an idea of Leviatan [5] so we will only give a sketch here.

Given a function $f\in C^k_{[-1,1]}$ with

$$\|\phi^k f^{(k)}\|\le 1\,, \tag{2}$$

we find an ordinary polynomial $s_n(x)=\sum_{j=0}^{n} a_j x^j$ such that

$$\|f-s_n\| \le C(k)n^{-k}\,.$$

Then by applying (1), we have a polynomial $q_n \in \Pi_n^k$ such that

$$\|x^k-q_n(x)\| \le C(k)n^{-k}\,.$$

Set

$$Q_n(x)=\sum_{j=0,\ j\ne k}^{n} a_j x^j + a_k q_n(x)\,.$$

Then $Q_n \in \Pi_n^k$ and

$$\|f-Q_n\| \le C(k)n^{-k} \tag{3}$$

since $|a_k|=O(1)$ (by (2)). For any $f\in C_{[-1,1]}$ and $n\ge 1$, using the Peetre kernel K_ϕ^k we deduce that there exists a function $g\in C_{[-1,1]}^k$ with the properties

$$\|f-g\| \le C(k)\omega_k^\phi(f,n^{-1}) \quad \text{and} \quad n^{-k}\|\phi^k g^{(k)}\| \le C(k)\omega_k^\phi(f,n^{-1})\,.$$

Combining this with (3), we obtain the required result. ∎

Proof of Theorem 2. Let $0<\varrho<1$. Define

$$A_n=\Big\{f\in C_{[-1,1]} : \text{there is an } N\ge n \text{ such that } |\Delta_{N^{-1}}^k f(0)|<n^{-1}N^{-k}\,,\\ \frac{E_{N,k}(f)}{N^\varrho \omega_{k+1}(f,N^{-1})}>n \text{ and } \omega_{k+1}(f,t)>0 \text{ for } t>0\Big\}\,.$$

Then $A=\bigcap_{n=1}^\infty A_n$.

Since for any $g\in C_{[-1,1]}$,

$$E_{N,k}(g)=\|g-p_N^k(g)\| \ge \|f-p_N^k(g)\|-\|g-f\| \ge E_{N,k}(f)-\|g-f\|\,,$$

and evidently,

$$\omega_{k+1}(g,N^{-1}) \le \omega_{k+1}(f,N^{-1})+2^{k+1}\|g-f\|\,,$$

we see that obviously g is in A_n if $f\in A_n$ and g and f are close enough, and thus A_n is open for every $n=1,2,\dots$

Let

$$h_n^*(x)=\begin{cases} x^k\left(\exp\left(\dfrac{n^{-4}}{x^2-n^{-4}}\right)-1\right), & |x|<n^{-2},\\ -x^k\,, & n^{-2}\le |x|\le 1.\end{cases}$$

By calculation,

$$\|h_n^*\|=O(1)\,, \tag{4}$$

$$\|h_n^*(x)+x^k\| \approx n^{-2k}\,, \tag{5}$$

where $w_n \approx v_n$ indicates that there is a positive constant c independent of n such that $c^{-1}w_n \le v_n \le cw_n$. Hence for any $t(x) \in \Pi_n^k$ (that is, $t^{(k)}(0) = 0$), by (5) and applying a Bernstein type inequality we obtain

$$\begin{aligned} n^{-2k} &\le C\|h_n^*(x) + x^k\| \le C(k)n^{-2k}|k! + t^{(k)}(0)| \\ &\le C(k)n^{-2k}n^k\|t(x) + x^k\| \le C(k)n^{-k}(\|h_n^*(x) + x^k\| + \|h_n^* - t\|)\,, \end{aligned}$$

or in other words,

$$E_{n,k}(h_n^*) \ge C(k)n^{-k}\,. \tag{6}$$

For any given $f \in C_{[-1,1]}$, $0 < \varepsilon \le 1$, and sufficiently large $N \ge n$, we find a polynomial $p_N^k(f,x) \in \Pi_N^k$ such that

$$\|f - p_N^k(f)\| \le \varepsilon \tag{7}$$

(since polynomials p with $p^{(k)}(0) = 0$ are dense in the space of continuous functions on $[-1,1]$ by Theorem 1). Define

$$h_N(x) = \varepsilon h_{m_N}^*(x) + p_N^k(f,x)\,,$$

where

$$m_N = \max\{1, \|p_N^k(f)\|^{1/\theta}\}\varepsilon^{-1/\theta}N^{(2k+2)/\theta}\,, \qquad \theta = \frac{1-\varrho}{2}\,.$$

From (4), (7),

$$\|f - h_N\| \le \|f - p_N^k(f)\| + \varepsilon\|h_{m_N}^*\| = O(\varepsilon)\,. \tag{8}$$

It is easy to see that

$$E_{m_N,k}(h_N) = \varepsilon E_{m_N,k}(h_{m_N}^*)$$

since $p_N^k(f) \in \Pi_N^k$. By (6) it follows that

$$E_{m_N,k}(h_N) \ge C(k)\varepsilon m_N^{-k}\,. \tag{9}$$

On the other hand,

$$\begin{aligned} \omega_{k+1}(h_N, m_N^{-1}) &\le \varepsilon\omega_{k+1}(h_{m_N}^*, m_N^{-1}) + \omega_{k+1}(p_N^k(f), m_N^{-1}) \\ &= \varepsilon\omega_{k+1}(h_{m_N}^*(x) + x^k, m_N^{-1}) + \omega_{k+1}(p_N^k(f), m_N^{-1}) \\ &\le 2^{k+1}\varepsilon\|h_{m_N}^*(x) + x^k\| + C(k)N^{2k+2}\|p_N^k(f)\|m_N^{-k-1} \\ &\le 2^{k+1}\varepsilon m_N^{-2k} + C(k)\varepsilon m_N^{-k-1+\theta} = O(m_N^{-k-\varrho-\theta})\,. \end{aligned} \tag{10}$$

Estimates (9) and (10) give

$$\frac{E_{m_N,k}(h_N)}{m_N^{\varrho}\omega_{k+1}(h_N, m_N^{-1})} \ge C(k)m_N^{\theta}\,. \tag{11}$$

It is now obvious that

$$h_N^{(k)}(0) = 0\,, \tag{12}$$

$$\omega_{k+1}(h_N, t) > 0 \quad \text{for } t > 0\,. \tag{13}$$

So (11)–(13) imply that $h_N \in A_n$ for large enough N. By (8), we have proved that A_n is dense in $C_{[-1,1]}$.

The proof is complete. ■

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Reçu par la Rédaction le 27.3.1992