COLLOQUIUM MATHEMATICUM

VOL. LXIV 1993 FASC. 1

A CLASS OF FOURIER SERIES

BY

JAVAD NAMAZI (MADISON, NEW JERSEY)

Let $T = [0, 2\pi]$. Let f be a periodic function with period 2π in $L^1(T)$. Define $f_s(t) = f(t - s)$. We say that f satisfies the *L^p-Dini condition* if

$$\int_0^1 \frac{\omega(s)}{s}\, ds < \infty\,,$$

where

$$\omega(s) = \|f - f_s\|_{L^p(T)} = \left(\frac{1}{2\pi}\int_0^{2\pi} |f_s(t) - f(t)|^p\, dt\right)^{1/p}.$$

f is said to be a Lipschitz function ($f \in \mathrm{Lip}_\alpha(T)$) if

$$\omega^*(s) = \sup_{t\in T} |f_s(t) - f(t)| \le C s^\alpha\,,$$

for some $\alpha > 0$, and a constant C. The kth Fourier coefficient of f is $\widehat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} f(t) e^{-ikt}\, dt$, and the Fourier series of f is defined as $\sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ikt}$. Let $1 < p \le 2$, and let q be its conjugate exponent, that is, $q = p/(p-1)$. If $f \in L^p(T)$, then by the Hausdorff–Young theorem

$$\Big(\sum_{k=-\infty}^{\infty} |\widehat{f}(k)|^q\Big)^{1/q} \le \|f\|_{L^p(T)}\,.$$

Bernstein (see [1]) has shown that if $f \in \mathrm{Lip}_\alpha(T)$, for some $\alpha > 1/2$, then $\sum_{k=-\infty}^{\infty} |\widehat{f}(k)| < \infty$. However, for $\alpha = 1/2$, there exist functions whose Fourier series are not absolutely convergent. A classical example is the Hardy–Littlewood series

$$\sum_{n=1}^{\infty} \frac{e^{in\log n}}{n} e^{int}$$

1991 *Mathematics Subject Classification*: Primary 42A16.

(see [1], vol. 1, p. 197). A weaker condition holds for functions that satisfy an L^p-Dini condition.

THEOREM. *Suppose f satisfies the L^p-Dini condition*, $1 < p \leq 2$. *Then*

$$\sum_{k\neq 0} \frac{|\widehat{f}(k)|}{|k|^{1/p}} < \infty.$$

Proof. Let m be a non-negative integer. If $s_m = 2\pi/(3 \cdot 2^m)$ and $2^m \leq |k| < 2^{m+1}$, then

$$|e^{-iks_m} - 1| \geq \sqrt{3}\,. \tag{1}$$

Also the Fourier series of $f_{s_m} - f$ is $\sum_{k=-\infty}^{\infty} \widehat{f}(k)(e^{-iks_m} - 1)e^{ikt}$. Now

$$\begin{aligned}\sum_{2^m \leq |k| < 2^{m+1}} \frac{|\widehat{f}(k)|}{|k|^{1/p}} &\leq 2^{-(m+1)/p} \sum_{2^m \leq |k| < 2^{m+1}} |\widehat{f}(k)| \\ &\leq 2^{-(m+1)/p} \cdot 2^{(m+1)/p} \Big(\sum_{2^m \leq |k| < 2^{m+1}} |\widehat{f}(k)|^q \Big)^{1/q} \\ &= \Big(\sum_{2^m \leq |k| < 2^{m+1}} |\widehat{f}(k)|^q \Big)^{1/q},\end{aligned}$$

by Hölder's inequality together with the fact that there are never more than 2^{m+1} integers k with $2^m \leq |k| < 2^{m+1}$. Hence,

$$\begin{aligned}\sum_{2^m \leq |k| < 2^{m+1}} \frac{|\widehat{f}(k)|}{|k|^{1/p}} &\leq \Big(\sum_{2^m \leq |k| < 2^{m+1}} |\widehat{f}(k)(e^{-iks_m} - 1)|^q \Big)^{1/q} \\ &\leq \Big(\sum_{k=-\infty}^{\infty} |\widehat{f}(k)(e^{-iks_m} - 1)|^q \Big)^{1/q} \\ &\leq \|f_{s_m} - f\|_{L^p(T)} = \omega(s_m)\,,\end{aligned}$$

by the Hausdorff–Young theorem and (1). Therefore,

$$\sum_{k\neq 0} \frac{|\widehat{f}(k)|}{|k|^{1/p}} \leq \sum_{m\geq 0} \omega(s_m) = \sum_{m \geq 0} \frac{\omega(s_m)}{s_m} s_m \approx \int_0^1 \frac{\omega(s)}{s}\, ds < \infty\,,$$

since the last sum is a limit of Riemann sums for $\int_0^1 (\omega(s)/s)\, ds$.

We also note that since $\omega(s) \leq \omega^*(s)$, it follows that if f is Lipschitz then it is L^2-Dini, therefore it satisfies the conclusion of the theorem.

REFERENCES

[1] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge University Press, 1959.

FAIRLEIGH DICKINSON UNIVERSITY
MADISON, NEW JERSEY 07940
U.S.A.

Reçu par la Rédaction le 7.1.1992;
en version modifiée le 5.5.1992