COLLOQUIUM MATHEMATICUM

VOL. LXIV 1993 FASC. 1

ON MANIFOLDS ADMITTING METRICS WHICH ARE LOCALLY CONFORMAL TO COSYMPLECTIC METRICS: THEIR CANONICAL FOLIATIONS, BOOTHBY–WANG FIBERINGS, AND REAL HOMOLOGY TYPE

BY

MAURO CAPURSI (BARI) AND
SORIN DRAGOMIR (STONY BROOK, NEW YORK)

1. Introduction. The present paper builds on work by Z. Olszak [16]. There, locally conformal cosymplectic (l.c.c.) manifolds are defined to be almost contact metric (a.ct.m.) manifolds whose almost contact and fundamental forms η, Θ are subject to $d\eta = \frac{1}{2}\omega \wedge \eta$, $d\Theta = \omega \wedge \Theta$ for some closed 1-form ω and with a $(1,1)$-structure tensor φ integrable. The reason for which such manifolds are termed l.c.c. is that the metric of the underlying a.ct.m. structure appears to be conformal to a (local) cosymplectic metric in some neighborhood of each point of the manifold. Our results are organized as follows. Totally geodesic orientable real hypersurfaces M^{2n+1} of a locally conformal Kaehler (l.c.K.) manifold M^{2n+2} are shown to carry a naturally induced l.c.c. structure, provided the Lee field B_0 of M^{2n+2} is tangent to M^{2n+1}. The same conclusion occurs if M^{2n+1} is totally umbilical and its mean curvature vector is given by $H = -\frac{1}{2}\operatorname{nor}(B_0)$ (cf. our Theorem 7). In Section 3 we show that odd-dimensional real Hopf manifolds $\mathbb{R}H^{2n+1} \approx S^{2n} \times S^1$, $n \geq 2$, thought of as local similarity (l.s.) manifolds carrying the metric discovered by C. Reischer and I. Vaisman [19] turn out to be l.c.c. manifolds in a natural way, yet admit no globally defined cosymplectic metrics, by a result of D. E. Blair and S. Goldberg [3]. Leaving definitions momentarily aside, we may also state

THEOREM 1. *Each leaf of the canonical foliation Σ of a strongly non-cosymplectic l.c.c. manifold M^{2n+1} carries an induced (f, g, u, v, λ)-structure whose 1-form v is closed. If the characteristic 1-form ω of M^{2n+1} is parallel, then Σ has totally geodesic leaves. If moreover the local cosymplectic metrics g_i, $i \in I$, of M^{2n+1} are flat then the leaves of Σ are Riemannian manifolds of constant sectional curvature. If additionally M^{2n+1} is normal, then each complete leaf of Σ is holomorphically isometric to $\mathbb{C}P^n(c^2)$, $c = \frac{1}{2}\|\omega\|$.*

THEOREM 2. *Let M^{2n+1} be a compact normal l.c.c. manifold. If the structure vector ξ is regular then*:

(i) *M^{2n+1} is a principal S^1-bundle over $M^{2n} = M^{2n+1}/\xi$,*
(ii) *the almost contact 1-form η yields a flat connection 1-form on M^{2n+1},*
(iii) *the base manifold M^{2n} has a natural structure of Kaehlerian manifold.*

THEOREM 3. *Let M^{2n+1} be a connected compact orientable (strongly non-cosymplectic) l.c.c. manifold with a parallel characteristic 1-form ω and flat Weyl connection. Then the Betti numbers of M^{2n+1} are given by*:

$$b_0(M^{2n+1}) = b_{2n+1}(M^{2n+1}) = 1\,, \qquad b_1(M^{2n+1}) = b_{2n}(M^{2n+1}) = 1\,,$$
$$b_p(M^{2n+1}) = 0\,, \qquad 2 \le p \le 2n-1\,,$$

i.e. M^{2n+1} is a real homology real Hopf manifold.

In addition to (odd-dimensional) real Hopf manifolds, several examples of l.c.c. manifolds (such as real hypersurfaces of a complex Inoue surface endowed with the l.c.K. metric discovered by F. Tricerri [23]) are discussed in Section 7.

2. Conformal changes of almost contact metric structures. Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be an almost contact metric (a.ct.m.) manifold of (real) dimension $2n+1$ (cf. D. E. Blair [2], pp. 19–20). It is said to be *normal* if $N^1 = 0$, where $N^1 = [\varphi, \varphi] + 2d\eta \otimes \xi$. An a.ct.m. manifold is *cosymplectic* if it is normal and both the almost contact and fundamental forms are closed. See D. E. Blair [1], Z. Olszak [15], S. Tanno [22] for general properties of cosymplectic manifolds.

Let M^{2n+1} be an a.ct.m. manifold. Then M^{2n+1} is said to be *locally conformal cosymplectic* (*l.c.c.*) if there exists an open covering $\{U_i\}_{i\in I}$ of M^{2n+1} and a family $\{f_i\}_{i\in I}$, $f_i \in C^\infty(U_i)$, of real-valued smooth functions such that $(U_i, \varphi_i, \xi_i, \eta_i, g_i)$ is a cosymplectic manifold, where $\varphi_i = \varphi_{|U_i}$, $\xi_i = \exp(f_i/2)\xi_{|U_i}$, $\eta_i = \exp(-f_i/2)\eta_{|U_i}$, $g_i = \exp(-f_i)g_{|U_i}$, $i \in I$. Clearly, if M^{2n+1} is l.c.c. then φ is integrable.

Let M^{2n+1} be an a.ct.m. manifold and $f \in C^\infty(M^{2n+1})$ a smooth real-valued function on M^{2n+1}. A *conformal change* of the a.ct.m. structure (cf. I. Vaisman [25]) is a transformation of the form

$$\varphi_f = \varphi\,, \quad \xi_f = \exp\left(\frac{f}{2}\right)\xi\,, \quad \eta_f = \exp\left(-\frac{f}{2}\right)\eta\,, \quad g_f = \exp(-f)g\,. \tag{1}$$

The Riemannian connections of g, g_f are related by

$$\nabla^f_X Y = \nabla_X Y - \tfrac{1}{2}[X(f)Y + Y(f)X - g(X,Y)\,\mathrm{grad}(f)]\,, \tag{2}$$

where $\mathrm{grad}(f) = (df)^\sharp$ and $\sharp$ denotes raising of indices with respect to g.

Clearly $(M^{2n+1}, \varphi, \xi_f, \eta_f, g_f)$ is an a.ct.m. manifold and is cosymplectic iff $d\eta = \frac{1}{2} df \wedge \eta$, $d\Theta = df \wedge \Theta$, $[\varphi, \varphi] = 0$, where $\Theta(X, Y) = g(X, \varphi Y)$. We may establish the following:

LEMMA 4. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a cosymplectic manifold, $n \geq 1$. If the cosymplectic property is invariant by the transformation* (1) *then $df \equiv 0$ on M^{2n+1}.*

Proof. Note that (2) yields

$$(3)\qquad (\nabla^f_X \varphi)Y = (\nabla_X \varphi)Y + \tfrac{1}{2}[Y(f)\varphi X - (\varphi Y)(f)X + \Theta(X, Y)\,\mathrm{grad}(f) - g(X, Y)\varphi(\mathrm{grad}(f))]\,.$$

Since M^{2n+1} is cosymplectic it is normal, so that $N^1 = 0$. This yields $N^2 = 0$, where $N^2 = (\mathcal{L}_{\varphi X}\eta)Y - (\mathcal{L}_{\varphi Y}\eta)X$ (cf. [2], p. 50). Here $\mathcal{L}$ denotes the Lie derivative. Then $\nabla\varphi = 0$, by [2], p. 53. Now, by (3) we obtain

$$(4)\qquad Y(f)\varphi X + \Theta(X, Y)\,\mathrm{grad}(f) = (\varphi Y)(f)X + g(X, Y)\varphi(\mathrm{grad}(f))\,.$$

Let $X = Y = \xi$ in (4). Then $\varphi(\mathrm{grad}(f)) = 0$. Use this to modify (4) and apply φ to the resulting equation. This yields $Y(f)\varphi^2 X = (\varphi Y)(f)\varphi X$. Take the inner product with $\varphi^2 X$ to get $Y(f)\|\varphi^2 X\|^2 = 0$. Finally, replace X by φX; as φ is an f-structure (in the sense of [26], p. 379), $\mathrm{rank}(\varphi) = 2n$, $n \geq 1$, so that $Y(f) = 0$ for any Y. ∎

THEOREM 5. *Let $(M^{2n+1}, \varphi, \xi, \eta, g)$ be a l.c.c. manifold. Then for any $i, j \in I$, $i \neq j$, with $U_i \cap U_j \neq \emptyset$, one has $df_i = df_j$ on $U_i \cap U_j$; therefore the* (*local*) *1-forms df_i glue up to a globally defined* (*closed*) *1-form ω. Also the Riemannian connections ∇^{f_i} of (U_i, g_i), $i \in I$, glue up to a globally defined torsion-free linear connection D on M^{2n+1} expressed by*

$$(5)\qquad D_X Y = \nabla_X Y - \tfrac{1}{2}[\omega(X)Y + \omega(Y)X - g(X, Y)B]\,,$$

where $B = \omega^\sharp$ and ∇ is the Levi-Civita connection of (M^{2n+1}, g).

Proof. Let $U_{ij} = U_i \cap U_j$, $i \neq j$, $i, j \in I$, $U_{ij} \neq \emptyset$. Then both $(\varphi, \xi_i, \eta_i, g_i)$, $(\varphi, \xi_j, \eta_j, g_j)$ are cosymplectic structures on U_{ij} and are related by a conformal transformation (1) with $f = f_j - f_i$; thus one may apply Lemma 4.

The 1-form ω furnished by Theorem 5 is referred to as the *characteristic 1-form* of M^{2n+1}; also B is the *characteristic field* and D the *Weyl connection*. Since $d\eta_i = 0$, $d\Theta_i = 0$, $i \in I$, where Θ_i denotes the fundamental 2-form of $(\varphi, \xi_i, \eta_i, g_i)$, it follows that

$$(6)\qquad d\eta = \tfrac{1}{2}\omega \wedge \eta\,, \qquad d\Theta = \omega \wedge \Theta\,.$$

Also, for any l.c.c. manifold, $[\varphi, \varphi] = 0$. Conversely, any a.ct.m. manifold M^{2n+1} satisfying (6) for some closed 1-form ω and with φ integrable is l.c.c.

If $\omega \equiv 0$ then M^{2n+1} is a cosymplectic manifold. If ω has no singular points, M^{2n+1} is termed *strongly non-cosymplectic*.

3. Odd-dimensional real Hopf manifolds. A *similarity transformation* of $\mathbb{R}^n$ is given by

$$x'^i = \varrho a^i_j x^j + b^i \,, \tag{7}$$

where $\varrho > 0$ and $[a^i_j] \in O(n)$. A manifold M^n is a *local similarity* (*l.s.*) manifold if it possesses a smooth atlas whose transition functions have the form (7) (see [19]). Let $0 < \lambda < 1$ be fixed. Let Δ_λ be the cyclic group generated by the transformation $x'^i = \lambda x^i$ of $\mathbb{R}^n - \{0\}$. Then $\mathbb{R}H^n = (\mathbb{R}^n - \{0\})/\Delta_\lambda$ is the *real Hopf manifold*. Define a diffeomorphism $f : \mathbb{R}H^n \to S^{n-1} \times S^1$ by setting:

$$f([x]) = \left(\frac{x^1}{|x|}, \dots, \frac{x^n}{|x|}, \exp\left(\sqrt{-1}\frac{2\pi \log|x|}{\log\lambda}\right)\right)$$

for any $[x] \in \mathbb{R}H^n$. Here $[x] = \pi(x)$, $x = (x^1, \dots, x^n)$, $x \in \mathbb{R}^n - \{0\}$, $|x|^2 = \sum_{i=1}^n (x^i)^2$ and $\pi : \mathbb{R}^n - \{0\} \to \mathbb{R}H^n$ denotes the natural projection. Then $\mathbb{R}H^n$, $n > 1$, is a compact connected l.s. manifold (with transition functions $x'^i = \lambda x^i$). Let us endow $\mathbb{R}^{2n+1} - \{0\}$ with the metric

$$ds^2 = (|x|^2 + t^2)^{-1}\{\delta_{ij} dx^i \otimes dx^j + dt^2\} \tag{8}$$

where (x^i, t), $1 \le i \le 2n$, are the natural coordinates (cf. (4.4) in [19], p. 287). As (8) is invariant under any transformation

$$x'^i = \lambda^m x^i \,, \quad m \in \mathbb{Z}\,, \tag{9}$$

it gives a globally defined metric g_0 on $\mathbb{R}H^{2n+1}$. We organize $\mathbb{R}H^{2n+1}$ into a l.c.c. manifold as follows. Let $\sigma = \log\{|x|^2 + t^2\}$. One may endow $\mathbb{R}^{2n+1} = \mathbb{R}^{2n} \times \mathbb{R}^1$ with a cosymplectic structure (cf. Z. Olszak [15], p. 241). Namely, let $g = \delta_{ij} dx^i \otimes dx^j + dt^2$ be the product metric on $\mathbb{R}^{2n+1}$. Let $\varphi(X + f\partial/\partial t) = JX$, where X is tangent to $\mathbb{R}^{2n}$ and $f \in C^\infty(\mathbb{R}^{2n+1})$. Here J denotes the canonical complex structure of $\mathbb{R}^{2n} \approx \mathbb{C}^n$. Also set $\eta(X + f\partial/\partial t) = f$. Then (φ, ξ, η, g), $\xi = \partial/\partial t$, is a cosymplectic structure on $\mathbb{R}^{2n+1}$. Note that $e^{\sigma/2}\xi$, $e^{-\sigma/2}\eta$ and (as noticed above) $e^{-\sigma} g$ are invariant under any transformation (9). Therefore $\mathbb{R}H^{2n+1}$ inherits a l.c.c. structure $(\varphi_0, \xi_0, \eta_0, g_0)$. Furthermore, by Proposition 3.5 in [19], p. 286, any orientable compact l.s. manifold of dimension $m \ge 3$ is a real homology real Hopf manifold, i.e. it has the Betti numbers $b_0 = b_1 = b_{m-1} = b_m = 1$ and $b_p = 0$ for $2 \le p \le m - 2$. By a theorem of D. E. Blair and S. Goldberg (Th. 2.4, in [3], p. 351), the Betti numbers of a compact cosymplectic manifold are non-zero. Combining the above statements one obtains in particular

THEOREM 6. *Any odd-dimensional real Hopf manifold $\mathbb{R}H^{2n+1}$, $n \geq 2$, has a natural structure of l.c.c. manifold but admits no globally defined cosymplectic metrics. The Weyl connection of $\mathbb{R}H^{2n+1}$ is flat and its characteristic form $\omega = d\sigma$ is parallel with respect to the Levi-Civita connection of $(\mathbb{R}H^{2n+1}, g_0)$.*

4. Real hypersurfaces of a locally conformal Kaehler manifold. Let (M^{2n+2}, g_0, J) be a locally conformal Kaehler (l.c.K.) manifold, with the complex structure J and the Hermitian metric g_0 (cf. e.g. P. Libermann [14]). Let M^{2n+1} be an orientable real hypersurface of M^{2n+2}. Given a unit normal field N on M^{2n+1}, we put as usual $\xi = -JN$. Set $\varphi X = \tan(JX)$, $FX = \mathrm{nor}(JX)$, for any tangent vector field X on M^{2n+1}. Here $\tan_x$, nor_x denote the natural projections associated with the direct sum decomposition $T_x(M^{2n+2}) = T_x(M^{2n+1}) \oplus E_x$, $x \in M^{2n+1}$. Also $E \to M^{2n+1}$ is the normal bundle of $\iota : M^{2n+1} \subset M^{2n+2}$. Let $\eta(X) = g_0(FX, N)$. Let $g = \iota^* g_0$ be the induced metric. By a result of [2], p. 30, (φ, ξ, η, g) is an a.ct.m. structure on M^{2n+1}. Let $\omega_0 = (1/n) i(\Omega) d\Omega$. Here $i(\Omega)$ denotes the adjoint (with respect to g_0) of $e(\Omega)$, where $e(\Omega)\lambda = \Omega \wedge \lambda$, for any differential form λ on M^{2n+2}, while Ω is the Kaehler 2-form of M^{2n+2}. Then $d\omega_0 = 0$, $d\Omega = \omega_0 \wedge \Omega$ (see e.g. [24]). Let $\omega = \iota^* \omega_0$. Let Θ be the fundamental form of the a.ct.m. structure (φ, ξ, η, g). Clearly $\Theta = \iota^* \Omega$. Thus

$$d\Theta = \omega \wedge \Theta\,, \qquad d\omega = 0\,. \tag{10}$$

We recall the Gauss–Weingarten formulae:

$$\nabla^0_X Y = \nabla_X Y + g(AX, Y) N\,, \qquad \nabla^0_X N = -AX\,, \tag{11}$$

where A denotes the shape operator of ι, while ∇ is the induced connection. Then (11) leads to

$$\begin{aligned}(\nabla_X \varphi) Y = {} & \eta(Y) AX - g(AX, Y)\xi \\ & + \tfrac{1}{2}\{\omega(\varphi Y) X - \omega(Y)\varphi X + g(X, Y)\varphi B - \Theta(X, Y) B \\ & + \omega_0(N)[\eta(Y) X - g(X, Y)\xi]\}\,.\end{aligned} \tag{12}$$

Here $B = \tan(B_0)$, $B_0 = \omega_0^\sharp$ (indices being raised with respect to g_0). Moreover,

$$\begin{aligned}(\nabla_X \eta) Y = {} & -\Theta(AX, Y) \\ & + \tfrac{1}{2}[g(X, Y)\omega(\xi) - \Theta(X, Y)\omega_0(N) - \eta(X)\omega(Y)]\,.\end{aligned} \tag{13}$$

As ∇ is torsion free, (13) leads to

$$\begin{aligned}2(d\eta)(X, Y) = {} & (\omega \wedge \eta)(X, Y) - \Theta(AX, Y) \\ & - \Theta(X, AY) - \Theta(X, Y)\omega_0(N)\,.\end{aligned} \tag{14}$$

Also (12) gives

$$[\varphi, \varphi](X, Y) = \eta(Y)[A, \varphi]X - \eta(X)[A, \varphi]Y \\ - \{g((A\varphi + \varphi A)X, Y) - \Theta(X, Y)\omega_0(N)\}\xi\,. \tag{15}$$

As an application of (14)–(15) one obtains

Theorem 7. *Let M^{2n+1} be a real hypersurface of the l.c.K. manifold M^{2n+2}, and assume that either M^{2n+1} is totally umbilical and its mean curvature vector satisfies $H = -\frac{1}{2}B^{\perp}$, $B^{\perp} = \mathrm{nor}(B_0)$, or M^{2n+1} is totally geodesic and tangent to the Lee field B_0 of M^{2n+2}. Then (φ, ξ, η, g) is a l.c.c. structure on M^{2n+1}.*

Let $\mathbb{C}H^{2n+1} \approx S^{2n+1} \times S^1$ be the complex Hopf manifold (cf. [13], Vol. II, p. 137) carrying the l.c.K. metric g_0 induced by the (G_d-invariant) metric $ds^2 = |x|^{-2}\delta_{ij}dx^i \otimes dx^j$, where $(x^1, \dots, x^{2n+2})$ are the natural (real-analytic) coordinates on $\mathbb{C}^{n+1}$. Here $G_d = \{d^m I : m \in \mathbb{Z}\}$, $d \in \mathbb{C} - \{0\}$, $|d| \neq 1$, while I is the identical transformation of $\mathbb{C}^{n+1} - \{0\}$. Let $\pi : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}H^{n+1}$ be the natural map (a local diffeomorphism). Let $\iota : M^{2n+1} \to (\mathbb{C}^{n+1} - \{0\}, \delta_{ij})$ be an orientable totally geodesic real hypersurface. Then $\psi : M^{2n+1} \to \mathbb{C}H^{n+1}$, $\psi = \pi \circ \iota$, is totally umbilical. Indeed, let h, h' be the second fundamental forms of M^{2n+1} in $(\mathbb{C}^{n+1}, |x|^{-2}\delta_{ij})$ and $(\mathbb{C}^{n+1}, \delta_{ij})$, respectively. Let g be the metric induced on M^{2n+1} by $|x|^{-2}\delta_{ij}$. Then ψ *is an isometric immersion of* (M^{2n+1}, g) *in* $(\mathbb{C}H^{n+1}, g_0)$. Let $B^{\perp}$ be the normal component of $-2x^i\partial/\partial x^i$ (with respect to M^{2n+1}). Then

$$2h' = 2h + g \otimes B^{\perp}\,. \tag{16}$$

Now (16) and $h' = 0$ give $h = g \otimes H$, $2H = -B^{\perp}$, i.e. $M^{2n+1} \to (\mathbb{C}^{n+1} - \{0\}, |x|^{-2}\delta_{ij})$ is totally umbilical. Let $\overline{\nabla}$ be the Riemannian connection of $|x|^{-2}\delta_{ij}$. For any tangent vector fields X, Y on $\mathbb{C}^{n+1}$ one has $\nabla^0_{\pi_* X}\pi_* Y = \pi_* \overline{\nabla}_X Y$ (cf. [13], Vol. I, p. 161). Thus $h_\psi = \pi_* h$, where h_ψ is the second fundamental form of ψ. Also (16) yields

$$H' = \exp(f)\{H + \tfrac{1}{2}B^{\perp}\}\,, \tag{17}$$

where f is the restriction to M^{2n+1} of $\log|x|^{-2}$. Thus (17) gives $h_\psi = g \otimes H_\psi$, i.e. ψ *is totally umbilical*. We may apply Theorem 7 to $M^{2n+1} \to \mathbb{C}H^{n+1}$ to conclude that M^{2n+1} inherits a l.c.c. structure.

5. The canonical foliation of a locally conformal cosymplectic manifold. Let M^{2n} be a real $2n$-dimensional differentiable manifold. An (f, g, u, v, λ)-structure on M^{2n} consists of a $(1,1)$-tensor field F, a Riemannian metric G, two 1-forms u, v and a smooth real-valued function

$\lambda \in C^\infty(M^{2n})$ subject to:

$$
(18)\qquad \begin{gathered}
f^2 = -I + u \otimes U + v \otimes V\,, \\
u \circ f = \lambda v\,, \quad v \circ f = -\lambda u\,, \quad fU = -\lambda V\,, \quad fV = \lambda U\,, \\
u(V) = v(U) = 0\,, \quad u(U) = v(V) = 1 - \lambda^2\,, \\
g(fX, fY) = g(X,Y) - u(X)u(Y) - v(X)v(Y)\,,
\end{gathered}
$$

where $U = u^\sharp$, $V = v^\sharp$ (raising of indices is performed with respect to g) (see [26], p. 386).

Let $(M^{2n+1}, \varphi, \xi, \eta, g_0)$ be a strongly non-cosymplectic manifold with characteristic 1-form ω. Then M^{2n+1} admits a canonical foliation Σ whose leaves are the maximal connected integral manifolds of the Pfaffian equation $\omega = 0$.

Now we may prove Theorem 1. To this end, let M^{2n} be a leaf of Σ. Let $B_0 = \omega^\sharp$ be the characteristic field of M^{2n+1}. Then $C = \|\omega\|^{-1}B_0$ is a unit normal vector field on M^{2n}. Let X be tangential and set $fX = \tan(\varphi X)$, $u(X) = g_0(\varphi X, C)$, $v(X) = \eta(X)$, $\lambda = \eta(C)$. Then M^{2n} inherits an obvious (f, g, u, v, λ)-structure, where g is the induced metric, while $V = \tan(\xi)$, $U = -\varphi C$. Since $\omega = 0$ on $T(M^{2n})$ by (6) one has $dv = 0$.

Let D^0 be the Weyl connection of M^{2n+1} and K_0 its curvature tensor field. As a consequence of (5) one has

$$
(19)\qquad \begin{aligned}
K_0(X,Y)Z &= R_0(X,Y)Z - \tfrac14\|\omega\|^2(X\wedge Y)Z \\
&\quad -\tfrac12\{L(X,Z)Y - L(Y,Z)X + g_0(X,Z)L(Y,\cdot)^\sharp - g_0(Y,Z)L(X,\cdot)^\sharp\}\,.
\end{aligned}
$$

Here R_0 denotes the curvature of (M^{2n+1}, g_0) and

$$
\begin{aligned}
L(X,Y) &= (\nabla^0_X\omega)Y + \tfrac12\omega(X)\omega(Y)\,, \\
(X\wedge Y)Z &= g_0(Y,Z)X - g_0(X,Z)Y\,.
\end{aligned}
$$

Let $K_0 = 0$; apply (19) and the Gauss equation of $M^{2n} \to M^{2n+1}$ to obtain

$$
(20)\qquad \begin{aligned}
R(X,Y)Z = {}& \tfrac14\|\omega\|^2(X\wedge Y)Z + (AX\wedge AY)Z \\
&+ \tfrac12\{\omega(h(Y,Z))X - \omega(h(X,Z))Y\} \\
&+ \tfrac12\|\omega\|\{g(Y,Z)AX - g(X,Z)AY\}\,.
\end{aligned}
$$

As Σ has codimension 1 and ω is parallel, $h = 0$ and (20) gives $R(X,Y) = c^2 X\wedge Y$, $c = \frac12\|\omega\|$, i.e. M^{2n} is an elliptic space-form. To prove the last statement in Theorem 1, assume M^{2n+1} is normal. Then $\omega = 2\lambda c\eta$; as $\eta(C) = \lambda$, this yields $\lambda^2 = 1$. Then (18) gives $u = 0$, $v = 0$, $f^2 = -I$ and M^{2n} turns out to be an almost Hermitian manifold. Moreover, $[\varphi,\varphi] = 0$, $u = 0$ lead to $[f,f] = 0$. Let Ω be the Kaehler 2-form of M^{2n}. By (6), $d\Omega = 0$, i.e. M^{2n} is Kaehlerian. Suppose M^{2n} is complete. Then $\pi_1(M^{2n}) = 0$, by a classical result in [20] and one may apply Th. 7.9 in [13], Vol. II, p. 170. ■

6. Regular locally conformal cosymplectic manifolds. A l.c.c. manifold M^{2n+1} with the characteristic 1-form ω is *normal* iff

$$\omega = \omega(\xi)\eta\,. \tag{21}$$

The structure vector ξ is *regular* if it defines a regular foliation (i.e. each point of M^{2n+1} admits a flat coordinate neighborhood, say (U, x^i, t), $1 \le i \le 2n$, which intersects the orbits of ξ in at most one slice $x^i = \text{const.}$, cf. [18]). By (21), if M^{2n+1} is strongly non-cosymplectic, then ξ is regular iff $B = \omega^\sharp$ is regular.

Let M^{2n+1} be compact; then ξ is complete (cf. [13], Vol. I, p. 14). Let $P(\xi)$ be the period function of ξ, $P(\xi)_x \neq 0$, $x \in M^{2n+1}$ (see [5], pp. 722–723). The global 1-parameter transformation group of $P(\xi)^{-1}\xi$, $P(P(\xi)^{-1}\xi) = 1$, induces a free action of S^1 on M^{2n+1}. By standard arguments (cf. [5], p. 725, [4], p. 178, and [2], p. 15), $M^{2n+1}(M^{2n}, \pi, S^1)$ is a principal S^1-bundle over the space of orbits $M^{2n} = M^{2n+1}/\xi$. By a result in [21], p. 236, as $\eta(\xi) = 1$ and $\mathcal{L}_\xi \eta = 0$ it follows that $P(\xi) = \text{const.}$ Thus $\mathcal{L}_{P(\xi)^{-1}\xi}\eta = 0$ and therefore η is invariant under the action of S^1. Now we may prove Theorem 2. Clearly ξ is vertical, i.e. tangent to the fibres of π. Let $A \in L(S^1)$ be the unique left invariant vector field on S^1 with $A^* = \xi$. (Here A^* denotes the fundamental vector field on M^{2n+1} associated with A, cf. [13], Vol. I, p. 51). Let $\overline{\eta} = \eta \otimes A$. Then $\overline{\eta}$ is a connection 1-form on M^{2n+1}. Let $H = \operatorname{Ker}(\overline{\eta})$. By normality $N^3 = 0$, where $N^3 = \mathcal{L}_\xi \varphi$ (see [2], p. 50). Thus φ commutes with right translations. Consequently, $J_p Z_p = (d_x\pi)\varphi_x Z_x^H$, $x \in \pi^{-1}(p)$, $p \in M^{2n}$, $Z \in T_p(M^{2n})$, is a well defined complex structure on M^{2n}. (Here Z^H denotes the horizontal lift of Z (with respect to $\overline{\eta}$).) Let $\overline{g}(Z, W) = g(Z^H, W^H)$. By (21), $\omega = 0$ on H and thus $(M^{2n}, \overline{g}, J)$ is Kaehlerian. ■

Remark. M^{2n} carries the Riemannian metric $\overline{g}$, so it is paracompact. By [13], Vol. I, p. 92, as $\overline{\eta}$ is flat, if $\pi_1(M^{2n}) = 0$ then $M^{2n+1} \approx M^{2n} \times S^1$ (i.e. M^{2n+1} is the trivial S^1-bundle).

7. Submanifolds of complex Inoue surfaces. Let $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ be the upper half of the complex plane. Let (z, w) be the natural complex coordinates on $\mathbb{C}^+ \times \mathbb{C}$. We endow $\mathbb{C}^+ \times \mathbb{C}$ with the Hermitian metric

$$ds^2 = y^{-2} dz \otimes d\overline{z} + y\, dw \otimes d\overline{w}\,, \tag{22}$$

where $z = x + iy$, $i = \sqrt{-1}$. Then (22) makes $\mathbb{C}^+ \times \mathbb{C}$ into a globally conformal Kaehlerian manifold with the Lee form $\omega = y^{-1}dy$. Let $A \in \mathrm{SL}(3, \mathbb{Z})$ with a real eigenvalue $\alpha > 0$ and two complex eigenvalues

$\beta \neq \overline{\beta}$. Let (a_1, a_2, a_3), (b_1, b_2, b_3) be respectively a real eigenvector and an eigenvector corresponding to α, β. Let G_A be the discrete group generated by the transformations f_α, $\alpha = 0, 1, 2, 3$, where $f_0(z, w) = (\alpha z, \beta w)$, $f_i(z, w) = (z + a_i, w + b_i)$, $i = 1, 2, 3$. Then G_A acts freely and properly discontinuously on $\mathbb{C}^+ \times \mathbb{C}$ so that $\mathbb{C}I^2 = (\mathbb{C}^+ \times \mathbb{C})/G_A$ becomes a (compact) complex surface. This is the *Inoue surface* (cf. [12]). It was observed in [23], p. 84, that (22) is G_A-invariant. Thus $\mathbb{C}I^2$ turns out to be a l.c.K. manifold with a non-parallel Lee form (see Prop. 2.4 of [23], p. 85). Let $\pi : \mathbb{C}^+ \times \mathbb{C} \to \mathbb{C}I^2$ be the natural projection. Let $\iota : M \subset \mathbb{C}^+ \times \mathbb{C}$ be a submanifold and g the metric induced by (22). Then $\psi : M \to \mathbb{C}I^2$, $\psi = \pi \circ \iota$, is an isometric immersion of (M, g) into $\mathbb{C}I^2$.

It is our purpose to build examples of (immersed) submanifolds of $\mathbb{C}I^2$ (and motivate the results in Section 4). Let $w = a + ib$; we set $X = \partial/\partial x$, $Y = \partial/\partial y$, $A = \partial/\partial a$, $B = \partial/\partial b$. The real components of (22) are:

$$g_0 : \quad \begin{pmatrix} y^{-2} & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & y^{-2} & 0 \\ 0 & 0 & 0 & y \end{pmatrix} .$$

Thus the non-zero Christoffel symbols of the Levi-Civita connection ∇^0 of $\mathbb{C}I^2$ are

$$\begin{gathered} \Gamma^1_{13} = \Gamma^3_{33} = -\Gamma^3_{11} = -y^{-1} , \\ \Gamma^2_{23} = \Gamma^4_{34} = \tfrac{1}{2} y^{-1} , \qquad \Gamma^3_{22} = \Gamma^3_{44} = -\tfrac{1}{2} y^2 . \end{gathered} \tag{23}$$

The Lee field of $\mathbb{C}I^2$ is (locally) given by $\mathcal{L} = yY$. Let $L^h = \{z \in \mathbb{C}^+ : \mathrm{Im}(z) = 1\}$ and $\iota : L^h \times \mathbb{C} \to \mathbb{C}^+ \times \mathbb{C}$ the natural inclusion. The tangent space at a point of $L^h \times \mathbb{C}$ is spanned by X, A and B. Then $N = yY$ is a unit normal vector field on $L^h \times \mathbb{C}$. By (23) one obtains

$$\nabla^0_X N = -X , \quad \nabla^0_A N = \tfrac{1}{2} A , \quad \nabla^0_B N = \tfrac{1}{2} B . \tag{24}$$

Let a_N be the shape operator of $\psi : L^h \times \mathbb{C} \to \mathbb{C}I^2$, $\psi = \pi \circ \iota$. Then $\mathrm{Trace}(a_N) = 0$, i.e. ψ *is minimal*. Clearly $L^h \times \mathbb{C}$ is a maximal connected integral manifold of the Pfaff equation $y^{-1} dy = 0$, i.e. a leaf of the canonical foliation of the (strongly non-Kaehler) l.c.K. manifold $\mathbb{C}I^2$, and therefore normal to $\mathcal{L}$.

Let $L^v = \{z \in \mathbb{C}^+ : \mathrm{Re}(z) = 0\}$ and $\iota : L^v \times \mathbb{C} \to \mathbb{C}^+ \times \mathbb{C}$ the inclusion. Tangent spaces at points of $L^v \times \mathbb{C}$ are spanned by A, Y, B, and $N = yX$ is a unit normal. By (23),

$$\begin{gathered} \nabla^0_A A = -\tfrac{1}{2} y^2 Y , \quad \nabla^0_A Y = \tfrac{1}{2} y^{-1} A , \quad \nabla^0_A B = 0 , \\ \nabla^0_Y Y = -y^{-1} Y , \quad \nabla^0_Y B = \tfrac{1}{2} y^{-1} B , \quad \nabla^0_B B = -\tfrac{1}{2} y^2 Y . \end{gathered} \tag{25}$$

Consequently, $\psi : L^v \times \mathbb{C} \to \mathbb{C}I^2$, $\psi = \pi \circ \iota$, *is a totally geodesic immersion.*

Clearly $L^v \times \mathbb{C}$ is tangent to $\mathcal{L}$ and inherits a l.c.c. structure (via our Theorem 7). Both $L^h \times \mathbb{C}$ and $L^v \times \mathbb{C}$ are generic, as real hypersurfaces of $\mathbb{C}I^2$.

8. Betti numbers of locally conformal cosymplectic manifolds. Let M^{2n+1} be a l.c.c. manifold with $\nabla\omega = 0$, $K = 0$ (i.e. having a flat Weyl connection). Set $\|\omega\| = 2c$, $c > 0$. By (19) the curvature of M^{2n+1} has the expression

$$\begin{aligned} R^m_{ijk} &= c^2\{g_{jk}\delta^m_i - g_{ik}\delta^m_j\} \\ &\quad + \tfrac{1}{4}\{(\omega_i\delta^m_j - \omega_j\delta^m_i)\omega_k + (g_{ik}\omega_j - g_{jk}\omega_i)B^m\}\,. \end{aligned} \tag{26}$$

Suitable contraction of indices in (26) gives the Ricci curvature

$$R_{jk} = (2n-1)\{c^2 g_{jk} - \tfrac{1}{4}\omega_j\omega_k\}\,. \tag{27}$$

If $\alpha = (1/p!)\alpha_{i_1\ldots i_p}dx^{i_1} \wedge \ldots \wedge dx^{i_p}$ is a differential p-form on M^{2n+1}, we consider the quadratic form

$$F_p(\alpha) = R_{ij}\alpha^{ii_2\ldots i_p}\alpha^j_{i_2\ldots i_p} - \tfrac{1}{2}(p-1)R_{ijkm}\alpha^{iji_3\ldots i_p}\alpha^{km}_{i_3\ldots i_p}$$

(cf. [10], p. 88). Then (26)–(27) lead to

$$R_{ij}\alpha^{ii_2\ldots i_p}\alpha^j_{i_2\ldots i_p} = (2n-1)\{c^2 p!\|\alpha\|^2 - \tfrac{1}{4}(p-1)!\|\iota_B\alpha\|^2\}\,, \tag{28}$$

$$R_{ijkm}\alpha^{iji_3\ldots i_p}\alpha^{km}_{i_3\ldots i_p} = 2c^2 p!\|\alpha\|^2 - (p-1)!\|\iota_B\alpha\|^2\,, \tag{29}$$

where ι_B denotes interior product with B.

Now we may prove our Theorem 3. Let α be a harmonic p-form on M^{2n+1}. By (3.2.9) in [10], p. 88, it follows that

$$\int_M \{pF_p(\alpha) + \nabla_i\alpha_{i_1\ldots i_p}\nabla^i\alpha^{i_1\ldots i_p}\} * 1 = 0\,. \tag{30}$$

On the other hand, by (28)–(29),

$$F_p(\alpha) = c^2\{p!(2n-p)\|\alpha\|^2 + (p-1)!(2p-2n-1)\|\iota_U\alpha\|^2\}\,, \tag{31}$$

where $U = \|\omega\|^{-1}B$. Hence, if $n+1 \le p \le 2n-1$, then $b_p(M^{2n+1}) = 0$ (cf. our (30)–(31)). By Poincaré duality one also has $b_p(M^{2n+1}) = 0$ when $2 \le p \le n$. Since ω is parallel, it is harmonic. Thus $b_1(M^{2n+1}) = b_{2n}(M^{2n+1}) \ge 1$ (as $c \ne 0$). To compute the first Betti number of M^{2n+1}, let σ be a harmonic 1-form. Then $*\sigma$ is a harmonic $2n$-form, where $*$ denotes the Hodge operator. Then (31) leads to

$$F_{2n}(*\sigma) = c^2(2n-1)!(2n-1)\|\iota_U(*\sigma)\|^2$$

and thus $\iota_U(*\sigma) = 0$, by (30). By applying once more the Hodge operator one has $u \wedge \sigma = 0$ or $\sigma = fu$ for some nowhere vanishing $f \in C^\infty(M^{2n+1})$. Here $u = \|\omega\|^{-1}\omega$. As σ is harmonic, it is closed, so that $df \wedge u = 0$ or $df = \lambda v$ for some $\lambda \in C^\infty(M^{2n+1})$. But σ is also coclosed, so that

$(df,\sigma)=(f,\delta\sigma)=0$ (by (2.9.3) in [10], p. 74), i.e. df and σ are orthogonal. Thus $0=(df,\sigma)=\lambda f\,\mathrm{vol}(M^{2n+1})$ yields $\lambda=0$. As M^{2n+1} is connected one obtains $f=\mathrm{const.}$, i.e. $b_1(M^{2n+1})=1$.

REFERENCES

[1] D. E. Blair, *The theory of quasi-Sasakian structures*, J. Differential Geom. 1 (1967), 331–345.

[2] —, *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Math. 509, Springer, 1976.

[3] D. E. Blair and S. Goldberg, *Topology of almost contact manifolds*, J. Differential Geom. 1 (1967), 347–354.

[4] D. E. Blair, C. D. Ludden and K. Yano, *Differential geometric structures on principal toroidal bundles*, Trans. Amer. Math. Soc. 181 (1973), 175–184.

[5] W. M. Boothby and H. C. Wang, *On contact manifolds*, Ann. of Math. (3) 68 (1958), 721–734.

[6] S. Dragomir, *On submanifolds of Hopf manifolds*, Israel J. Math. (2) 61 (1988), 98–110.

[7] —, *Cauchy–Riemann submanifolds of locally conformal Kaehler manifolds. I–II*, Geom. Dedicata 28 (1988), 181–197, Atti Sem. Mat. Fis. Univ. Modena 37 (1989), 1–11.

[8] S. Dragomir and L. M. Abatangelo, *Principal toroidal bundles over Cauchy–Riemann products*, Internat. J. Math. Math. Sci. (2) 13 (1990), 299–310.

[9] S. Dragomir and R. Grimaldi, *Isometric immersions of Riemann spaces in a real Hopf manifold*, J. Math. Pures Appl. 68 (1989), 355–364.

[10] S. I. Goldberg, *Curvature and Homology*, Academic Press, New York 1962.

[11] —, *Totally geodesic hypersurfaces of Kaehler manifolds*, Pacific J. Math. (2) 27 (1968), 275–281.

[12] M. Inoue, *On surfaces of class* VII_0, Invent. Math. 24 (1974), 269–310.

[13] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Interscience, Vols. I–II, New York 1963, 1969.

[14] P. Libermann, *Sur les structures presque complexes et autres structures infinitésimales régulières*, Bull. Soc. Math. France 83 (1955), 195–224.

[15] Z. Olszak, *On almost cosymplectic manifolds*, Kodai Math. J. 1 (1981), 239–250.

[16] —, *Locally conformal almost cosymplectic manifolds*, Colloq. Math. 57 (1989), 73–87.

[17] —, *Normal almost contact metric manifolds of dimension* 3, Ann. Polon. Math. 47 (1986), 41–50.

[18] R. S. Palais, *A global formulation of the Lie theory of transformation groups*, Mem. Amer. Math. Soc. 22 (1957).

[19] C. Reischer and I. Vaisman, *Local similarity manifolds*, Ann. Mat. Pura Appl. 135 (1983), 279–292.

[20] J. L. Synge, *On the connectivity of spaces of positive curvature*, Quart. J. Math. Oxford Ser. 7 (1936), 316–320.

[21] S. Tanno, *A theorem on regular vector fields and its applications to almost contact structures*, Tôhoku Math. J. 17 (1965), 235–243.

[22] —, *Quasi-Sasakian structures of rank* $2p+1$, J. Differential Geom. 5 (1971), 317–324.

[23] F. Tricerri, *Some examples of locally conformal Kaehler manifolds*, Rend. Sem. Mat. Univ. Politec. Torino 40 (1982), 81–92.
[24] I. Vaisman, *Locally conformal Kaehler manifolds with parallel Lee form*, Rend. Mat. 12 (1979), 263–284.
[25] —, *Conformal change of almost contact metric structures*, in: Proc. Conference on Differential Geometry, Haifa 1979, Lecture Notes in Math. 792, Springer, 1980.
[26] K. Yano and M. Kon, *Structures on Manifolds*, Ser. Pure Math., World Sci., 1984.

UNIVERSITÀ DEGLI STUDI DI BARI
DIPARTIMENTO DI MATEMATICA
I-70125 BARI, ITALY

MATHEMATICS DEPARTMENT
STATE UNIVERSITY OF NEW YORK
AT STONY BROOK
STONY BROOK, NEW YORK 11794-3651
U.S.A.

Reçu par la Rédaction le 21.1.1991