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ON VECTOR-VALUED INEQUALITIES FOR SIDON SETS AND SETS OF INTERPOLATION

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Let E be a Sidon subset of the integers and suppose X is a Banach space. Then Pisier has shown that E-spectral polynomials with values in X behave like Rademacher sums with respect to L_p-norms. We consider the situation when X is a quasi-Banach space. For general quasi-Banach spaces we show that a similar result holds if and only if E is a set of interpolation (I_0-set). However, for certain special classes of quasi-Banach spaces we are able to prove such a result for larger sets. Thus if X is restricted to be "natural" then the result holds for all Sidon sets. We also consider spaces with plurisubharmonic norms and introduce the class of analytic Sidon sets.

1. Introduction. Suppose G is a compact abelian group. We denote by μ_G normalized Haar measure on G and by Γ the dual group of G. We recall that a subset E of Γ is called a *Sidon set* if there is a constant M such that for every finitely nonzero map $a : E \to \mathbb{C}$ we have

$$\sum_{\gamma\in E} |a(\gamma)| \le M \max_{g\in G} \Big|\sum_{\gamma\in E} a(\gamma)\gamma(g)\Big| .$$

We define Δ to be the Cantor group, i.e. $\Delta = \{\pm 1\}^{\mathbb{N}}$. If $t \in \Delta$ we denote by $\varepsilon_n(t)$ the nth coordinate of t. The sequence (ε_n) is an example of a Sidon set. Of course the sequence (ε_n) is a model for the Rademacher functions on $[0, 1]$. Similarly we denote the coordinate maps on $\mathbb{T}^{\mathbb{N}}$ by η_n.

Suppose now that G is a compact abelian group. If X is a Banach space, or more generally a quasi-Banach space with a continuous quasinorm and $\phi : G \to X$ is a Borel map we define $\|\phi\|_p$ for $0 < p \le \infty$ to be the L_p-norm of ϕ, i.e. $\|\phi\|_p = (\int_G \|\phi(g)\|^p \, d\mu_G(g))^{1/p}$ if $0 < p < \infty$ and $\|\phi\|_\infty = \operatorname{ess\,sup}_{g\in G} \|\phi(g)\|$.

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It is a theorem of Pisier [12] that if E is a Sidon set then there is a constant M so that for every subset $\{\gamma_1,\dots,\gamma_n\}$ of E, every $x_1,\dots,x_n$ chosen from a Banach space X and every $1\le p\le\infty$ we have

$$(*)\qquad M^{-1}\Big\|\sum_{k=1}^n x_k\varepsilon_k\Big\|_p\le\Big\|\sum_{k=1}^n x_k\gamma_k\Big\|_p\le M\Big\|\sum_{k=1}^n x_k\varepsilon_k\Big\|_p.$$

Thus a Sidon set behaves like the Rademacher sequence for Banach space valued functions. The result can be similarly stated for (η_n) in place of (ε_n). Recently Asmar and Montgomery-Smith [1] have taken Pisier's ideas further by establishing distributional inequalities in the same spirit.

It is natural to ask whether Pisier's inequalities can be extended to arbitrary quasi-Banach spaces. This question was suggested to the author by Asmar and Montgomery-Smith. For convenience we suppose that every quasi-Banach space is r-normed for some $r<1$, i.e. the quasinorm satisfies $\|x+y\|^r\le\|x\|^r+\|y\|^r$ for all x, y; an r-norm is necessarily continuous. We can then ask, for fixed $0<p\le\infty$, for which sets E inequality $(*)$ holds, if we restrict X to belong to some class of quasi-Banach spaces, for some constant $M=M(E,X)$.

It turns out Pisier's results do not in general extend to the non-locally convex case. In fact, we show that if we fix $r<1$ and ask that a set E satisfies $(*)$ for some fixed p and every r-normable quasi-Banach space X then this condition precisely characterizes sets of interpolation as studied in [2]–[5], [8], [9], [13] and [14]. We recall that E is called a *set of interpolation* (*set of type* (I_0)) if it has the property that every $f\in\ell_\infty(E)$ (the collection of all bounded complex functions on E) can be extended to a continuous function on the Bohr compactification $b\Gamma$ of Γ.

However, in spite of this result, there are specific classes of quasi-Banach spaces for which $(*)$ holds for a larger class of sets E. If we restrict X to be a natural quasi-Banach space then $(*)$ holds for all Sidon sets E. Here a quasi-Banach space is called *natural* if it is linearly isomorphic to a closed linear subspace of a (complex) quasi-Banach lattice Y which is q-convex for some $q>0$, i.e. such that for a suitable constant C we have

$$\Big\|\Big(\sum_{k=1}^n|y_k|^q\Big)^{1/q}\Big\|\le C\Big(\sum_{k=1}^n\|y_k\|^q\Big)^{1/q}$$

for every $y_1,\dots,y_n\in Y$. Natural quasi-Banach spaces form a fairly broad class including almost all function spaces which arise in analysis. The reader is referred to [6] for a discussion of examples. Notice that, of course, the spaces L_q for $q<1$ are natural so that, in particular, $(*)$ holds for all p and all Sidon sets E for every $0<p\le\infty$. The case $p=q$ here would be a direct consequence of Fubini's theorem, but the other cases, including $p=\infty$, are less obvious.

A quasi-Banach lattice X is natural if and only if it is *A-convex*, i.e. it has an equivalent plurisubharmonic quasi-norm. Here a quasinorm is *plurisubharmonic* if it satisfies

$$\|x\| \leq \int_0^{2\pi} \|x + e^{i\theta}y\| \frac{d\theta}{2\pi}$$

for every $x, y \in X$. There are examples of A-convex spaces which are not natural, namely the Schatten ideals S_p for $p < 1$ [7]. Of course, it follows that S_p cannot be embedded in any quasi-Banach lattice which is A-convex when $0 < p < 1$. Thus we may ask for what sets E $(*)$ holds for every A-convex space. Here, we are unable to give a precise characterization of the sets E such that $(*)$ holds. In fact, we define E to be an analytic Sidon set if $(*)$ holds, for $p = \infty$ (or, equivalently for any other $0 < p < \infty$), for every A-convex quasi-Banach space X. We show that any finite union of Hadamard sequences in $\mathbb{N} \subset \mathbb{Z}$ is an analytic Sidon set. In particular, a set such as $\{3^n\} \cup \{3^n + n\}$ is an analytic Sidon set but not a set of interpolation. However, we have no example of a Sidon set which is not an analytic Sidon set.

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2. The results. Suppose G is a compact abelian group and Γ is its dual group. Let E be a subset of Γ. Suppose X is a quasi-Banach space and that $0 < p \leq \infty$; then we will say that E has *property* $\mathcal{C}_p(X)$ if there is a constant M such that for any finite subset $\{\gamma_1, \ldots, \gamma_n\}$ of E and any $x_1, \ldots, x_n$ of X we have $(*)$, i.e.

$$M^{-1}\Big\|\sum_{k=1}^n x_k\varepsilon_k\Big\|_p \leq \Big\|\sum_{k=1}^n x_k\gamma_k\Big\|_p \leq M\Big\|\sum_{k=1}^n x_k\varepsilon_k\Big\|_p .$$

(Note that in contrast to Pisier's result $(*)$, we here assume p fixed.) We start by observing that E is a Sidon set if and only if E has property $\mathcal{C}_\infty(\mathbb{C})$. It follows from the results of Pisier [12] that a Sidon set has property $\mathcal{C}_p(X)$ for every Banach space X and for every $0 < p < \infty$. See also Asmar and Montgomery-Smith [1] and Pełczyński [11].

Note that for any $t \in \Delta$ we have $\|\sum \varepsilon_k(t)x_k\varepsilon_k\|_p = \|\sum x_k\varepsilon_k\|_p$. Now any real sequence $(a_1, \ldots, a_n)$ with $\max|a_k| \leq 1$ can be written in the form $a_k = \sum_{j=1}^\infty 2^{-j}\varepsilon_k(t_j)$ and it follows quickly by taking real and imaginary parts that there is a constant $C = C(r, p)$ so that for any complex $a_1, \ldots, a_n$ and any r-normed space X we have

$$\Big\|\sum_{k=1}^n a_kx_k\varepsilon_k\Big\|_p \leq C\|a\|_\infty\Big\|\sum_{k=1}^n x_k\varepsilon_k\Big\|_p .$$

From this it follows quickly that $\|\sum_{k=1}^{n} x_k\eta_k\|_p$ is equivalent to $\|\sum_{k=1}^{n} x_k\varepsilon_k\|_p$. In particular, we can replace ε_k by η_k in the definition of property $\mathcal{C}_p(X)$.

We note that if E has property $\mathcal{C}_p(X)$ then it is immediate that E has property $\mathcal{C}_p(\ell_p(X))$ and further that E has property $\mathcal{C}_p(Y)$ for any quasi-Banach space finitely representable in X (or, of course, in $\ell_p(X)$).

For a fixed quasi-Banach space X and a fixed subset E of Γ we let $\mathcal{P}_E(X)$ denote the space of X-valued E-polynomials, i.e. functions $\phi : G \to X$ of the form $\phi = \sum_{\gamma\in E} x(\gamma)\gamma$ where $x(\gamma)$ is only finitely nonzero. If $f \in \ell_\infty(E)$ we define $T_f : \mathcal{P}_E(X) \to \mathcal{P}_E(X)$ by

$$T_f\Big(\sum x(\gamma)\gamma\Big) = \sum f(\gamma)x(\gamma)\gamma\,.$$

We then define $\|f\|_{\mathcal{M}_p(E,X)}$ to be the operator norm of T_f on $\mathcal{P}_E(X)$ for the L_p-norm (and to be ∞ if this operator is unbounded).

LEMMA 1. *In order that E has property $\mathcal{C}_p(X)$ it is necessary and sufficient that there exists a constant C such that*

$$\|f\|_{\mathcal{M}_p(E,X)} \le C\|f\|_\infty \quad \textit{for all } f \in \ell_\infty(E)\,.$$

Proof. If E has property $\mathcal{C}_p(X)$ then it also satisfies $(*)$ for (η_n) in place of (ε_n) for a suitable constant M. Thus if $f \in \ell_\infty(E)$ and $\phi \in \mathcal{P}_E(X)$ then

$$\|T_f\phi\|_p \le M^2\|f\|_\infty\|\phi\|_p\,.$$

For the converse direction, we consider the case $p < \infty$. Suppose $\{\gamma_1, \ldots, \gamma_n\}$ is a finite subset of E. Then for any $x_1, \ldots, x_n$

$$\begin{aligned} C^{-p}\int_{\mathbb{T}^{\mathbb{N}}} \Big\|\sum_{k=1}^{n} x_k\eta_k\Big\|^p d\mu_{\mathbb{T}^{\mathbb{N}}} &= C^{-p}\int_{\mathbb{T}^{\mathbb{N}}}\int_G \Big\|\sum_{k=1}^{n} x_k\eta_k(s)\gamma_k(t)\Big\|^p d\mu_{\mathbb{T}^{\mathbb{N}}}(s)\,d\mu_G(t) \\ &\le \int_G \Big\|\sum_{k=1}^{n} x_k\gamma_k\Big\|^p d\mu_G \\ &\le C^p\int_{\mathbb{T}^{\mathbb{N}}}\int_G \Big\|\sum_{k=1}^{n} x_k\eta_k(s)\gamma_k(t)\Big\|^p d\mu_{\mathbb{T}^{\mathbb{N}}}(s)\,d\mu_G(t) \\ &\le C^p\int_{\mathbb{T}^{\mathbb{N}}} \Big\|\sum_{k=1}^{n} x_k\eta_k\Big\|^p d\mu_{\mathbb{T}^{\mathbb{N}}}\,. \end{aligned}$$

This estimate together with a similar estimate in the opposite direction gives the conclusion. The case $p = \infty$ is similar. ∎

If E is a subset of Γ, $N \in \mathbb{N}$ and $\delta > 0$ we let $AP(E, N, \delta)$ be the set of $f \in \ell_\infty(E)$ such that there exist $g_1, \ldots, g_N \in G$ (not necessarily distinct) and $\alpha_1, \ldots, \alpha_N \in \mathbb{C}$ with $\max_{1\le j\le N} |\alpha_j| \le 1$ and

$$\Big| f(\gamma) - \sum_{j=1}^{N} \alpha_j \gamma(g_j) \Big| \le \delta$$

for $\gamma \in E$.

The following theorem improves slightly on results of Kahane [5] and Méla [8]. Perhaps also, our approach is slightly more direct. We write $B_{\ell_\infty(E)}$ $= \{f \in \ell_\infty(E) : \|f\|_\infty \le 1\}$.

THEOREM 2. *Let G be a compact abelian group and let Γ be its dual group. Suppose E is a subset of Γ. Then the following conditions on E are equivalent*:

(1) *E is a set of interpolation.*

(2) *There exists an integer N so that $B_{\ell_\infty(E)} \subset AP(E, N, 1/2)$.*

(3) *There exists M and $0 < \delta < 1$ so that if $f \in B_{\ell_\infty(E)}$ then there exist complex numbers $(c_j)_{j=1}^\infty$ with $|c_j| \le M\delta^j$ and $(g_j)_{j=1}^\infty$ in G with*

$$f(\gamma) = \sum_{j=1}^{\infty} c_j \gamma(g_j)$$

for $\gamma \in E$.

Proof. (1)⇒(2). It follows from the Stone–Weierstrass theorem that

$$\mathbb{T}^E \subset \bigcup_{m=1}^{\infty} AP(E, m, 1/5)\,.$$

Let $\mu = \mu_{\mathbb{T}^E}$. Since each $AP(E, m, 1/5) \cap \mathbb{T}^E$ is closed it is clear that there exists m so that $\mu(AP(E, m, 1/5) \cap \mathbb{T}^E) > 1/2$. Thus if $f \in \mathbb{T}^E$ we can find $f_1, f_2 \in AP(E, m, 1/5) \cap \mathbb{T}^E$ so that $f = f_1 f_2$. Hence $f \in AP(E, m^2, 1/2)$. This clearly implies (2) with $N = 2m^2$.

(2)⇒(3). We let $\delta = 2^{-1/N}$ and $M = 2$. Then given $f \in B_{\ell_\infty(E)}$ we can find $(c_j)_{j=1}^N$ and $(g_j)_{j=1}^N$ with $|c_j| \le 1 \le M\delta^j$ and

$$\Big| f(\gamma) - \sum_{j=1}^{N} c_j \gamma(g_j) \Big| \le 1/2$$

for $\gamma \in E$. Let $f_1(\gamma) = 2(f(\gamma) - \sum_{j=1}^N c_j \gamma(g_j))$ and iterate the argument.

(3)⇒(1). Obvious. ■

THEOREM 3. *Suppose G is a compact abelian group*, *E is a subset of the dual group Γ and that $0 < r < 1$, $0 < p \le \infty$. In order that E satisfies $\mathcal{C}_p(X)$ for every r-normable quasi-Banach space X it is necessary and sufficient that E be a set of interpolation.*

Proof. First suppose that E is a set of interpolation so that it satisfies (3) of Theorem 2. Suppose X is an r-normed quasi-Banach space. Suppose $f \in B_{\ell_\infty(E)}$. Then there exist $(c_j)_{j=1}^\infty$ and $(g_j)_{j=1}^\infty$ so that $|c_j| \le M\delta^j$ and $f(\gamma) = \sum c_j\gamma(g_j)$ for $\gamma \in E$. Now if $\phi \in \mathcal{P}_E(X)$ it follows that

$$T_f\phi(h) = \sum_{j=1}^\infty c_j\phi(g_jh)$$

and so

$$\|T_f\phi\|_p \le M\Big(\sum_{j=1}^\infty \delta^{js}\Big)^{1/s}\|\phi\|_p$$

where $s = \min(p, r)$. Thus $\|f\|_{\mathcal{M}_p(E,X)} \le C$ where $C = C(p, r, E)$ and so by Lemma 1, E has property $\mathcal{C}_p(X)$.

Now, conversely, suppose that $0 < r < 1$, $0 < p \le \infty$ and that E has property $\mathcal{C}_p(X)$ for every r-normable space X. It follows from consideration of ℓ_∞-products that there exists a constant C so that for every r-normed space X we have $\|f\|_{\mathcal{M}_p(E,X)} \le C\|f\|_\infty$ for $f \in \ell_\infty(E)$.

Suppose F is a finite subset of E. We define an r-norm $\|\ \|_A$ on $\ell_\infty(F)$ by setting $\|f\|_A$ to be the infimum of $(\sum |c_j|^r)^{1/r}$ over all $(c_j)_{j=1}^\infty$ and $(g_j)_{j=1}^\infty$ such that

$$f(\gamma) = \sum_{j=1}^\infty c_j\gamma(g_j)$$

for $\gamma \in F$. Notice that $\|f_1f_2\|_A \le \|f_1\|_A\|f_2\|_A$ for all $f_1, f_2 \in A = \ell_\infty(F)$.

For $\gamma \in F$ let e_γ be defined by $e_\gamma(\gamma) = 1$ if $\gamma = \chi$ and 0 otherwise. Then for $f \in A$, with $\|f\|_\infty \le 1$,

$$\Big(\int_G \Big\|\sum_{\gamma\in F} f(\gamma)e_\gamma\gamma\Big\|_A^p\,d\mu_G\Big)^{1/p} \le C\Big(\int_G \Big\|\sum_{\gamma\in F} e_\gamma\gamma\Big\|_A^p\,d\mu_G\Big)^{1/p}.$$

But for any $g \in G$, $\|\sum \gamma(g)e_\gamma\|_A \le 1$. Define H to be the subset of $h \in G$ such that $\|\sum_{\gamma\in F} f(\gamma)\gamma(h)e_\gamma\|_A \le 3^{1/p}C$. Then $\mu_G(H) \ge 2/3$. Thus there exist $h_1, h_2 \in H$ such that $h_1h_2 = 1$ (the identity in G). Hence by the algebra property of the norm

$$\|f\|_A \le 3^{2/p}C^2$$

and so if we fix an integer $C_0 > 3^{2/p}C^2$ we can find c_j and g_j so that $\sum |c_j|^r \le C_0^r$ and

$$f(\gamma) = \sum c_j\gamma(g_j)$$

for $\gamma \in F$. We can suppose $|c_j|$ is decreasing and hence that $|c_j| \le C_0j^{-1/r}$.

Choose N_0 so that $C_0 \sum_{j=N_0+1}^{\infty} j^{-1/r} \le 1/2$. Thus

$$\Big| f(\gamma) - \sum_{j=1}^{N_0} c_j \gamma(g_j) \Big| \le 1/2$$

for $\gamma \in F$. Since each $|c_j| \le C_0$ this implies that $B_{\ell_\infty(F)} \subset AP(F, N, 1/2)$ where $N = C_0 N_0$.

As this holds for every finite set F it follows by an easy compactness argument that $B_{\ell_\infty(E)} \subset AP(E, N, 1/2)$ and so by Theorem 2, E is a set of interpolation. ■

THEOREM 4. *Let X be a natural quasi-Banach space and suppose* $0 < p \le \infty$. *Then any Sidon set has property $\mathcal{C}_p(X)$.*

Proof. Suppose E is a Sidon set. Then there is a constant C_0 so that if $f \in \ell_\infty(E)$ then there exists $\nu \in C(G)^*$ such that $\widehat{\mu}(\gamma) = f(\gamma)$ for $\gamma \in E$ and $\|\mu\| \le C_0 \|f\|_\infty$. We will show the existence of a constant C such that $\|f\|_{\mathcal{M}_p(E,X)} \le C\|f\|_\infty$. If no such constant exists then we may find a sequence E_n of finite subsets of E such that $\lim C_n = \infty$ where C_n is the least constant such that $\|f\|_{\mathcal{M}_p(E_n,X)} \le C_n \|f\|_\infty$ for all $f \in \ell_\infty(E_n)$.

Now the spaces $\mathcal{M}_p(E_n, X)$ are each isometric to a subspace of $\ell_\infty(L_p(G, X))$ and hence so is $Y = c_0(\mathcal{M}_p(E_n, X))$. In particular, Y is natural. Notice that Y has a finite-dimensional Schauder decomposition. We will calculate the Banach envelope Y_c of Y. Clearly $Y_c = c_0(Y_n)$ where Y_n is the finite-dimensional space $\mathcal{M}_p(E_n, X)$ equipped with its envelope norm $\|f\|_c$.

Suppose $f \in \ell_\infty(E_n)$. Then clearly $\|f\|_\infty \le \|f\|_{\mathcal{M}_p(E,X)}$ and so $\|f\|_\infty \le \|f\|_c$. Conversely, if $f \in \ell_\infty(E_n)$ there exists $\nu \in C(G)^*$ with $\|\nu\| \le C_0 \|f\|_\infty$ and such that $\int \gamma \, d\nu = f(\gamma)$ for $\gamma \in E_n$. In particular, $C_0^{-1} \|f\|_\infty^{-1} f$ is in the absolutely closed convex hull of the set of functions $\{\widetilde{g} : g \in G\}$ where $\widetilde{g}(\gamma) = \gamma(g)$ for $\gamma \in E_n$. Since $\|\widetilde{g}\|_{\mathcal{M}_p(E,X)} = 1$ for all $g \in G$ we see that $\|f\|_\infty \le \|f\|_c \le C_0 \|f\|_\infty$.

This implies that Y_c is isomorphic to c_0. Since Y has a finite-dimensional Schauder decomposition and is natural we can apply Theorem 3.4 of [6] to deduce that $Y = Y_c$ is already locally convex. Thus there is a constant C_0' independent of n so that $\|f\|_{\mathcal{M}_p(E,X)} \le C_0' \|f\|_\infty$ whenever $f \in \ell_\infty(E_n)$. This contradicts the choice of E_n and proves the theorem. ■

We now consider the case of A-convex quasi-Banach spaces. For this notion we will introduce the concept of an analytic Sidon set. We say a subset E of Γ is an *analytic Sidon set* if E satisfies $\mathcal{C}_\infty(X)$ for every A-convex quasi-Banach space X.

PROPOSITION 5. *Suppose* $0 < p < \infty$. *Then E is an analytic Sidon set if and only if E satisfies $\mathcal{C}_p(X)$ for every A-convex quasi-Banach space X.*

Proof. Suppose first E is an analytic Sidon set, and that X is an A-convex quasi-Banach space (for which we assume the quasinorm is plurisubharmonic). Then $L_p(G,X)$ also has a plurisubharmonic quasinorm and so E satisfies $(*)$ for X replaced by $L_p(G,X)$ and p replaced by ∞ with constant M. Now suppose $x_1,\ldots,x_n \in X$ and $\gamma_1,\ldots,\gamma_n \in E$. Define $y_1,\ldots,y_n \in L_p(G,X)$ by $y_k(g)=\gamma_k(g)x_k$. Then

$$\max_{g\in G}\Big\|\sum_{k=1}^n y_k\gamma_k(g)\Big\|_{L_p(G,X)} = \Big\|\sum_{k=1}^n x_k\gamma_k\Big\|_p$$

and a similar statement holds for the characters ε_k on the Cantor group. It follows quickly that E satisfies $(*)$ for p and X with constant M.

For the converse direction suppose E satisfies $\mathcal{C}_p(X)$ for every A-convex space X. Suppose X has a plurisubharmonic quasinorm. We show that $\mathcal{M}_\infty(E,X)=\ell_\infty(E)$. In fact, $\mathcal{M}_\infty(F,X)$ can be isometrically embedded in $\ell_\infty(X)$ for every finite subset F of E. Thus $(*)$ holds for X replaced by $\mathcal{M}_\infty(F,X)$ for some constant M, independent of F. Denoting by e_γ the canonical basis vectors in $\ell_\infty(E)$ we see that if $F=\{\gamma_1,\ldots,\gamma_n\}\subset E$ then

$$\Big(\int_\Delta \Big\|\sum_{k=1}^n \varepsilon_k(t)e_{\gamma_k}\Big\|^p_{\mathcal{M}_\infty(F,X)}\,d\mu_\Delta(t)\Big)^{1/p} \le M\max_{g\in G}\Big\|\sum_{k=1}^n \gamma_k(g)e_{\gamma_k}\Big\|_{\mathcal{M}_\infty(F,X)} = M\,.$$

Thus the set K of $t\in\Delta$ such that $\|\sum_{k=1}^n\varepsilon_k(t)e_{\gamma_k}\|_{\mathcal{M}_\infty(F,X)}\le 3^{1/p}M$ has measure at least $2/3$. Arguing that $K\cdot K=\Delta$ we obtain

$$\Big\|\sum_{k=1}^n \varepsilon_k(t)e_{\gamma_k}\Big\|_{\mathcal{M}_\infty(F,X)}\le 3^{2/p}M^2$$

for every $t\in\Delta$. It follows quite simply that there is a constant C so that for every real-valued $f\in\ell_\infty(F)$ we have $\|f\|_{\mathcal{M}_\infty(E,X)}\le C\|f\|_\infty$. In fact, this is proved by writing each such f with $\|f\|_\infty=1$ in the form $f(\gamma_k)=\sum_{j=1}^\infty 2^{-j}\varepsilon_k(t_j)$ for a suitable sequence $t_j\in\Delta$. A similar estimate for complex f follows by estimating real and imaginary parts. Finally, since these estimates are independent of F we conclude that $\ell_\infty(E)=\mathcal{M}_\infty(E,X)$. ∎

Of course any set of interpolation is an analytic Sidon set and any analytic Sidon set is a Sidon set. The next theorem will show that not every analytic Sidon set is a set of interpolation. If we take $G=\mathbb{T}$ and $\Gamma=\mathbb{Z}$, we recall that a *Hadamard gap sequence* is a sequence $(\lambda_k)_{k=1}^\infty$ of positive integers such that for some $q>1$ we have $\lambda_{k+1}/\lambda_k\ge q$ for $k\ge 1$. It is shown in [10] and [14] that a Hadamard gap sequence is a set of interpolation. However, the union of two such sequences may fail to be a set of interpolation; for example $(3^n)_{n=1}^\infty \cup (3^n+n)_{n=1}^\infty$ is not a set of interpolation, since the closures of (3^n) and (3^n+n) in $b\mathbb{Z}$ are not disjoint.

THEOREM 6. *Let $G=\mathbb{T}$ so that $\Gamma=\mathbb{Z}$. Suppose $E\subset\mathbb{N}$ is a finite union of Hadamard gap sequences. Then E is an analytic Sidon set.*

Proof. Suppose $E=(\lambda_k)_{k=1}^\infty$ where (λ_k) is increasing. We start with the observation that E is the union of m Hadamard sequences if and only if there exists $q>1$ so that $\lambda_{m+k}\ge q^m\lambda_k$ for every $k\ge 1$.

We will prove the theorem by induction on m. Note first that if $m=1$ then E is a Hadamard sequence and hence [14] a set of interpolation. Thus by Theorem 2 above, E is an analytic Sidon set.

Suppose now that E is the union of m Hadamard sequences and that the theorem is proved for all unions of l Hadamard sequences where $l<m$. We assume that $E=(\lambda_k)$ and that there exists $q>1$ such that $\lambda_{k+m}\ge q^m\lambda_k$ for $k\ge 1$. We first decompose E into at most m Hadamard sequences. To do this let us define $E_1=\{\lambda_1\}\cup\{\lambda_k : k\ge 2,\ \lambda_k\ge q\lambda_{k-1}\}$. We will write $E_1=(\tau_k)_{k\ge 1}$ where τ_k is increasing. Of course E_1 is a Hadamard sequence.

For each k let $D_k=E\cap[\tau_k,\tau_{k+1})$. It is easy to see that $|D_k|\le m$ for every k. Further, if $n_k\in D_k$ then $n_{k+1}\ge\tau_{k+1}\ge qn_k$ so that (n_k) is a Hadamard sequence. In particular, $E_2=E\setminus E_1$ is the union of at most $m-1$ Hadamard sequences and so E_2 is an analytic Sidon set by the inductive hypothesis.

Now suppose $w\in\mathbb{T}$. We define $f_w\in\ell_\infty(E)$ by $f_w(n)=w^{n-\tau_k}$ for $n\in D_k$. We will show that f_w is uniformly continuous for the Bohr topology on $\mathbb{Z}$; equivalently we show that f_w extends to a continuous function on the closure $\widetilde{E}$ of E in the Bohr compactification $b\mathbb{Z}$ of $\mathbb{Z}$. Indeed, if this is not the case there exists $\xi\in\widetilde{E}$ and ultrafilters $\mathcal{U}_0$ and $\mathcal{U}_1$ on E both converging to ξ so that $\lim_{n\in\mathcal{U}_0} f_w(n)=\zeta_0$ and $\lim_{n\in\mathcal{U}_1} f_w(n)=\zeta_1$ where $\zeta_1\ne\zeta_0$. We will let $\delta=\frac{1}{3}|\zeta_1-\zeta_0|$.

We can partition E into m sets $A_1,\dots,A_m$ so that $|A_j\cap D_k|\le 1$ for each k. Clearly $\mathcal{U}_0$ and $\mathcal{U}_1$ each contain exactly one of these sets. Let us suppose $A_{j_0}\in\mathcal{U}_0$ and $A_{j_1}\in\mathcal{U}_1$.

Next define two ultrafilters $\mathcal{V}_0$ and $\mathcal{V}_1$ on $\mathbb{N}$ by $\mathcal{V}_0=\{V:\bigcup_{k\in V}D_k\in\mathcal{U}_0\}$ and $\mathcal{V}_1=\{V:\bigcup_{k\in V}D_k\in\mathcal{U}_1\}$. We argue that $\mathcal{V}_0$ and $\mathcal{V}_1$ coincide. If not we can pick $V\in\mathcal{V}_0\setminus\mathcal{V}_1$. Consider the set $A=(A_{j_0}\cap\bigcup_{k\in V}D_k)\cup(A_{j_1}\cap\bigcup_{k\notin V}D_k)$. Then A is a Hadamard sequence and hence a set of interpolation. Thus for the Bohr topology the sets $A_{j_0}\cap\bigcup_{k\in V}D_k$ and $A_{j_1}\cap\bigcup_{k\notin V}D_k$ have disjoint closures. This is a contradiction since of course ξ must be in the closure of each. Thus $\mathcal{V}_0=\mathcal{V}_1$.

Since both $\mathcal{U}_0$ and $\mathcal{U}_1$ converge to the same limit for the Bohr topology we can find sets $H_0\in\mathcal{U}_0$ and $H_1\in\mathcal{U}_1$ so that if $n_0\in H_0$, $n_1\in H_1$ then $|w^{n_1}-w^{n_0}|<\delta$ and further $|f_w(n_0)-\zeta_0|<\delta$ and $|f_w(n_1)-\zeta_1|<\delta$.

Let $V_0=\{k\in\mathbb{N}:D_k\cap H_0\neq\emptyset\}$ and $V_1=\{k\in\mathbb{N}:D_k\cap H_1\neq\emptyset\}$. Then $V_0\in\mathcal{V}_0$ and $V_1\in\mathcal{V}_1$. Thus $V=V_0\cap V_1\in\mathcal{V}_0=\mathcal{V}_1$. If $k\in V$ there exists $n_0\in D_k\cap H_0$ and $n_1\in D_k\cap H_1$. Then

$$\begin{aligned}3\delta=|\zeta_1-\zeta_0|&<|f_w(n_1)-f_w(n_0)|+2\delta\\&=|w^{n_1}-w^{n_0}|+2\delta<3\delta\,.\end{aligned}$$

This contradiction shows that each f_w is uniformly continuous for the Bohr topology.

Now suppose that X is an r-normed A-convex quasi-Banach space where the quasi-norm is plurisubharmonic. Since both E_1 and E_2 are analytic Sidon sets we can introduce a constant C so that if $f\in\ell_\infty(E_j)$ where $j=1,2$ then $\|f\|_{\mathcal{M}_\infty(E_j,X)}\le C\|f\|_\infty$. Pick a constant $0<\delta<1$ so that $3\cdot4^{1/r}\delta<C$.

Let $K_l=\{w\in\mathbb{T}:f_w\in AP(E,l,\delta)\}$. It is easy to see that each K_l is closed and since each f_w is uniformly continuous by the Bohr topology it follows from the Stone–Weierstrass theorem that $\bigcup K_l=\mathbb{T}$. If we pick l_0 so that $\mu_{\mathbb{T}}(K_{l_0})>1/2$ then $K_{l_0}K_{l_0}=\mathbb{T}$ and hence, since the map $w\to f_w$ is multiplicative, $f_w\in AP(E,l_0^2,3\delta)$ for every $w\in\mathbb{T}$.

Let F be an arbitrary finite subset of E. Then there is a least constant β so that $\|f\|_{\mathcal{M}_\infty(F,X)}\le\beta\|f\|_\infty$. The proof is completed by establishing a uniform bound on β.

For $w\in\mathbb{T}$ we can find c_j with $|c_j|\le1$ and $\zeta_j\in\mathbb{T}$ for $1\le j\le l_0^2$ such that

$$\Big|f_w(n)-\sum_{j=1}^{l_0^2}c_j\zeta_j^n\Big|\le3\delta$$

for $n\in E$. If $\widetilde{\zeta}_j$ is defined by $\widetilde{\zeta}_j(n)=\zeta_j^n$ then of course $\|\widetilde{\zeta}_j\|_{\mathcal{M}_\infty(E,X)}=1$. Restricting to F we see that

$$\|f_w\|^r_{\mathcal{M}_\infty(F,X)}\le l_0^2+\beta^r(3\delta)^r\,.$$

Define $H:\mathbb{C}\to\mathcal{M}_\infty(F,X)$ by $H(z)(n)=z^{n-\tau_k}$ if $n\in D_k$. Note that H is a polynomial. As in Theorem 5, $\mathcal{M}_\infty(F,X)$ has a plurisubharmonic norm. Hence

$$\|H(0)\|^r\le\max_{|w|=1}\|H(w)\|^r\le l_0^2+(3\delta)^r\beta^r\,.$$

Thus, if χ_A is the characteristic function of A,

$$\|\chi_{E_1\cap F}\|^r_{\mathcal{M}_\infty(F,X)}\le l_0^2+(3\delta)^r\beta^r\,.$$

It follows that

$$\|\chi_{E_2\cap F}\|^r_{\mathcal{M}_\infty(F,X)}\le l_0^2+(3\delta)^r\beta^r+1\,.$$

Now suppose $f\in\ell_\infty(F)$ and $\|f\|_\infty\le1$. Then

$$\|f\chi_{E_j\cap F}\|_{\mathcal{M}_\infty(F,X)}\le\|f\chi_{E_j\cap F}\|_{\mathcal{M}_\infty(E_j\cap F,X)}\|\chi_{E_j\cap F}\|_{\mathcal{M}_\infty(F,X)}$$

for $j = 1, 2$. Thus

$$\|f\|^r_{\mathcal{M}_\infty(F,X)} \le C^r(1 + 2l_0^2 + 2(3\delta)^r\beta^r)\,.$$

By maximizing over all f this implies

$$\beta^r \le C^r(1 + 2l_0^2 + 2(3\delta)^r\beta^r)\,,$$

which gives an estimate

$$\beta^r \le 2C^r(1 + 2l_0^2)$$

in view of the original choice of δ. This estimate, which is independent of F, implies that E is an analytic Sidon set. ∎

Remark. We know of no example of a Sidon set which is not an analytic Sidon set.

Added in proof. In a forthcoming paper with S. C. Tam (*Factorization theorems for quasi-normed spaces*) we show that Theorem 4 holds for a much wider class of spaces.

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