COLLOQUIUM MATHEMATICUM

VOL. LXV 1993 FASC. 1

# SOME NEW HARDY SPACES ON LOCALLY COMPACT VILENKIN GROUPS

BY

SHANZHEN LU AND DACHUN YANG (BEIJING)

**1. Introduction and notations.** Let $G$ denote a locally compact Vilenkin group. In Section 1 of this paper we define some new Hardy spaces $HK^q_{p,\alpha}(G)$ associated with the Herz spaces $K^q_{p,\alpha}(G)$ on $G$, where $0 < q \le 1 < p < \infty$ and $-1 < \alpha \le 0$. Section 2 establishes their atomic decomposition theorem and some interpolation results. The molecular characterization can be found in Section 3. In Section 4, we give some application. Now, let us introduce some basic notations; for more details we refer to [1]–[11].

Throughout this paper $G$ will denote a locally compact Abelian group containing a strictly decreasing sequence of open compact subgroups $\{G_n\}_{n=-\infty}^{\infty}$ such that

(i) $\bigcup_{n=-\infty}^{\infty} G_n = G$ and $\bigcap_{n=-\infty}^{\infty} G_n = \{0\}$,
(ii) $\sup\{\mathrm{order}(G_n/G_{n+1}) : n \in \mathbb{Z}\} < \infty$.

Let $\Gamma$ denote the dual group of $G$ and for each $n \in \mathbb{Z}$, let

$$\Gamma_n := \{\gamma \in \Gamma : \gamma(x) = 1 \text{ for all } x \in G_n\}.$$

Then $\{\Gamma_n\}_{n=-\infty}^{\infty}$ is a strictly increasing sequence of open compact subgroups of $\Gamma$ and

(i)* $\bigcup_{n=-\infty}^{\infty} \Gamma_n = \Gamma$ and $\bigcap_{n=-\infty}^{\infty} \Gamma_n = \{1\}$,
(ii)* $\mathrm{order}(\Gamma_{n+1}/\Gamma_n) = \mathrm{order}(G_n/G_{n+1})$.

We choose Haar measures $\mu$ on $G$ and $\lambda$ on $\Gamma$ so that $\mu(G_0) = \lambda(\Gamma_0) = 1$. Then $\mu(G_n) = (\lambda(\Gamma_n))^{-1} =: (m_n)^{-1}$ for each $n \in \mathbb{Z}$. For each $\alpha > 0$ and $k \in \mathbb{Z}$, we have

$$\sum_{n=k}^{\infty} (m_n)^{-\alpha} \le C(m_k)^{-\alpha}, \tag{1.1}$$

$$\sum_{n=-\infty}^{k} (m_n)^{\alpha} \le C(m_k)^{\alpha} \tag{1.2}$$

Supported by the National Science Foundation of China.

(see [5] and [8]). Here, like elsewhere in this paper, $C$ denotes a constant whose value may change from one occurrence to the next.

There exists a metric $d$ on $G \times G$ defined by $d(x,x) = 0$ and $d(x,y) = (m_l)^{-1}$ if $x - y \in G_l \setminus G_{l+1}$, for $l \in \mathbb{Z}$. Then the topology on $G$ determined by $d$ coincides with the original topology. For $x \in G$, we set $|x| = d(x,0)$, and define the function $v_\alpha$ on $G$ by $v_\alpha(x) = |x|^\alpha$ for each $\alpha \in \mathbb{R}$; the corresponding measure $v_\alpha d\mu = |x|^\alpha d\mu$ is denoted by $d\mu_\alpha$. Moreover, $dx$ will sometimes be used in place of $d\mu$. It is easy to note that $\mu_\alpha(G_l) \le C(m_l)^{-(\alpha+1)}$ if $\alpha > -1$, and if $l < n$ and $x \in G_l \setminus G_{l+1}$, then $\mu_\alpha(x + G_n) = (m_l)^{-\alpha}(m_n)^{-1}$. Similarly to $G$, we can define a metric $\delta$ on $\Gamma \times \Gamma$ such that the topology on $\Gamma$ induced by $\delta$ coincides with the original topology. Furthermore, we write $\|\gamma\| = \delta(\gamma, 1) = m_n$ if $\gamma \in \Gamma_{n+1} \setminus \Gamma_n$ and $\langle\gamma\rangle = \max\{1, \|\gamma\|\}$.

The symbols $\wedge$ and $\vee$ will be used to denote the Fourier transform and inverse Fourier transform respectively. A simple computation shows that

$$(\chi_{G_n})^\wedge = (\lambda(\Gamma_n))^{-1}\chi_{\Gamma_n} = (m_n)^{-1}\chi_{\Gamma_n}\,,$$

and hence,

$$(\chi_{\Gamma_n})^\vee = (\mu(G_n))^{-1}\chi_{G_n} = m_n\chi_{G_n} := \Delta_n\,.$$

In this paper, $S(G)$ (or $S(\Gamma)$) and $S'(G)$ (or $S'(\Gamma)$) denote the spaces of test functions and distributions on $G$ (or $\Gamma$) respectively. For details, see [6], [8] and [10].

**2. The spaces $HK^q_{p,\alpha}(G)$.** First of all, we introduce some Herz spaces defined by Onneweer in [4].

DEFINITION 2.1. Let $0 < q \le 1 < p < \infty$ and $-1 < \alpha \le 0$. The *Herz space* $K^q_{p,\alpha}(G)$ is defined by

$$K^q_{p,\alpha}(G) := \{f : f \text{ is a measurable function on } G \text{ and } \|f\|_{K^q_{p,\alpha}(G)} < \infty\}$$

where

$$\|f\|_{K^q_{p,\alpha}(G)} := \Big(\sum_{l=-\infty}^{\infty} \mu_\alpha(G_l)^{1-q/p}\|f\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)}\Big)^{1/q}.$$

Here we write $L^p_\alpha(G) = \{f : f$ is a measurable function on $G$ and $(\int_G |f(x)|^p\,d\mu_\alpha(x))^{1/p} < \infty\}$. Obviously, $K^q_{p,\alpha}(G) \subset L^q_\alpha(G)$.

For $f \in S'(G)$, we define $f_n(x) = f * \Delta_n(x)$. Then $f_n$ is a function on $G$ which is constant on the cosets of $G_n$ in $G$. Moreover, $\lim_{n\to\infty} f_n = f$ in $S'(G)$ (see [10]). For $f \in S'(G)$ we define its maximal function $f^*(x)$ by

$$f^*(x) = \sup_{n\in\mathbb{Z}} |f * \Delta_n(x)| = \sup_{n\in\mathbb{Z}} \Big|\mu(G_n)^{-1} \int_{x+G_n} f(y)\,d\mu(y)\Big|\,.$$

Now, we define some new Hardy spaces $HK^q_{p,\alpha}(G)$ associated with the Herz spaces $K^q_{p,\alpha}(G)$.

DEFINITION 2.2. Suppose $0 < q \le 1 < p < \infty$ and $-1 < \alpha \le 0$. We define $HK^q_{p,\alpha}(G)$ by

$$HK^q_{p,\alpha}(G) := \{f \in S'(G) : f^* \in K^q_{p,\alpha}(G)\},$$

and set

$$\|f\|_{HK^q_{p,\alpha}(G)} := \|f^*\|_{K^q_{p,\alpha}(G)}.$$

Clearly $HK^q_{p,\alpha}(G) \subset H^q_\alpha(G)$, where $H^q_\alpha(G)$ are weighted Hardy spaces on $G$ (see [1], [2], [6] and [7]).

Remark 2.3. Let $1 < p < \infty$ and $-1 < \alpha \le 0$. Because $K^1_{p,\alpha}(G) \subset L^1_\alpha(G)$, if $f^* \in K^1_{p,\alpha}(G)$ then $f \in L^1_\alpha(G)$ by Lemma 3.5 of Kitada [1]. Therefore we can redefine $HK^1_{p,\alpha}(G)$ by

$$HK^1_{p,\alpha}(G) = \{f \in L^1_\alpha(G) : f^* \in K^1_{p,\alpha}(G)\}.$$

We now characterize the spaces $HK^q_{p,\alpha}(G)$ in terms of atoms. First, we have

DEFINITION 2.4. Let $0 < q \le 1 < p < \infty$ and $-1 < \alpha \le 0$. A function $a$ on $G$ is said to be a *central* $(q,p)_\alpha$*-atom* if

(1) $\operatorname{supp} a \subset G_{n_j}$ for some $n_j \in \mathbb{Z}$,
(2) $(\int_{G_{n_j}} |a(x)|^p \, d\mu_\alpha(x))^{1/p} \le \mu_\alpha(G_{n_j})^{1/p-1/q}$,
(3) $\int a(x)\, d\mu(x) = 0$.

THEOREM 2.5. *Assume $0 < q \le 1 < p < \infty$ and $-1 < \alpha \le 0$. A distribution $f$ on $G$ is in $HK^q_{p,\alpha}(G)$ if and only if $f = \sum \lambda_j a_j$ in $S'(G)$ where $a_j$'s are central $(q,p)_\alpha$-atoms and $\sum |\lambda_j|^q < \infty$. Then*

$$\|f\|_{HK^q_{p,\alpha}(G)} \sim \inf\Big\{\Big(\sum |\lambda_j|^q\Big)^{1/q}\Big\}$$

*where the infimum is taken over all atomic decompositions of $f$. Moreover, for $q = 1$ the identity $f(x) = \sum \lambda_j a_j(x)$ holds pointwise.*

Proof. We first show the necessity for $q = 1$. Let $f \in HK^1_{p,\alpha}(G)$. Noting that $f$ is a function in this case, we can write

$$\begin{aligned} f(x) &= \sum_{l=-\infty}^{\infty} f(x)\chi_{G_l \setminus G_{l+1}}(x) \\ &= \sum_{l=-\infty}^{\infty} \Big\{ f(x)\chi_{G_l\setminus G_{l+1}}(x) - \Big( \int\limits_{G_l \setminus G_{l+1}} f(y)\,dy \Big) \frac{\chi_{G_{l+1}\setminus G_{l+2}}(x)}{\mu(G_{l+1} \setminus G_{l+2})} \Big\} \end{aligned}$$

$$
\begin{aligned}
&+\sum_{l=-\infty}^{\infty}\Big(\int\limits_{G_l\setminus G_{l+1}} f(y)\,dy\Big)\frac{\chi_{G_{l+1}\setminus G_{l+2}}(x)}{\mu(G_{l+1}\setminus G_{l+2})}\\
=:\,& I_1+I_2\,.
\end{aligned}
$$

Let

$$
b_l(x):=f(x)\chi_{G_l\setminus G_{l+1}}(x)-\Big(\int\limits_{G_l\setminus G_{l+1}} f(y)\,dy\Big)\frac{\chi_{G_{l+1}\setminus G_{l+2}}(x)}{\mu(G_{l+1}\setminus G_{l+2})}\,.
$$

Clearly, $\int b_l(x)\,dx=0$, $\operatorname{supp} b_l\subset G_l\setminus G_{l+2}\subset G_l$ and from Lemma 1 of [5] and (1) in [8], for $\alpha>-1$, we easily deduce that

$$
\mu_\alpha(G_{l+1}\setminus G_{l+2})\approx \mu_\alpha(G_{l+1})\approx (m_{l+1})^{-(\alpha+1)}\approx (m_l)^{-(\alpha+1)}\,.
$$

From this, we have

$$
\begin{aligned}
\Big(\int\limits_{G_l}|b_l(x)|^p\,d\mu_\alpha(x)\Big)^{1/p}&\le \Big(\int\limits_{G_l\setminus G_{l+1}}|f(x)|^p\,d\mu_\alpha(x)\Big)^{1/p}\\
&\quad+\Big(\int\limits_{G_l\setminus G_{l+1}}|f(x)|\,dx\Big)\frac{\mu_\alpha(G_{l+1}\setminus G_{l+2})^{1/p}}{\mu(G_{l+1}\setminus G_{l+2})}\\
&\le \Big(\int\limits_{G_l\setminus G_{l+1}}|f(x)|^p\,d\mu_\alpha(x)\Big)^{1/p}\\
&\quad+\Big(\int\limits_{G_l\setminus G_{l+1}}|f(x)|^p|x|^\alpha\,dx\Big)^{1/p}\\
&\quad\times\frac{(m_l)^{\alpha/p}\mu(G_l\setminus G_{l+1})^{1-1/p}\mu_\alpha(G_{l+1}\setminus G_{l+2})^{1/p}}{\mu(G_{l+1}\setminus G_{l+2})}\\
&\le C_0\Big(\int\limits_{G_l\setminus G_{l+1}}|f(x)|^p\,d\mu_\alpha(x)\Big)^{1/p}.
\end{aligned}
$$

Thus, it is easy to see that

$$
a_l(x):=\Big\{C_0\Big(\int\limits_{G_l\setminus G_{l+1}}|f(y)|^p\,d\mu_\alpha(y)\Big)^{1/p}\Big\}^{-1}\mu_\alpha(G_l)^{1/p-1}b_l(x)
$$

is a central $(1,p)_\alpha$-atom. If we write

$$
\lambda_l:=C_0\mu_\alpha(G_l)^{1-1/p}\Big(\int\limits_{G_l\setminus G_{l+1}}|f(x)|^p\,d\mu_\alpha(x)\Big)^{1/p},
$$

then $I_1 = \sum_{l=-\infty}^{\infty} \lambda_l a_l(x)$, and

$$\sum_{l=-\infty}^{\infty} \lambda_l \le C_0 \sum_{l=-\infty}^{\infty} \mu_\alpha(G_l)^{1/p'} \|f\chi_{G_l\setminus G_{l+1}}\|_{L^p_\alpha(G)}$$
$$\le C_0\|f\|_{K^1_{p,\alpha}(G)} \le C_0\|f^*\|_{K^1_{p,\alpha}(G)} = C_0\|f\|_{HK^1_{p,\alpha}(G)}\,.$$

To estimate $I_2$, we write $\widetilde{\chi}_{G_l\setminus G_{l+1}}(x) = \{\mu(G_l\setminus G_{l+1})\}^{-1}\chi_{G_l\setminus G_{l+1}}(x)$; then

$$\begin{aligned}
I_2 &= \sum_{l=-\infty}^{\infty} \Big(\int_{G_l\setminus G_{l+1}} f(y)\,dy\Big)\widetilde{\chi}_{G_{l+1}\setminus G_{l+2}}(x)\\
&= \sum_{l=-\infty}^{\infty} \Big\{\sum_{j=l}^{\infty}\int_{G_j\setminus G_{j+1}} f(y)\,dy\Big\}(\widetilde{\chi}_{G_{l+1}\setminus G_{l+2}}(x) - \widetilde{\chi}_{G_l\setminus G_{l+1}}(x))\\
&=: \sum_{l=-\infty}^{\infty} h_l(x)\,,
\end{aligned}$$

where

$$\begin{aligned}
h_l(x) &:= \Big(\sum_{j=l}^{\infty}\int_{G_j\setminus G_{j+1}} f(y)\,dy\Big)(\widetilde{\chi}_{G_{l+1}\setminus G_{l+2}}(x) - \widetilde{\chi}_{G_l\setminus G_{l+1}}(x))\\
&= \Big(\int_{G_l} f(y)\,dy\Big)(\widetilde{\chi}_{G_{l+1}\setminus G_{l+2}}(x) - \widetilde{\chi}_{G_l\setminus G_{l+1}}(x))\,.
\end{aligned}$$

Therefore $\operatorname{supp} h_l \subset G_l\setminus G_{l+2} \subset G_l$ and $\int h_l(x)\,dx = 0$. In addition, by Lemma 1(d) of [5] for $\alpha = 0$ and (1) in [8], we have

$$\begin{aligned}
|h_l(x)| &\le C_1\mu(G_l)f^*(x)\chi_{G_l}(x)\\
&\quad\times(\mu(G_l)^{-1}\chi_{G_l\setminus G_{l+1}}(x) + \mu(G_{l+1})^{-1}\chi_{G_{l+1}\setminus G_{l+2}}(x))\\
&\le C_1 f^*(x)\chi_{G_l\setminus G_{l+1}}(x) + C_1 f^*(x)\chi_{G_{l+1}\setminus G_{l+2}}(x))\,,
\end{aligned}$$

so

$$\|h_l\|_{L^p_\alpha(G)} \le C_1\|f^*\chi_{G_l\setminus G_{l+1}}\|_{L^p_\alpha(G)} + C_1\|f^*\chi_{G_{l+1}\setminus G_{l+2}}\|_{L^p_\alpha(G)}\,.$$

Thus

$$\begin{aligned}
a_l(x) &:= \{C_1\|f^*\chi_{G_l\setminus G_{l+1}}\|_{L^p_\alpha(G)} + C_1\|f^*\chi_{G_{l+1}\setminus G_{l+2}}\|_{L^p_\alpha(G)}\}^{-1}\\
&\quad\times\mu_\alpha(G_l)^{1/p-1}h_l(x)
\end{aligned}$$

is a central $(1,p)_\alpha$-atom. If we write

$$\lambda_l := \{C_1\|f^*\chi_{G_l\setminus G_{l+1}}\|_{L^p_\alpha(G)} + C_1\|f^*\chi_{G_{l+1}\setminus G_{l+2}}\|_{L^p_\alpha(G)}\}\mu_\alpha(G_l)^{1-1/p}\,,$$

then $I_2 = \sum \lambda_l a_l$, and

$$\sum_{l=-\infty}^{\infty} |\lambda_l| \le C_1 \|f^*\|_{K^1_{p,\alpha}(G)} = C_1 \|f\|_{HK^1_{p,\alpha}(G)} .$$

It remains to verify that $f = \sum \lambda_l a_l$ in $S'(G)$. We first prove

$$\lambda_l \int_{G_{l+1}\setminus G_{l+2}} a_l(x)\,dx \to 0 \quad \text{as } l \to \infty . \tag{1}$$

This can be deduced from

$$\int_{G_{l+1}\setminus G_{l+2}} b_l(x)\,dx \to 0 \quad \text{and} \quad \int_{G_{l+1}\setminus G_{l+2}} h_l(x)\,dx \to 0$$

as $l \to \infty$. By the definitions of $b_l$ and $h_l$, (1) is easily reduced to

$$\int_{G_l\setminus G_{l+1}} f(y)\,dy \to 0 \quad \text{and} \quad \int_{G_l} f(y)\,dy \to 0 \tag{2}$$

as $l \to \infty$. If $\alpha = 0$, since $f \in L^1(G)$ and $0 < \mu(G_l \setminus G_{l+1}) \le \mu(G_l) \to 0$ as $l \to \infty$, (2) holds. If $-1 < \alpha < 0$, since $f \in L^1_\alpha(G)$, noting that $|y| \le (m_l)^{-1}$ for $y \in G_l$, and $|y| = (m_l)^{-1}$ for $y \in G_l \setminus G_{l+1}$, we have

$$\Big|\int_{G_l} f(y)\,dy\Big| \le (m_l)^\alpha \int_{G_l} |f(y)||y|^\alpha\,dy \le (m_l)^\alpha \|f\|_{L^1_\alpha(G)} \to 0$$

as $l \to \infty$, and

$$\Big|\int_{G_l\setminus G_{l+1}} f(y)\,dy\Big| \le \int_{G_l\setminus G_{l+1}} |f(y)|\,dy \le \int_{G_l} |f(y)|\,dy \to 0$$

as $l \to \infty$, thus (2) also holds.

Now, assume $\varphi \in S(G)$, $\operatorname{supp}\varphi \subset G_m$ and $\varphi$ is constant on the cosets of $G_{t_0}$ in $G$ but not on the cosets of $G_{t_0-1}$ (unless $\varphi(x) \equiv 0$). Obviously, $t_0 \ge m$. Because $f(x) = \sum \lambda_l a_l(x)$ pointwise, from (1) and $\int_{G_l\setminus G_{l+2}} a_l(x)\,d\mu(x) = 0$, it follows that

$$\begin{aligned}
\langle f, \varphi\rangle &= \int \Big(\sum_{l=-\infty}^{\infty} \lambda_l a_l(x)\Big)\varphi(x)\,dx \\
&= \sum_{t=m}^{\infty} \int_{G_t\setminus G_{t+1}} \Big(\sum_{l=t-1}^{t} \lambda_l a_l(x)\Big)\varphi(x)\,dx \\
&= \sum_{t=m}^{\infty} \Big(\lambda_{t-1} \int_{G_t\setminus G_{t+1}} a_{t-1}\varphi + \lambda_t \int_{G_t\setminus G_{t+1}} a_t\varphi\Big)
\end{aligned}$$

$$= \begin{cases} \lambda_{m-1} \int_{G_m \setminus G_{m+1}} a_{m-1}\varphi, & t_0 = m, \\ \lambda_{m-1} \int_{G_m \setminus G_{m+1}} a_{m-1}\varphi + \sum_{t=m}^{t_0-1} \lambda_t \int a_t\varphi, & t_0 > m, \end{cases}$$

$$= \lim_{\substack{m_1\to\infty \\ m_2\to\infty}} \int \Big( \sum_{l=-m_1}^{m_2} \lambda_l a_l \Big) \varphi \,,$$

that is, $f = \sum \lambda_l a_l$ in $S'(G)$. Thus, we have shown the necessity of Theorem 2.5 for $q = 1$.

If $0 < q < 1$, we have $\lim_{n\to\infty} f_n = \lim_{n\to\infty} f * \varDelta_n = f$ in $S'(G)$ and $f_n(x) = f * \varDelta_n(x)$ is a function on $G$ which is constant on the cosets of $G_n$ in $G$. It is easy to show the following facts:

$$|f_n(x)| \le f^*(x)\,, \tag{3}$$

$$(f_n)^*(x) \le f^*(x)\,, \tag{4}$$

$$\int\limits_{G_l \setminus G_{l+1}} f_n(x)\,dx \to 0 \quad \text{and} \quad \int\limits_{G_l} f_n(x)\,dx \to 0 \quad \text{as } l \to \infty\,. \tag{5}$$

Using (3)–(5) for $f_n$, and repeating the above process, we obtain

$$f_n(x) = \sum_{l=-\infty}^{\infty} \lambda_l a_l^n(x) \tag{6}$$

in $S'(G)$ and pointwise, where $\sum_{l=-\infty}^{\infty} |\lambda_l|^q \le C\|f\|^q_{HK^q_{p,\alpha}(G)}$, and each $a_l^n(x)$ is a central $(q,p)_\alpha$-atom supported in $G_l$, in fact, $\operatorname{supp} a_l^n \subset G_l \setminus G_{l+2}$. Since

$$\sup_{n\in\mathbb{N}} \|a_0^n(x)\|_{L^p_\alpha(G)} \le \mu_\alpha(G_0)^{1/p-1/q}\,,$$

the Banach–Alaoglu theorem implies that there exists a subsequence $\{a_0^{n_{v_0}}\}$ of $\{a_0^n\}$ converging in the weak* topology of $L^p_\alpha(G)$ to some $a_0 \in L^p_\alpha(G)$. It is easy to verify that $a_0$ is a central $(q,p)_\alpha$-atom supported in $G_0$. Next, since

$$\sup_{n_{v_0}} \|a_1^{n_{v_0}}(x)\|_{L^p_\alpha(G)} \le \mu_\alpha(G_1)^{1/p-1/q}\,,$$

another application of the Banach–Alaoglu theorem yields a subsequence $\{a_1^{n_{v_1}}\}$ of $\{a_1^{n_{v_0}}\}$ and a central $(q,p)_\alpha$-atom $a_1$ with $\operatorname{supp} a_1 \subset G_1$ which converges weak* in $L^p_\alpha(G)$ to $a_1$. Furthermore, because

$$\sup_{n_{v_1}} \|a_{-1}^{n_{v_1}}(x)\|_{L^p_\alpha(G)} \le \mu_\alpha(G_{-1})^{1/p-1/q}\,,$$

similarly, by the Banach–Alaoglu theorem, we obtain a subsequence $\{a_{-1}^{n_{v_{-1}}}\}$ of $\{a_{-1}^{n_{v_1}}\}$ which converges weak* in $L^p_\alpha(G)$ to some $a_{-1} \in L^p_\alpha(G)$, and $a_{-1}$ is a central $(q,p)_\alpha$-atom supported in $G_{-1}$. Repeating the above process, for each $l \in \mathbb{Z}$, we can find a subsequence $\{a_l^{n_{v_l}}\}$ of $\{a_l^n\}$ converging weak* in $L^p_\alpha(G)$ to some $a_l \in L^p_\alpha(G)$, and $a_l$ is a central $(q,p)_\alpha$-atom supported in $G_l$. By the usual diagonal method we obtain a sequence $\{n_v\}$ of natural numbers such that for each $l \in \mathbb{Z}$, $\lim_{v\to\infty} a_l^{n_v} = a_l$ in the weak* topology of $L^p_\alpha(G)$, and therefore, in $S'(G)$.

We shall prove that

$$f = \sum_{l=-\infty}^{\infty} \lambda_l a_l \tag{7}$$

in $S'(G)$. To do this, take any $\varphi \in S(G)$ and suppose $\operatorname{supp}\varphi \subset G_m$ and $\varphi$ is constant on the cosets of $G_{t_0}$ in $G$ but not on the cosets of $G_{t_0-1}$. Then, similarly to the proof of (6), we have

$$\begin{aligned}
\langle f,\varphi\rangle &= \lim_{n_v\to\infty}\langle f_{n_v},\varphi\rangle = \lim_{n_v\to\infty}\Big\langle \sum_{l=-\infty}^{\infty}\lambda_l a_l^{n_v},\varphi\Big\rangle\\
&= \begin{cases} \lim_{n_v\to\infty}\lambda_{m-1}\int a_{m-1}^{n_v}\varphi, & t_0=m,\\ \lim_{n_v\to\infty}\{\lambda_{m-1}\int a_{m-1}^{n_v}\varphi + \sum_{l=m}^{t_0-1}\lambda_l\int a_l^{n_v}\varphi\}, & t_0>m,\end{cases}\\
&= \begin{cases} \lambda_{m-1}\int a_{m-1}\varphi, & t_0=m,\\ \lambda_{m-1}\int a_{m-1}\varphi + \sum_{l=m}^{t_0-1}\lambda_l\int a_l\varphi, & t_0>m,\end{cases}\\
&= \lim_{\substack{m_1\to\infty\\ m_2\to\infty}} \int\Big(\sum_{l=-m_1}^{m_2}\lambda_l a_l\Big)\varphi\,,
\end{aligned}$$

that is, (7) holds in $S'(G)$. The necessity of Theorem 2.5 has been shown.

Conversely, suppose $f=\sum_{j=-\infty}^{\infty}\lambda_j a_j$ in $S'(G)$, where $a_j$ is a central $(q,p)_\alpha$-atom. Then $f^*(x)\le\sum_{j=-\infty}^{\infty}|\lambda_j|a_j^*(x)$ and

$$\|f^*\|^q_{K^q_{p,\alpha}(G)} \le \sum_{j=-\infty}^{\infty}|\lambda_j|^q\|a_j^*\|^q_{K^q_{p,\alpha}(G)}\,.$$

It remains to verify that

$$\|a_j^*\|_{K^q_{p,\alpha}(G)}\le C\,,$$

where $C$ is independent of $a_j$. Assuming $\operatorname{supp}a_j\subset G_{n_j}$, we first prove that $\operatorname{supp}a_j^*\subset G_{n_j}$. We have

$$a_j * \Delta_n(x) = m_n\int_{G_{n_j}\cap(x+G_n)} a_j(y)\,dy\,.$$

Thus, if $x\notin G_{n_j}$ then for $n_j\ge n$, $G_{n_j}\subseteq G_n$ and $G_{n_j}\cap(x+G_n)\ne\emptyset$ implies $G_{n_j}\cap(x+G_n)\subseteq G_n$, so $a_j*\Delta_n(x)=0$. For $n_j<n$, we have $G_n\subseteq G_{n_j}$ and thus $G_{n_j}\cap(x+G_n)=\emptyset$. Note $a_j^*(x)=\sup_{n\in\mathbb{Z}}(a_j*\Delta_n)(x)$

and $\operatorname{supp} a_j^* \subset G_{n_j}$; thus

$$\begin{aligned}
\|a_j^*\|^q_{K^q_{p,\alpha}(G)} &= \sum_{l=-\infty}^{\infty} \mu_\alpha(G_l)^{1-q/p} \|a_j^* \chi_{G_l \setminus G_{l+1}}\|^q_{L^p_\alpha(G)} \\
&\le C \sum_{l=n_j}^{\infty} \mu_\alpha(G_l)^{1-q/p} \|a_j\|^q_{L^p_\alpha(G)} \\
&\le C \mu_\alpha(G_{n_j})^{q/p-1} \sum_{l=n_j}^{\infty} (m_l)^{-(\alpha+1)(1-q/p)} \le C\,.
\end{aligned}$$

Here we use the fact that $\|a_j^*\|_{L^p_\alpha(G)} \le C\|a_j\|_{L^p_\alpha(G)}$ (see [2]) together with (1.1) and $(\alpha+1)(1-q/p) > 0$.

Thus the proof of Theorem 2.5 is complete.

Using Theorem 2.5, it is easy to deduce the following interpolation theorems.

THEOREM 2.6. *Let* $0 < q_1 \le 1 < p < \infty$ *and* $-1 < \alpha \le 0$. *If a linear operator* $T$ *is* $(L^p_\alpha(G), L^p_\alpha(G))$*-type and* $(HK^{q_1}_{p,\alpha}(G), HK^{q_1}_{p,\alpha}(G))$*-type, then* $T$ *is* $(HK^q_{p,\alpha}(G), HK^q_{p,\alpha}(G))$*-type, where* $q_1 \le q \le 1$.

The proof is easy, and we omit it.

THEOREM 2.7. *Suppose* $0 < q_1 < q_2 \le 1 < p < \infty$, $0 < \theta_1 < \theta_2 \le 1$, $q_i \le \theta_i$, $i = 1, 2$, *and* $-1 < \alpha \le 0$. *If a linear operator* $T$ *is* $(HK^{q_i}_{p,\alpha}(G), HK^{\theta_i}_{p,\alpha}(G))$*-type*, $i = 1, 2$, *then* $T$ *is* $(HK^q_{p,\alpha}(G), HK^\theta_{p,\alpha}(G))$*-type, where* $1/q = t/q_1 + (1-t)/q_2$, $1/\theta = t/\theta_1 + (1-t)/\theta_2$, $0 < t < 1$.

Proof. By Theorem 2.5 and $q \le \theta$, we only need to prove that if $a$ is a central $(q,p)_\alpha$-atom supported in $G_{l_0}$, then

$$\|(Ta)^*\|^\theta_{K^\theta_{p,\alpha}(G)} \le C\,,$$

where $C$ is independent of $a$.

First of all, because $a$ is a central $(q,p)_\alpha$-atom with support $G_{l_0}$, it is easy to verify that $\mu_\alpha(G_{l_0})^{1/q-1/q_i} a$ is a central $(q_i,p)_\alpha$-atom for $i = 1, 2$. So,

$$\|a\|_{HK^{q_i}_{p,\alpha}} \le \mu_\alpha(G_{l_0})^{1/q_i - 1/q} \qquad \text{for } i = 1, 2\,.$$

Next, we write $\beta_0 = (1/q_1 - 1/q_2)/(1/\theta_1 - 1/\theta_2)$. Noting that $\mu_\alpha(G_{l+1}) < \mu_\alpha(G_l)$ for $l \in \mathbb{Z}$, and $\mu_\alpha(G_l) \to \infty$ (or 0) as $l \to -\infty$ (or $\infty$), we can choose $l_1$ satisfying

$$\mu_\alpha(G_{l_1+1}) < \mu_\alpha(G_{l_0})^{\beta_0} \le \mu_\alpha(G_{l_1})\,.$$

On the other hand,

$$\begin{aligned}\|(Ta)^*\|^\theta_{K^\theta_{p,\alpha}(G)} &= \sum_{l=-\infty}^{\infty} \mu_\alpha(G_l)^{1-\theta/p}\|(Ta)^*\chi_{G_l\setminus G_{l+1}}\|^\theta_{L^p_\alpha(G)}\\ &= \sum_{l=-\infty}^{l_1}\dots + \sum_{l=l_1+1}^{\infty}\dots =: I_1+I_2\,,\end{aligned}$$

where

$$\begin{aligned}I_1 &\le \mu_\alpha(G_{l_1})^{1-\theta/\theta_1}\sum_{l=-\infty}^{l_1}\mu_\alpha(G_l)^{(1-\theta_1/p)\theta/\theta_1}\|(Ta)^*\chi_{G_l\setminus G_{l+1}}\|^{\theta_1\theta/\theta_1}_{L^p_\alpha(G)}\\ &\le \mu_\alpha(G_{l_1})^{1-\theta/\theta_1}\|Ta\|^\theta_{HK^{\theta_1}_{p,\alpha}(G)}\\ &\le C\mu_\alpha(G_{l_1})^{1-\theta/\theta_1}\|a\|^\theta_{HK^{q_1}_{p,\alpha}(G)}\\ &\le C\mu_\alpha(G_{l_0})^{\beta_0(1-\theta/\theta_1)+\theta(1/q_1-1/q)}\,.\end{aligned}$$

To estimate $I_2$, noting that $\alpha>-1$, we first have

$$\sum_{l=l_0+1}^{\infty}\mu_\alpha(G_l)\le C\sum_{l=l_1+1}^{\infty}(m_l)^{-(\alpha+1)}\le C(m_{l_1+1})^{-(\alpha+1)}\le C\mu_\alpha(G_{l_1+1})$$

by (1.1) in Section 1 of this paper and Lemma 1(a) of [5]. Therefore, using $\theta<\theta_2$ and Hölder's inequality, we have

$$\begin{aligned}I_2 &\le \Big(\sum_{l=l_1+1}^{\infty}\mu_\alpha(G_l)\Big)^{1-\theta/\theta_2}\\ &\quad\times\Big(\sum_{l=l_1+1}^{\infty}\mu_\alpha(G_l)^{1-\theta_2/p}\|(Ta)^*\chi_{G_l\setminus G_{l+1}}\|^{\theta_2}_{L^p_\alpha(G)}\Big)^{\theta/\theta_2}\\ &\le C\mu_\alpha(G_{l_1+1})^{1-\theta/\theta_2}\|Ta\|^\theta_{HK^{\theta_2}_{p,\alpha}(G)}\\ &\le C\mu_\alpha(G_{l_1+1})^{1-\theta/\theta_2}\|a\|^\theta_{HK^{q_2}_{p,\alpha}(G)}\\ &\le C\mu_\alpha(G_{l_0})^{\beta_0(1-\theta/\theta_2)+\theta(1/q_2-1/q)}\,.\end{aligned}$$

Since $1/q=t/q_1+(1-t)/q_2$ and $1/\theta=t/\theta_1+(1-t)/\theta_2$, we have $\beta_0=(1/q_1-1/q)/(1/\theta_1-1/\theta)=(1/q_2-1/q)/(1/\theta_2-1/\theta)$. Thus,

$$\|(Ta)^*\|^\theta_{K^\theta_{p,\alpha}(G)}\le C\,,$$

where $C$ is independent of $a$. This finishes the proof of Theorem 2.7.

Next, we consider the dual spaces of $HK^q_p(G):=HK^q_{p,0}(G)$, where $0<q\le 1<p<\infty$. We first define some spaces $\mathrm{CMO}^q_p(G)$ of central mean oscillation functions.

Definition 2.8. Let $0<q\le 1<p<\infty$. A function $f\in L^p_{\mathrm{loc}}(G)$ will be said to belong to $\mathrm{CMO}^q_p(G)$ if and only if for every $n\in\mathbb{Z}$, there exists a constant $C_n$ such that

$$\sup_{n\in\mathbb{Z}} \mu(G_n)^{1-1/q}\Big(\mu(G_n)^{-1}\int_{G_n} |f(x)-C_n|^p\,dx\Big)^{1/p} < \infty\,.$$

It is easy to verify that we can take $C_n = m_{G_n}(f) = \mu(G_n)^{-1}\int_{G_n} f(x)\,dx$; set

$$\|f\|_{\mathrm{CMO}_p^q(G)} := \sup_{n\in\mathbb{Z}} \mu(G_n)^{1-1/q}\Big(\mu(G_n)^{-1}\int_{G_n} |f(x)-m_{G_n}(f)|^p\,dx\Big)^{1/p}.$$

For the space $HK_p^q(G)$, we have the following duality theorem.

Theorem 2.9. *Let $0<q\le 1<p<\infty$, and $1/p+1/p'=1$. Then*

$$(HK_p^q(G))^* = \mathrm{CMO}_{p'}^q(G)$$

*in the following sense. Given $g\in\mathrm{CMO}_{p'}^q(G)$, the functional $\Lambda_g$ defined for finite combinations of atoms $f=\sum_{\text{finite}}\lambda_j a_j\in HK_p^q(G)$ by*

$$\Lambda_g(f) = \int_G f(x)g(x)\,dx$$

*extends uniquely to a continuous linear functional $\Lambda_g\in(HK_p^q(G))^*$ whose $(HK_p^q(G))^*$-norm satisfies*

$$\|\Lambda_g\|\le C\|g\|_{\mathrm{CMO}_{p'}^q(G)}\,.$$

*Conversely, given $\Lambda\in(HK_p^q(G))^*$, there exists a unique (up to constants) $g\in\mathrm{CMO}_{p'}^q(G)$ such that $\Lambda=\Lambda_g$. Further,*

$$\|g\|_{\mathrm{CMO}_{p'}^q(G)}\le C\|\Lambda\|\,.$$

Proof. Take $g\in\mathrm{CMO}_{p'}^q(G)$. If $a$ is a central $(q,p)$-atom (i.e., $(q,p)_0$-atom) supported in $G_n$, then

$$\begin{aligned}|\Lambda_g(a)| &= \Big|\int_G g(x)a(x)\,dx\Big| = \Big|\int_{G_n} a(x)(g(x)-m_{G_n}(g))\,dx\Big|\\ &\le \Big(\int_{G_n}|a(x)|^p\,dx\Big)^{1/p}\Big(\int_{G_n}|g(x)-m_{G_n}(g)|^{p'}\,dx\Big)^{1/p'}\\ &\le \mu(G_n)^{1/p-1/q}\Big(\int_{G_n}|g(x)-m_{G_n}(g)|^{p'}\,dx\Big)^{1/p'}\\ &\le \|g\|_{\mathrm{CMO}_{p'}^q(G)}\,.\end{aligned}$$

Thus, if $f=\sum_{\text{finite}}\lambda_j a_j\in HK^q_p(G)$, where each $a_j$ is a central $(q,p)$-atom, then

$$\Big|\int f(x)g(x)\,dx\Big|\le\sum_{\text{finite}}|\lambda_k|\Big|\int a_k(x)g(x)\,dx\Big|$$
$$\le\sum|\lambda_k|\|g\|_{\mathrm{CMO}^q_{p'}(G)}\le\Big(\sum|\lambda_k|^q\Big)^{1/q}\|g\|_{\mathrm{CMO}^q_{p'}(G)}\,,$$

that is, $|\Lambda_g(f)|\le C\|f\|_{HK^q_p(G)}\|g\|_{\mathrm{CMO}^q_{p'}(G)}$.

Obviously, the class of finite combinations of atoms is dense in $HK^q_p(G)$, so $\Lambda_g$ can be extended to a continuous linear functional on $HK^q_p(G)$, and $\|\Lambda_g\|\le C\|g\|_{\mathrm{CMO}^q_{p'}(G)}$.

Conversely, given $\Lambda\in(HK^q_p(G))^*$, we must prove that there exists a unique (up to constants) $g\in\mathrm{CMO}^q_{p'}(G)$ such that $\Lambda=\Lambda_g$, and $\|g\|_{\mathrm{CMO}^q_{p'}(G)}\le C\|\Lambda\|$.

Fixing $n\in\mathbb{Z}$, let $L^p_0(G_n):=\{f\in L^p(G_n):\int_{G_n}f(x)\,dx=0\}$. For each $f\in L^p_0(G_n)$, it is easy to see that $g(x)=\mu(G_n)^{1/p-1/q}\|f\|_p^{-1}f(x)$ is a central $(q,p)$-atom, where $\|f\|_p=(\int|f(x)|^p\,dx)^{1/p}$ for $1<p<\infty$. Therefore,

$$f(x)=\mu(G_n)^{1/q-1/p}\|f\|_p g(x)\in HK^q_p(G)\,.$$

Moreover, we have

$$\|f\|_{HK^q_p(G)}\le\mu(G_n)^{1/q-1/p}\|f\|_p\,.$$

Thus, if $\Lambda\in(HK^q_p(G))^*$, it follows that

$$|\Lambda(f)|\le\|\Lambda\|\,\|f\|_{HK^q_p(G)}\le(\mu(G_n)^{1/q-1/p}\|\Lambda\|)\|f\|_p\,.$$

That is, $\Lambda\in(L^p_0(G_n))^*$. From this, we know that there exists a $g_n\in L^{p'}_0(G_n)\subset L^{p'}_{\mathrm{loc}}(G)$ such that

$$\Lambda(f)=\int_{G_n}f(x)g_n(x)\,dx$$

for any $f\in L^p_0(G_n)$, where $1/p+1/p'=1$. In the following, we need to construct a function $g\in L^{p'}_{\mathrm{loc}}(G)$ such that

$$\Lambda(f)=\int_G f(x)g(x)\,dx$$

for any $f\in\bigcup_{n=-\infty}^{\infty}L^p_0(G_n)$.

Let $f\in L^p_0(G_0)$. From the above argument, we know that there exists a $g_0\in L^{p'}_{\mathrm{loc}}(G)$ such that

$$\Lambda(f)=\int_{G_0}f(x)g_0(x)\,dx\,.$$

But, since $L_0^p(G_n) \subset L_0^p(G_0),\ n \in \mathbb{N}$, we have

$$\Lambda(f) = \int_G f(x) g_0(x)\, dx$$

for any $f \in \bigcup_{n=0}^{\infty} L_0^p(G_n) = L_0^p(G_0)$. In addition, because $L_0^p(G_0) \subset L_0^p(G_{-1})$, from the above argument, there exists a $g_{-1} \in L_{\mathrm{loc}}^{p'}(G)$ such that

$$\Lambda(f) = \int_{G_{-1}} f(x) g_{-1}(x)\, dx = \int_{G_0} f(x) g_0(x)\, dx\,.$$

It follows that

$$\int_{G_0} f(x)\big\{g_0(x) - g_{-1}(x)\big\}\, dx = 0$$

for any $f \in L_0^p(G_0)$. For each $f \in L^p(G_0)$, if we write $h(x) = f(x) - m_{G_0}(f)$, then $h \in L_0^p(G_0)$. From this and the above equality, we have

$$\int_{G_0} f(x)\big\{(g_0(x) - g_{-1}(x)) - m_{G_0}(g_0 - g_{-1})\big\}\, dx = 0$$

for any $f \in L^p(G_0)$. Thus, if $x \in G_0$, then $g_0(x) = g_{-1}(x) + C_{-1}$, where $C_{-1}$ is a constant. Define

$$g(x) = \begin{cases} g_0(x), & x \in G_0, \\ g_{-1}(x) + C_{-1}, & x \in G_{-1} \setminus G_0. \end{cases}$$

Then $g(x) = g_{-1}(x) + C_{-1}$ for $x \in G_{-1}$. It is easy to see that

$$\Lambda(f) = \int_G f(x) g(x)\, dx$$

for each $f \in \bigcup_{n=-1}^{\infty} L_0^p(G_n)$. Continuing, we obtain the desired function $g$ defined on $G$. It remains to verify $g \in \mathrm{CMO}_{p'}^q(G)$. For each $n \in \mathbb{Z}$, we have

$$\Big(\int_{G_n} |g(x) - m_{G_n}(g)|^{p'}\, dx\Big)^{1/p'}$$
$$= \sup\Big\{\Big|\int_G (g(x) - m_{G_n}(g)) h(x)\, dx\Big| : \|h\|_{L^p(G_n)} = 1,\ \operatorname{supp} h \subset G_n\Big\}.$$

From

$$\int_{G_n} (g(x) - m_{G_n}(g)) h(x)\, dx = \int_{G_n} g(x)(h(x) - m_{G_n}(h))\, dx$$

and

$$\|(h - m_{G_n}(h))\chi_{G_n}\|_{HK_p^q(G)} \le C|\mu(G_n)|^{1/q - 1/p}\,,$$

it follows that

$$\Big(\int\limits_{G_n} |g(x)-m_{G_n}(g)|^{p'}\,dx\Big)^{1/p'}$$
$$\le \sup\{|\Lambda(h)| : h\in L^p_0(G_n),\ \|h\|_{L^p(G_n)}\le C\}$$
$$\le C\|\Lambda\||\mu(G_n)|^{1/q-1/p}$$

for each $n\in\mathbb{Z}$, and therefore

$$\|g\|_{\mathrm{CMO}^q_{p'}(G)}\le C\|\Lambda\|\,.$$

We have finished the proof of Theorem 2.9.

**3. The molecular characterization.** We first define weighted central molecules.

DEFINITION 3.1. Assume $0<q\le 1<p<\infty$, $-1<\alpha\le 0$ and $b>\max\{(1+\alpha)(1/q-1/p),\,1-(1+\alpha)/p\}$. A function $M(x)$ on $G$ is said to be a *central* $(q,p,b)_\alpha$*-molecule* if

(1) $\int M(x)\,dx=0$,

(2) $\Re_{p,\alpha}(M):=\|M\|^{1-\theta}_{L^p_\alpha(G)}\||x|^bM\|^{\theta}_{L^p_\alpha(G)}<\infty$

where $\theta=(1/q-1/p)(\alpha+1)/b$.

We first point out that the central $(q,p,b)_\alpha$-molecule is indeed a generalization of the central $(q,p)_\alpha$-atom.

PROPOSITION 3.2. *Let* $0<q\le 1<p<\infty$ *and* $-1<\alpha\le 0$. *If* $a$ *is a central* $(q,p)_\alpha$*-atom, then* $a$ *is a central* $(q,p,b)_\alpha$*-molecule and* $\Re_{p,\alpha}(a)\le C$, *where* $b>\max\{(1+\alpha)(1/q-1/p),1-(1+\alpha)/p\}$ *and* $C$ *is independent of* $a$.

P r o o f. We only need to verify that $a$ satisfies (2) of Definition 3.1. Assume $\operatorname{supp} a\subset G_n$. Note that if $x\in G_n$, then $|x|\le (m_n)^{-1}$. Therefore, we have

$$\||x|^ba\|_{L^p_\alpha(G)}\le (m_n)^{-b}\|a\|_{L^p_\alpha(G)}\,.$$

It follows that

$$\begin{aligned}\Re_{p,\alpha}(a)&=\|a\|^{1-\theta}_{L^p_\alpha(G)}\||x|^ba\|^{\theta}_{L^p_\alpha(G)}\le (m_n)^{-b\theta}\|a\|_{L^p_\alpha(G)}\\&\le (m_n)^{-b\theta}\mu_\alpha(G_n)^{1/p-1/q}\le C\,,\end{aligned}$$

where $C$ is independent of $a$.

This completes the proof of the proposition.

We can now characterize the space $HK^q_{p,\alpha}(G)$ in terms of molecules.

THEOREM 3.3. *Suppose* $0<q\le 1<p<\infty$, $-1<\alpha\le 0$ *and* $b>\max\{(1+\alpha)(1/q-1/p),1-(1+\alpha)/p\}$. *A distribution* $f$ *on* $G$ *is in* $HK^q_{p,\alpha}(G)$

*if and only if* $f=\sum_k \lambda_k M_k$, *both in* $S'(G)$ *and pointwise, where each* $M_k$ *is a central* $(q,p,b)_\alpha$*-molecule,* $\Re_{p,\alpha}(M_k)\le C$ *and* $\sum|\lambda_k|^q<\infty$. *Further,*

$$\|f\|_{HK^q_{p,\alpha}(G)} \sim \inf\Big\{\Big(\sum|\lambda_k|^q\Big)^{1/q}\Big\}$$

*where the infimum is taken over all molecular decompositions of* $f$. *Moreover, for* $q=1$ *the identity* $f(x)=\sum\lambda_k M_k(x)$ *holds pointwise.*

The proof is immediately deduced from Proposition 3.2 and the following proposition.

Proposition 3.4. *If* $M$ *is a central* $(q,p,b)_\alpha$*-molecule, then* $M\in HK^q_{p,\alpha}(G)$ *and*

$$\|M\|_{HK^q_{p,\alpha}(G)} \le C\Re_{p,\alpha}(M)$$

*where* $C$ *is independent of* $M$.

Proof. Without loss of generality, we can suppose $\Re_{p,\alpha}(M)=1$, therefore, $\||x|^bM\|^{\theta}_{L^p_\alpha(G)}=\|M\|^{\theta-1}_{L^p_\alpha(G)}$. Choose $n_0\in\mathbb{Z}$ satisfying

$$\mu_\alpha(G_{n_0})\le \|M\|^{-(1/q-1/p)^{-1}}_{L^p_\alpha(G)} < \mu_\alpha(G_{n_0-1})\,.$$

Write $B_{-1}:=\emptyset$ and $B_k:=G_{n_0-k}$ for $k\in\mathbb{Z}_+$. Moreover, for $k\in\mathbb{Z}_+$, define $D_k:=B_k\setminus B_{k-1}$. Let

$$M_k(x):=\mu(D_k)^{-1}\Big(\int\limits_{D_k} M(t)\,d\mu(t)\Big)\chi_{D_k}(x)\,.$$

Note that if $x\in D_0=B_0=G_{n_0},\ |x|\le(m_{n_0})^{-1}$, and $\alpha\le 0$, then

$$\|M_0\|^p_{L^p_\alpha(G)}\le C\int\limits_{D_0}|M(x)|^p|x|^\alpha\,dx$$

by Hölder's inequality. That is, $\|M_0\|_{L^p_\alpha(G)}\le C\|M\|_{L^p_\alpha(G)}$ with $C$ independent of $n_0$ and thus of $M$. If $k\ge1$, then

$$\|M_k\|_{L^p_\alpha(G)}=\Big(\int\limits_{D_k}|M_k(x)|^p\,d\mu_\alpha(x)\Big)^{1/p}\le C\mu(B_k)^{-b}\||x|^bM\|_{L^p_\alpha(G)}\,.$$

Now, we decompose $M(x)$ as follows:

$$\begin{aligned} M(x) &= \sum_{k=0}^{\infty}(M(x)-M_k(x))\chi_{D_k}(x)+\sum_{k=0}^{\infty}M_k(x)\chi_{D_k}(x)\\ &= \sum_{k=0}^{\infty}(M(x)-M_k(x))\chi_{D_k}(x)\end{aligned}$$

$$
+\sum_{k=1}^{\infty}\Big(\int\limits_{G\setminus B_{k-1}} M(t)\,d\mu(t)\Big)\Big(\frac{\chi_{D_k}(x)}{\mu(D_k)}-\frac{\chi_{D_{k-1}}(x)}{\mu(D_{k-1})}\Big)
$$

$$
=: \sum_{k=0}^{\infty} a_k(x)+\sum_{k=1}^{\infty} b_k(x)\,.
$$

Then

$$
\|a_0\|_{L^p_\alpha(G)} \le C\|M\|_{L^p_\alpha(G)}\,,
$$

and, for $k \ge 1$,

$$
\|a_k\|_{L^p_\alpha(G)} \le C\Big(\int\limits_{D_k} |M(x)|^p\,d\mu_\alpha(x)\Big)^{1/p} \le C\mu(B_k)^{-b}\||x|^b M\|_{L^p_\alpha(G)}\,.
$$

Moreover, $\operatorname{supp} a_0 \subset D_0 \subset B_0$, $\operatorname{supp} a_k \subset D_k = B_k \setminus B_{k-1} \subset B_k$ for $k \in \mathbb{N}$, and if $k \in \mathbb{Z}_+$, then $\int a_k(x)\,dx = 0$. If we write

$$
a_k^*(x) := \|a_k\|^{-1}_{L^p_\alpha(G)} a_k(x)\mu_\alpha(B_k)^{-(1/q-1/p)}\,,
$$

then $a_k^*$ is a central $(q,p)_\alpha$-atom, and $\operatorname{supp} a_k^* \subset B_k$. Furthermore, if $\lambda_k := \|a_k\|_{L^p_\alpha(G)}\mu_\alpha(B_k)^{1/q-1/p}$, then $\sum_{k=0}^\infty a_k(x) = \sum_{k=0}^\infty \lambda_k a_k^*(x)$ and

$$
\begin{aligned}
\sum_{k=0}^{\infty}|\lambda_k|^q &\le \|M\|^q_{L^p_\alpha(G)}\mu_\alpha(G_{n_0})^{(1/q-1/p)q} + \sum_{k=1}^{\infty}\mu_\alpha(B_k)^{1-q/p}\|a_k\|^q_{L^p_\alpha(G)} \\
&\le 1+2\sum_{k=1}^{\infty}\mu_\alpha(B_k)^{1-q/p}\mu(B_k)^{-bq}\||x|^b M\|^q_{L^p_\alpha(G)} \\
&\le 1+C\||x|^b M\|^q_{L^p_\alpha(G)}\sum_{k=1}^{\infty}(m_{n_0-k})^{q\{b-(1/q-1/p)(\alpha+1)\}} \\
&\le 1+C\|M\|^{q(\theta-1)/\theta+q(1-\theta)/\theta}_{L^p_\alpha(G)} \le C
\end{aligned}
$$

where $C$ is independent of $M$. In addition,

$$
\int\limits_{G\setminus B_{k-1}} |M(x)|\,d\mu(x) \le \||x|^b M\|_{L^p_\alpha(G)}\Big(\int\limits_{G\setminus B_{k-1}} |x|^{-(\alpha/p+b)p'}\,d\mu(x)\Big)^{1/p'}\,.
$$

Note that the integral in parentheses is

$$
\sum_{i=k}^{\infty}\int\limits_{B_i\setminus B_{i-1}} |x|^{-(\alpha/p+b)p'}\,dx \le C\mu(B_k)^{1-p'(b+\alpha/p)}\,,
$$

and thus

$$
\int\limits_{G\setminus B_{k-1}} |M(x)|\,d\mu(x) \le C\||x|^b M\|_{L^p_\alpha(G)}\mu(B_k)^{1/p'-(b+\alpha/p)}\,.
$$

On the other hand,

$$\|\mu(D_k)^{-1}\chi_{D_k}\|_{L^p_\alpha(G)} \le C\mu(B_k)^{-1+(\alpha+1)/p},$$

therefore

$$\|b_k\|_{L^p_\alpha(G)} \le C\||x|^b M\|_{L^p_\alpha(G)}\mu(B_k)^{-b}.$$

Note that $\operatorname{supp} b_k \subset D_k \cup D_{k-1} = B_k \setminus B_{k-2} \subset B_k$, $k \ge 1$, and $\int b_k(x)\,dx = 0$. Hence if we let $b_k^*(x) := \|b_k\|^{-1}_{L^p_\alpha(G)} b_k(x)\mu_\alpha(B_k)^{-(1/q-1/p)}$ and $\gamma_k := \|b_k\|_{L^p_\alpha(G)}\mu_\alpha(B_k)^{1/q-1/p}$, then $\sum_{k=1}^\infty b_k(x) = \sum_{k=1}^\infty \gamma_k b_k^*(x)$, and $b_k^*$ is a central $(q,p)_\alpha$-atom. Further,

$$\begin{aligned}\sum_{k=1}^\infty |\gamma_k|^q &\le C\||x|^b M\|^q_{L^p_\alpha(G)} \sum_{k=1}^\infty \mu(B_k)^{-bq+q(\alpha+1)(1/q-1/p)}\\ &\le C\|M\|^{q(\theta-1)/\theta}_{L^p_\alpha(G)} \sum_{k=1}^\infty (m_{n_0-k})^{q\{b-(\alpha+1)(1/q-1/p)\}} \le C,\end{aligned}$$

where $C$ is independent of $b$.

Finally, considering that $\operatorname{supp} a_k \subset \operatorname{supp} b_k \subset B_k \setminus B_{k-2}$ for $k \ge 1$ and $\operatorname{supp} a_0 \subset B_0 = G_{n_0}$, the pointwise equality $M(x) = \sum_{k=0}^\infty \lambda_k a_k(x) + \sum_{k=1}^\infty \gamma_k b_k(x)$ obviously holds in $S'(G)$. This finishes the proof of Proposition 3.4.

**4. Some application.** We establish some multiplier theorems on the space $HK^q_{p,\alpha}(G)$ in this section.

THEOREM 4.1. *Let $0 < q \le 1 < p < \infty$ and $-1 < \alpha \le 0$. Suppose $\varphi$ is a multiplier on $L^p_\alpha(G)$, and consider the condition*

$$(*) \qquad \sup_{l<n} \Big\{ \sum_{j=n}^\infty \|(D^\lambda \varphi^j)^\vee\|_{L^r(G_l \setminus G_{l+1})} \Big\} \le C(m_n)^{1/r'-\lambda},$$

*where $\varphi^j := \varphi\chi_{\Gamma_{j+1}\setminus\Gamma_j}$, $D^\lambda\varphi^j := (|x|^\lambda(\varphi^j)^\vee(x))^\wedge$, $1 < r < \infty$ and $1/r + 1/r' = 1$. If the condition $(*)$ is satisfied in the following two cases*:

(i) *for some $r \ge p$ and some $\lambda > (\alpha+1)/q - 1/r$,*

*or, in the case when $p' < p$ where $1/p + 1/p' = 1$,*

(ii) *for some $r$ with $p' \le r < p$ and some $\lambda > (\alpha+1)/q$,*

*then $\varphi$ is a multiplier on $HK^q_{p,\alpha}(G)$.*

Proof. Let $\varphi_k := \varphi\chi_{\Gamma_k}$ and $f := (\varphi_k)^\vee * a$. By Theorem 2.5, we need only prove that for any central $(q,p)_\alpha$-atom $a$,

$$\|f^*\|_{K^q_{p,\alpha}(G)} \le C,$$

where $C$ is independent of $k$ and $a$.

Suppose that $\operatorname{supp} a \subset G_n$. If $l \geq n$, then

$$\Big(\int_{G_l\setminus G_{l+1}} |f^*(x)|^p\,dx\Big)^{1/p} \leq \|f^*\|_{L^p_\alpha(G)} \leq C\|f\|_{L^p_\alpha(G)} \leq C\|a\|_{L^p_\alpha(G)} \leq C(m_n)^{-(\alpha+1)(1/p-1/q)}$$

by the results of Kitada [2]. Therefore,

$$\Big(\sum_{l=n}^{\infty}\mu(G_l)^{1-q/p}\|f^*\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)}\Big)^{1/q} \leq C(m_n)^{-(\alpha+1)(1/p-1/q)}\Big\{\sum_{l=n}^{\infty}(m_l)^{-(\alpha+1)(1-q/p)}\Big\}^{1/q} \leq C\,.$$

If $l < n$, then $(G_l\setminus G_{l+1})\cap G_n = \emptyset$ and $G_{n+1}\subset G_n\subset G_l$. By Kitada's proof of Theorem 2 in [2], we have

$$f^*(x) \leq \sum_{j=n}^{\infty}|(\varphi^j)^\vee * a(x)|$$

and

$$\begin{aligned}\Big(\int_{G_l\setminus G_{l+1}} |f^*(x)|^p\,d\mu_\alpha(x)\Big)^{1/p} &\leq \sum_{j=n}^{\infty}\Big(\int_{G_l\setminus G_{l+1}}|(\varphi^j)^\vee * a(x)|^p\,d\mu_\alpha(x)\Big)^{1/p}\\ &\leq \sum_{j=n}^{\infty}(m_l)^{-\alpha/p}\|a\|_1\Big(\int_{G_l\setminus G_{l+1}}|(\varphi^j)^\vee(x)|^p\,d\mu(x)\Big)^{1/p}\\ &= \sum_{j=n}^{\infty}(m_l)^{-\alpha/p+\lambda}\|a\|_1\Big(\int_{G_l\setminus G_{l+1}}|(D^\lambda\varphi^j)^\vee(x)|^p\,d\mu(x)\Big)^{1/p},\end{aligned}$$

where $\|a\|_1 = \int|a(x)|\,dx$. If $p \leq r$, then

$$\Big(\int_{G_l\setminus G_{l+1}} |f^*(x)|^p\,d\mu_\alpha(x)\Big)^{1/p} \leq (m_l)^{-(\alpha+1)/p+\lambda+1/r}\|a\|_1\sum_{j=n}^{\infty}\|(D^\lambda\varphi^j)^\vee\|_{L^r(G_l\setminus G_{l+1})}\,.$$

For $\lambda > (\alpha+1)/q - 1/r$, from $(*)$ and

$$\|a\|_1 \leq \|a\|_p(m_n)^{-(1-1/p)} \leq \|a\|_{L^p_\alpha(G)}(m_n)^{(\alpha+1)/p-1} \leq C(m_n)^{(\alpha+1)/q-1}\,,$$

we have

$$\Big(\sum_{l=-\infty}^{n-1}\mu(G_l)^{1-q/p}\|f^*\chi_{G_l\setminus G_{l+1}}\|^q_{L^p_\alpha(G)}\Big)^{1/q}$$
$$\le C\|a\|_1\Big(\sum_{l=-\infty}^{n-1}(m_l)^{-(\alpha+1)+(\lambda+1/r)q}\Big)^{1/q}(m_n)^{1/r'-\lambda}$$
$$\le C\|a\|_1(m_n)^{1-(\alpha+1)/q}\le C\,.$$

This shows the conclusion of Theorem 4.1(i).

For (ii), suppose $p'<p$, where $1/p+1/p'=1$. If $x\in G_l\setminus G_{l+1}$ then

$$|(\varphi^j)^\vee * a(x)|\le\Big(\int_{G_n}|(\varphi^j)^\vee(x-t)|^r\,dt\Big)^{1/r}\|a\|_{r'}$$
$$\le\|(\varphi^j)^\vee\chi_{G_l\setminus G_{l+1}}\|_r\|a\|_p(m_n)^{1/p-1/r'}$$
$$\le(m_n)^{(\alpha+1)/q-1/r'}(m_l)^\lambda\|(D^\lambda\varphi^j)^\vee\chi_{G_l\setminus G_{l+1}}\|_r\,.$$

Therefore,

$$\Big(\int_{G_l\setminus G_{l+1}}|f^*(x)|^p\,d\mu_\alpha(x)\Big)^{1/p}$$
$$\le\sum_{j=n}^{\infty}(m_n)^{(\alpha+1)/q-1/r'}(m_l)^{\lambda-(\alpha+1)/p}\|(D^\lambda\varphi^j)^\vee\|_{L^r(G_l\setminus G_{l+1})}$$
$$\le C(m_n)^{(\alpha+1)/q-\lambda}(m_l)^{\lambda-(\alpha+1)/p}\,.$$

For $\lambda>(\alpha+1)/q$, we have

$$\Big(\sum_{l=-\infty}^{n-1}\mu(G_l)^{1-q/p}\|f^*\|^q_{L^p_\alpha(G_l\setminus G_{l+1})}\Big)^{1/q}$$
$$\le C\Big(\sum_{l=-\infty}^{n-1}(m_l)^{-(\alpha+1)+\lambda q}\Big)^{1/q}(m_n)^{(\alpha+1)/q-\lambda}\le C\,.$$

This shows the conclusion of Theorem 4.1(ii).

COROLLARY 4.2. *If* $0<q\le1$ *and there exist* $r\ge2$ *and* $\lambda>1/q-1/r$ *such that* $\varphi\in M(r',\lambda)$ (see [1] for definition), *where* $1/r+1/r'=1$, *then* $\varphi$ *is a multiplier on* $HK^q_2(G)$.

THEOREM 4.3. *Suppose* $0<q\le1$, $-1<\alpha\le0$ *and* $\sigma$ *is a multiplier on* $L^2_\alpha(G)$. *Assume* $\sigma\in S^{-\mu}_\varrho$ (see [7] for definition, and also [9]), *where* $0\le\varrho<1$ *and* $\mu>\max\{(1-\varrho)(1/q-1/2)+\alpha((1-\varrho)/q-\varrho/2),(1-\varrho)/2-\varrho\alpha/2\}$. *Then* $\sigma$ *is a multiplier on* $HK^q_{2,\alpha}(G)$.

Proof. Using Theorem 3.3, we only need to show that if $a$ is a central $(q,2)_\alpha$-atom with support $G_n$, then $M=(\sigma\widehat{a})^\vee$ is a central $(q,2,b)_\alpha$-molecule, where $b$ satisfies

$$(\mu+\varrho\alpha-\alpha/2)/(1-\varrho)\ge b>\max\{(\alpha+1)(1/q-1/2),1-(\alpha+1)/2\}\,.$$

Obviously, we need only estimate $\Re_{2,\alpha}(M)$. First, we have

$$\|M\|_{L^2_\alpha(G)}\le C\|a\|_{L^2_\alpha(G)}\le C\mu_\alpha(G_n)^{1/2-1/q}\le C(m_n)^{(\alpha+1)(1/q-1/2)}\,.$$

It remains to estimate $\||x|^bM\|_{L^2_\alpha(G)}$. Because $\operatorname{supp} a\subset G_n$, $\widehat{a}$ is constant on $\Gamma_n$. Write $\sigma=\sigma\chi_{\Gamma_n}+\sigma\chi_{\Gamma\setminus\Gamma_n}=:\sigma_1+\sigma_2$; then $M=(\sigma\widehat{a})^\vee=(\sigma_2\widehat{a})^\vee$, and

$$\begin{aligned}\||x|^b(\sigma_2\widehat{a})^\vee\|_{L^2_\alpha(G)}&=\|(\sigma_2\widehat{a})^\vee\|_{K(b+\alpha,2,2,G)}\\&\le\Big(\sum_{l=-\infty}^{\infty}(m_l)^{-2(b+\alpha)}\sup\{\|\tau_\xi(\sigma_2\widehat{a})-\sigma_2\widehat{a}\|_2^2:\xi\in\Gamma_l\}\Big)^{1/2}\end{aligned}$$

by the theorem of [4] (or [7]). We now consider two cases: $n\ge0$ and $n<0$.

(A) $n\ge0$. Define $[n\varrho]$ to satisfy

$$\mu(G_{[n\varrho]+1})\le\mu(G_n)^\varrho<\mu(G_{[n\varrho]})\,.$$

Clearly, $n\ge[n\varrho]$ for $n\ge0$. Write

$$\begin{aligned}\||x|^b(\sigma_2\widehat{a})^\vee\|_{L^2_\alpha(G)}&\le\Big(\sum_{l=-\infty}^{[n\varrho]}\dots+\sum_{l=[n\varrho]+1}^{\infty}\dots\Big)^{1/2}\\&=:(A_1+A_2)^{1/2}\,.\end{aligned}$$

For $A_1$, because $l\le[n\varrho]\le n$, we have $\xi\in\Gamma_l\subset\Gamma_n$, and therefore, for $\gamma\in\Gamma$,

$$(\sigma_2\widehat{a})(\gamma-\xi)-(\sigma_2\widehat{a})(\gamma)=(\sigma_2(\gamma-\xi)-\sigma_2(\gamma))\widehat{a}(\gamma)\,.$$

Thus,

$$\sup\{\|\tau_\xi(\sigma_2\widehat{a})-\sigma_2\widehat{a}\|_2^2:\xi\in\Gamma_l\}\le C_k^2(m_l)^{2k}(m_n)^{-2(\mu+\varrho k)}\|\widehat{a}\|_2^2\,,$$

where we take $k>b+\alpha$. We have

$$\begin{aligned}A_1&\le C_k^2(m_n)^{-2(\mu+\varrho k)}\|\widehat{a}\|_2^2\sum_{l=-\infty}^{[n\varrho]}(m_l)^{-2(b+\alpha-k)}\\&\le C_k^2(m_n)^{-2(\mu+\varrho b+\varrho\alpha)}\|a\|_2^2\,.\end{aligned}$$

In order to estimate $A_2$, note that $|\sigma_2(\gamma)|\le\langle\gamma\rangle^{-\mu}$ and $b+\alpha>0$. Then

$$A_2\le C\sum_{l=[n\varrho]+1}^{\infty}(m_l)^{-2(b+\alpha)}\|\sigma_2\widehat{a}\|_2^2\le C(m_n)^{-2(b+\alpha)\varrho}(m_n)^{-2\mu}\|a\|_2^2\,.$$

To sum up,

$$\begin{aligned}\||x|^b(\sigma_2\widehat{a})^\vee\|_{L^2_\alpha(G)} &\le C(m_n)^{-(\mu+\varrho b+\varrho\alpha)}\|a\|_2\\ &\le C(m_n)^{-(\mu+\varrho b+\varrho\alpha)+\alpha/2}\|a\|_{L^2_\alpha(G)}\\ &\le C(m_n)^{-(\mu+\varrho b+\varrho\alpha)+\alpha/2-(\alpha+1)(1/2-1/q)}\,.\end{aligned}$$

Therefore,

$$\begin{aligned}\Re_{2,\alpha}(M) &= \|M\|^{1-\theta}_{L^2_\alpha(G)}\||x|^bM\|^{\theta}_{L^2_\alpha(G)}\\ &\le C(m_n)^{b\theta-(\mu+\varrho b+\varrho\alpha)\theta+\alpha\theta/2}\le C\,,\end{aligned}$$

because $m_n \ge 1$ and $b-(\mu+\varrho b+\varrho\alpha)+\alpha/2\le 0$.

(B) $n<0$. Then

$$\begin{aligned}\||x|^b(\sigma_2\widehat{a})^\vee\|_{L^2_\alpha(G)} &\le \Big(\sum_{l=-\infty}^{n}\ldots+\sum_{l=n+1}^{\infty}\ldots\Big)^{1/2}\\ &=: (A_1+A_2)^{1/2}\,.\end{aligned}$$

For $A_1$, note that $\langle\gamma\rangle\ge 1$. Similarly to (A), we take $k>b+\alpha$. Then,

$$A_1\le C\sum_{l=-\infty}^{n}(m_l)^{2k-2(b+\alpha)}\|a\|_2^2\le C(m_n)^{-2(b+\alpha)}\|a\|_2^2\,.$$

On the other hand,

$$A_2\le C\sum_{l=n}^{\infty}(m_l)^{-2(b+\alpha)}\|a\|_2^2\le C(m_n)^{-2(b+\alpha)}\|a\|_2^2\,.$$

So,

$$\begin{aligned}\||x|^b(\sigma_2\widehat{a})^\vee\|_{L^2_\alpha(G)} &\le C(m_n)^{-(b+\alpha)}\|a\|_2^2\\ &\le C(m_n)^{-(b+\alpha)-1/2+(\alpha+1)/q}\,.\end{aligned}$$

Therefore,

$$\Re_{2,\alpha}(M)=\|M\|^{1-\theta}_{L^2_\alpha(G)}\||x|^bM\|^{\theta}_{L^2_\alpha(G)}\le C(m_n)^{-\alpha\theta/2}\le C\,.$$

This finishes the proof of Theorem 4.3.

COROLLARY 4.4. *Let* $0<q\le 1$, $0\le\varrho<1$, $\sigma\in S^{-\mu}_{\varrho}$ *and suppose* $\mu>(1-\varrho)(1/q-1/2)$. *Then* $\sigma$ *is a multiplier on* $HK^q_2(G)$.

**Acknowledgements.** We thank the referee for a number of penetrating comments which enabled us to make significant improvements to an earlier version of this paper.

*REFERENCES*


[1] T. Kitada, *Weighted $H^p$ multipliers on locally compact Vilenkin groups*, Monatsh. Math. 110 (1990), 283–295.

[2] —, *Multipliers on weighted Hardy spaces over certain totally disconnected groups*, Internat. J. Math. Math. Sci. 11 (1988), 665–674.

[3] S. Lu and D. Yang, *Some weak Hardy spaces on locally compact Vilenkin groups*, J. Beijing Normal Univ. (Natur. Sci.) 28 (1992), 409–419.

[4] C. W. Onneweer, *Generalized Lipschitz spaces and Herz spaces on certain totally disconnected groups*, in: Lecture Notes in Math. 939, Springer, 1981, 106–121.

[5] —, *Multipliers on weighted $L_p$-spaces over certain totally disconnected groups*, Trans. Amer. Math. Soc. 288 (1985), 347–362.

[6] C. W. Onneweer and T. S. Quek, *Multipliers on weighted Hardy spaces over locally compact Vilenkin groups*, *I*, J. Austral. Math. Soc. Ser. A 48 (1990), 472–496.

[7] —, —, *Multipliers on weighted Hardy spaces over locally compact Vilenkin groups, II*, Colloq. Math. 60/61 (1990), 305–314.

[8] C. W. Onneweer and W. Su, *Homogeneous Besov spaces on locally compact Vilenkin groups*, Studia Math. 93 (1989), 17–39.

[9] L. Saloff-Coste, *Opérateurs pseudo-différentiels sur certains groupes totalement discontinus*, ibid. 83 (1986), 205–228.

[10] M. H. Taibleson, *Fourier Analysis on Local Fields*, Math. Notes 15, Princeton Univ. Press, 1975.

[11] N. Ya. Vilenkin, *On a class of complete orthonormal systems*, Amer. Math. Soc. Transl. (2) 28 (1963), 1–35.


DEPARTMENT OF MATHEMATICS
BEIJING NORMAL UNIVERSITY
BEIJING, 100875
THE PEOPLE'S REPUBLIC OF CHINA

*Reçu par la Rédaction le 20.1.1992;*
*en version modifiée le 10.12.1992*