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CURVATURE PROPERTIES OF CERTAIN COMPACT PSEUDOSYMMETRIC MANIFOLDS

BY

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1. Introduction. Let (M,g) be a connected n-dimensional, $n\ge 3$, Riemannian manifold of class C^∞ with a not necessarily definite metric g. We define on M the endomorphisms $\widetilde{R}(X,Y)$ and $X\wedge Y$ by

$$\widetilde{R}(X,Y)Z=[\nabla_X,\nabla_Y]Z-\nabla_{[X,Y]}Z\,,$$
$$(X\wedge Y)Z=g(Y,Z)X-g(X,Z)Y\,,$$

respectively, where ∇ is the Levi-Civita connection of (M,g) and $X,Y,Z\in\varXi(M)$, $\varXi(M)$ being the Lie algebra of vector fields on M. Furthermore, we define the *Riemann–Christoffel curvature tensor* R and the *concircular tensor* $Z(R)$ of (M,g) by

$$R(X_1,X_2,X_3,X_4)=g(\widetilde{R}(X_1,X_2)X_3,X_4)\,,$$
$$Z(R)(X_1,X_2,X_3,X_4)=R(X_1,X_2,X_3,X_4)-\frac{K}{n(n-1)}G(X_1,X_2,X_3,X_4)\,,$$

respectively, where K is the scalar curvature of (M,g) and G is defined by

$$G(X_1,X_2,X_3,X_4)=g((X_1\wedge X_2)X_3,X_4)\,.$$

Now we define on M the $(0,6)$-tensors $R\cdot R$ and $Q(g,R)$ by

$$(R\cdot R)(X_1,\ldots,X_4;X,Y)=-R(\widetilde{R}(X,Y)X_1,X_2,X_3,X_4)-\ldots-R(X_1,X_2,X_3,\widetilde{R}(X,Y)X_4)\,,$$
$$Q(g,R)(X_1,\ldots,X_4;X,Y)=R((X\wedge Y)X_1,X_2,X_3,X_4)+\ldots+R(X_1,X_2,X_3,(X\wedge Y)X_4)\,,$$

respectively.

The Riemannian manifold (M,g) is said to be *pseudosymmetric* [15] if at every point of M the following condition is satisfied:

$(*)$ the tensors $R\cdot R$ and $Q(g,R)$ are linearly dependent.

The manifold (M,g) is pseudosymmetric if and only if

$$R \cdot R = LQ(g,R) \tag{1}$$

on the set $U = \{x \in M \mid Z(R) \neq 0 \text{ at } x\}$, where L is some function on U. It is clear that any semisymmetric manifold ($R \cdot R = 0$, [24]) is pseudosymmetric. The condition $(*)$ arose during the study of totally umbilical submanifolds of semisymmetric manifolds ([1], [10]) as well as during the consideration of geodesic mappings of semisymmetric manifolds ([22], [18], [5]).

There exist many examples of pseudosymmetric manifolds which are not semisymmetric ([15], [18], [9], [20]). The examples also include compact manifolds. In Section 4 we will present an example of a compact pseudosymmetric warped product manifold $S^p \times_F S^{n-p}$, $p \geq 2$, $n-p \geq 2$. We will prove that it cannot be realized as a hypersurface isometrically immersed in a manifold of constant curvature. At the end of that section we will give other examples of compact pseudosymmetric manifolds: $S^1 \times_F S^{n-1}$ and the n-dimensional torus T^n with a certain metric.

2. Warped products. Let $(M_1, \overline{g})$ and $(M_2, \widetilde{g})$, $\dim M_1 = p$, $\dim M_2 = n-p$, $1 \leq p < n$, be Riemannian manifolds covered by systems of charts $\{U; x^a\}$ and $\{V; y^\alpha\}$, respectively. Let F be a positive C^∞ function on M_1. The *warped product* $M_1 \times_F M_2$ of $(M_1, \overline{g})$ and $(M_2, \widetilde{g})$ ([21], [3]) is the product manifold $M_1 \times M_2$ with the metric $g = \overline{g} \times_F \widetilde{g}$,

$$\overline{g} \times_F \widetilde{g} = \Pi_1^* \overline{g} + (F \circ \Pi_1) \Pi_2^* \widetilde{g},$$

where $\Pi_i : M_1 \times M_2 \to M_i$ are the natural projections, $i = 1, 2$. Let $\{U \times V; x^1, \ldots, x^p, x^{p+1} = y^1, \ldots, x^n = y^{n-p}\}$ be a product chart for $M_1 \times M_2$. The local components of the metric $g = \overline{g} \times_F \widetilde{g}$ with respect to this chart are $g_{rs} = \overline{g}_{ab}$ if $r = a$ and $s = b$, $g_{rs} = F\widetilde{g}_{\alpha\beta}$ if $r = \alpha$ and $s = \beta$, and $g_{rs} = 0$ otherwise, where $a, b, c, \ldots \in \{1, \ldots, p\}$, $\alpha, \beta, \gamma, \ldots \in \{p+1, \ldots, n\}$ and $r, s, t, \ldots \in \{1, \ldots, n\}$. We denote by bars (resp., tildes) tensors formed from $\overline{g}$ (resp., $\widetilde{g}$).

The only possibly not identically vanishing local components of the tensors R and S of $M_1 \times_F M_2$ are the following ([6]):

$$R_{abcd} = \overline{R}_{abcd}, \tag{2}$$

$$R_{\alpha ab\beta} = -\frac{1}{2F} T_{ab} g_{\alpha\beta}, \tag{3}$$

$$R_{\alpha\beta\gamma\delta} = F\widetilde{R}_{\alpha\beta\gamma\delta} - \frac{\Delta_1 F}{4F^2} G_{\alpha\beta\gamma\delta}, \tag{4}$$

$$S_{ab} = \overline{S}_{ab} - \frac{n-p}{2F} T_{ab}, \tag{5}$$

$$S_{\alpha\beta} = \widetilde{S}_{\alpha\beta} - \frac{1}{2F}\left(\operatorname{tr}(T) + \frac{n-p-1}{2F}\Delta_1 F\right) g_{\alpha\beta}, \tag{6}$$

where

$$T_{ab} = \overline{\nabla}_b F_a - \frac{1}{2F} F_a F_b, \qquad \mathrm{tr}(T) = \overline{g}^{ab} T_{ab}\,, \tag{7}$$
$$\Delta_1 F = \Delta_{1\overline{g}} F = \overline{g}^{ab} F_a F_b\,,$$

and T is the $(0,2)$-tensor with the local components T_{ab}.

EXAMPLE 2.1. Let $(M_1, \overline{g}) = S^p(1/\sqrt{k})$ be the p-dimensional , $p \geq 2$, standard sphere of radius $1/\sqrt{k}$, $k > 0$. Let f be a non-constant function on M_1 satisfying the equality ([23])

$$\overline{\nabla}(df) + kfg = 0\,. \tag{8}$$

We put

$$F = (f + c)^2\,, \tag{9}$$

$$L = k(1 - c\tau), \qquad \tau = \frac{1}{\sqrt{F}}\,, \tag{10}$$

where c is a non-zero constant such that $f + c$ is either positive or negative on M_1. Now, using (7)–(10), we can easily verify that the tensor $\frac{1}{2}T + FL\overline{g}$ vanishes on M_1. Furthermore, from (8) we get

$$\Delta_1 f = -kf^2 + c_2, \qquad c_2 \in \mathbb{R}\,. \tag{11}$$

Combining (11) with (9) we can state that

$$\frac{1}{4F^2}\Delta_1 F = c_1\tau^2 + 2kc\tau - k, \qquad c_1 \in \mathbb{R}\,, \tag{12}$$

on M_1.

Remark 2.1. Let (M, g), $n \geq 4$, be a Riemannian manifold. For any $X, Y \in \Xi(M)$ we define the endomorphism $\widetilde{C}(X, Y)$ by

$$\widetilde{C}(X, Y) = \widetilde{R}(X, Y) - \frac{1}{n-2}(X \wedge \widetilde{S}Y + \widetilde{S}X \wedge Y) + \frac{K}{(n-1)(n-2)} X \wedge Y\,,$$

where $\widetilde{S}$ is the Ricci operator of (M, g) related to S by $S(X, Y) = g(X, \widetilde{S}Y)$. Further, we denote by C,

$$C(X_1, X_2, X_3, X_4) = g(\widetilde{C}(X_1, X_2)X_3, X_4)\,,$$

the Weyl conformal curvature tensor of (M, g). Now we define on M the $(0,6)$-tensor $C \cdot C$ by

$$(C \cdot C)(X_1, \ldots, X_4; X, Y) = -C(\widetilde{C}(X, Y)X_1, X_2, X_3, X_4) - \ldots - C(X_1, X_2, X_3, \widetilde{C}(X, Y)X_4)\,.$$

Moreover, we can also define on M the tensor $Q(g,C)$ in the same way as the tensor $Q(g,R)$.

In [13] (Theorem 2) it was proved that at every point of a warped product $M_1 \times_F M_2$, with $\dim M_1 = \dim M_2 = 2$, the following condition is satisfied:

$(**)$ the tensors $C \cdot C$ and $Q(g,C)$ are linearly dependent.

In the next section we will present an example of a Riemannian manifold of dimension ≥ 4 realizing $(**)$. Many examples of manifolds satisfying $(**)$ will be given in the subsequent paper [7].

Remark 2.2. Let (M,g), $n \geq 3$, be a Riemannian manifold. We define on M the $(0,6)$-tensor $Q(S,R)$ by

$$Q(S,R)(X_1,\ldots,X_4;X,Y) = R((X \wedge_S Y)X_1, X_2, X_3, X_4) + \ldots + R(X_1,X_2,X_3,(X \wedge_S Y)X_4)\,,$$

where $X \wedge_S Y$ is the endomorphism defined by

$$(X \wedge_S Y)Z = S(Y,Z)X - S(X,Z)Y\,.$$

The Riemannian manifold (M,g) is said to be *Ricci-generalized pseudosymmetric* [4] if at every point of M the following condition is satisfied:

$(***)$ the tensors $R \cdot R$ and $Q(S,R)$ are linearly dependent.

An important subclass of Ricci-generalized pseudosymmetric manifolds is formed by manifolds satisfying ([17], [4], [6])

$$R \cdot R = Q(S,R)\,. \tag{13}$$

Any 3-manifold (M,g) satisfies (13) ([12]). Moreover, so does any hypersurface M isometrically immersed in E^{n+1}, $n \geq 4$, ([19]).

Remark 2.3. As was proved in [19] any hypersurface M isometrically immersed in a manifold M^{n+1}, $n \geq 4$, of constant curvature satisfies at every point of M the following condition:

$(****)$ the tensors $R \cdot R - Q(S,R)$ and $Q(g,C)$ are linearly dependent.

Remark 2.4. It is easy to see that if $(*)$ holds on (M,g), $n \geq 4$, then at every point of M the following condition is satisfied:

$(*****)$ the tensors $R \cdot C$ and $Q(g,C)$ are linearly dependent.

Manifolds satisfying $(*****)$ have been studied in [16], [11] and [13].

Remark 2.5. Hypersurfaces isometrically immersed in a manifold of constant curvature and satisfying $(*)$ or $(*****)$ were considered in [8], [19] and [14].

Remark 2.6. A Riemannian manifold (M,g), $n \geq 4$, is said to be a *manifold with harmonic Weyl tensor* C ([2], p. 440) if the tensor $S - \frac{K}{2(n-1)}g$

is a *Codazzi tensor* on M, i.e. if

$$\nabla\left(S - \frac{K}{2(n-1)}g\right)(X, Y; Z) = \nabla\left(S - \frac{K}{2(n-1)}g\right)(X, Z; Y) \tag{14}$$

on M. It is well known that any conformally flat manifold of dimension ≥ 4 is a manifold with harmonic Weyl tensor C.

3. Examples

Example 3.1 ([20], Example 5). Let $(M_2, \widetilde{g})$ be a 1-dimensional manifold. Then the warped product $S^{n-1}(1/\sqrt{k}) \times_F M_2$, $n \geq 4$, $k > 0$, with F defined by (9), is a conformally flat manifold satisfying the condition $R \cdot R = (L \circ \Pi_1)Q(g, R)$, where L is the function defined by (10). In particular, the manifold

$$S^{n-1}\left(\frac{1}{\sqrt{k}}\right) \times_F S^1\left(\frac{1}{\sqrt{l}}\right), \qquad l > 0\,,$$

is pseudosymmetric.

Example 3.2. Let $M^{n-p}(l)$ be an $(n-p)$-dimensional manifold, $p \geq 2$, $n - p \geq 2$, of constant curvature l. We consider the warped product

$$S^p\left(\frac{1}{\sqrt{k}}\right) \times_F M^{n-p}(l)\,,$$

where F is defined by (9) and $k > 0$. Using (10) and (12) and the fact that the tensor $\frac{1}{2}T + FL\overline{g}$, defined in Example 2.1, is the zero tensor, we can write the formulas (2)–(6) in the following form:

$$R_{abcd} = kG_{abcd}\,, \tag{15}$$

$$R_{a\alpha\beta b} = k(1 - c\tau)G_{a\alpha\beta b}\,, \tag{16}$$

$$R_{\alpha\beta\gamma\delta} = ((l - c_1)\tau^2 - 2kc\tau + k)G_{\alpha\beta\gamma\delta}\,, \tag{17}$$

$$S_{ab} = k((n-1) - (n-p)c\tau)g_{ab}\,, \tag{18}$$

$$\begin{aligned} S_{\alpha\beta} = {} & ((n-p-1)(l-c_1)\tau^2 \\ & - (2n-p-2)kc\tau + (n-1)k)g_{\alpha\beta}\,. \end{aligned} \tag{19}$$

Next, by making use of (15)–(19) and the relations

$$\begin{aligned} C_{rstu} = {} & R_{rstu} + \frac{K}{(n-1)(n-2)}G_{rstu} \\ & - \frac{1}{n-2}(g_{ru}S_{ts} + g_{ts}S_{ru} - g_{rt}S_{us} - g_{us}S_{rt})\,, \end{aligned} \tag{20}$$

$$\begin{aligned} K &= g^{ab}S_{ab} + g^{\alpha\beta}S_{\alpha\beta} \\ &= (n-p)(n-p-1)(l-c_1)\tau^2 \\ &\quad - 2(n-1)(n-p)kc\tau + n(n-1)k\,, \end{aligned} \tag{21}$$

we find the non-zero components of C:

$$C_{abcd} = \frac{\varrho}{p(p-1)} G_{abcd}\,, \tag{22}$$

$$C_{a\alpha\beta b} = -\frac{\varrho}{p(n-p)} G_{a\alpha\beta b}\,, \tag{23}$$

$$C_{\alpha\beta\gamma\delta} = \frac{\varrho}{(n-p)(n-p-1)} G_{\alpha\beta\gamma\delta}\,, \tag{24}$$

where

$$\varrho = \frac{p(p-1)(n-p)(n-p-1)}{(n-1)(n-2)} (l-c_1)\tau^2\,. \tag{25}$$

Furthermore, applying (12), (15)–(19), (22)–(24), we can easily verify that the only components of $R\cdot R$, $Q(g,R)$, $C\cdot C$, $Q(g,C)$ and $Q(S,R)$ which are not identically zero are:

$$(R\cdot R)_{\alpha abcd\beta} = k^2 c\tau(1-c\tau) G_{dabc} g_{\alpha\beta}\,, \tag{26}$$

$$(R\cdot R)_{a\alpha\beta\gamma d\delta} = k\tau(kc + (c_1 - kc^2 - l)\tau + (l-c_1)c\tau^2) g_{ad} G_{\delta\alpha\beta\gamma}\,, \tag{27}$$

$$Q(g,R)_{\alpha abcd\beta} = kc\tau G_{dabc} g_{\alpha\beta}\,, \tag{28}$$

$$Q(g,R)_{a\alpha\beta\gamma d\delta} = (kc + (c_1 - l)\tau)\tau g_{ad} G_{\delta\alpha\beta\gamma}\,, \tag{29}$$

$$(C\cdot C)_{\alpha abcd\beta} = -\frac{(n-1)\varrho^2}{p^2(n-p)^2(p-1)} G_{dabc} g_{\alpha\beta}\,, \tag{30}$$

$$(C\cdot C)_{a\alpha\beta\gamma d\delta} = \frac{(n-1)\varrho^2}{p^2(n-p)^2(n-p-1)} g_{ad} G_{\delta\alpha\beta\gamma}\,, \tag{31}$$

$$Q(g,C)_{\alpha abcd\beta} = \frac{(n-1)\varrho}{p(p-1)(n-p)} G_{dabc} g_{\alpha\beta}\,, \tag{32}$$

$$Q(g,C)_{a\alpha\beta\gamma d\delta} = -\frac{(n-1)\varrho}{p(n-p)(n-p-1)} g_{ad} G_{\delta\alpha\beta\gamma}\,, \tag{33}$$

$$\begin{aligned} Q(S,R)_{\alpha abcd\beta} &= k(-(n-p-1)k + (2n-2p-1)ck\tau \\ &\quad + (n-p-1)(((l-kc)-(n-p)kc^2)\tau^2 - c_1\tau^3)) G_{dabc} g_{\alpha\beta}\,, \end{aligned} \tag{34}$$

$$\begin{aligned} Q(S,R)_{a\alpha\beta\gamma d\delta} &= k\tau(kc + ((p-2)kc^2 \\ &\quad - p(l-c_1))\tau + (l-c_1)c\tau^2) g_{ad} G_{\delta\alpha\beta\gamma}\,. \end{aligned} \tag{35}$$

4. Main results

THEOREM 4.1. *Let $(N,g) = S^p(1/\sqrt{k}) \times_F M^{n-p}(l)$ be the warped product of a sphere $S^p(1/\sqrt{k})$ and a manifold of constant curvature $M^{n-p}(l)$, $k>0$, $l\in\mathbb{R}$, $p\ge 2$, $n-p\ge 2$, with F defined by* (9). *Then*:

(i) *(N,g) is a non-semisymmetric pseudosymmetric manifold.*

(ii) *If $l \ne c_1$ then (N,g) is a non-conformally flat manifold satisfying $C\cdot C = L_C Q(g,C)$ on $U_C = \{x \in N \mid C(x) \ne 0\}$, where L_C is some* *function on* U_C *and* c_1 *is the constant defined by* (12). *If* $l = c_1$, *then* (N, g) *is conformally flat.*

(iii) *If* $l \neq c_1$ *then* (N, g) *is a manifold with non-harmonic Weyl tensor* C.

(iv) $R \cdot R - Q(S, R)$ *is a non-zero tensor on* N.

(v) $R \cdot R - Q(S, R)$ *and* $Q(g, C)$ *are not linearly dependent on* N.

Proof. (i) (resp., (ii)) is an immediate consequence of (26)–(29) (resp., (22)–(24) and (30)–(33)).

(iii) Using (18), (19) and (21) we get

$$\nabla_c S_{ab} - \nabla_b S_{ac} - \frac{1}{2(n-1)}((\nabla_c K) g_{ab} - (\nabla_b K) g_{ac}) = -\frac{(n-p)(n-p-1)}{n-1}(l - c_1)\tau((\nabla_c \tau) g_{ab} - (\nabla_b \tau) g_{ac}) \,.$$

Now Remark 2.6 completes the proof.

(iv) (26) and (34) yield

$$((R \cdot R) - Q(S, R))_{\alpha a b c d \beta} = (n-p-1)k(k - 2kc\tau + (kc - l + kc^2)\tau^2 + c_1\tau^3) g_{\alpha\beta} G_{dabc} \,.$$

Thus $R \cdot R - Q(S, R)$ is a non-zero tensor on N.

(v) Using (26), (27) and (32)–(35) we obtain the last assertion.

Our theorem is thus proved.

Combining the above theorem with Remarks 2.2 and 2.3 we obtain the following corollary.

Corollary 4.1. *The manifold*

$$S^p\left(\frac{1}{\sqrt{k}}\right) \times_F S^{n-p}\left(\frac{1}{\sqrt{l}}\right),$$

$p \geq 2$, $n - p \geq 2$, $k > 0$, $l > 0$, *with* F *defined by* (9), *satisfies* (i)–(iii) *of Theorem* 4.1. *Moreover*, *this manifold cannot be realized as a hypersurface isometrically immersed in a manifold of constant curvature.*

Example 4.1. The manifold $\mathbb{R} \times_F S^{n-1}(l)$, $n \geq 3$, $l > 0$, with $\overline{g}_{11} = \varepsilon$, $\varepsilon \in \{-1, 1\}$, and a periodic positive C^∞ function F on $\mathbb{R}$, is a conformally flat pseudosymmetric manifold (cf. [9], Lemma 3.1). A periodic function on $\mathbb{R}$ can be considered as a function on the circle S^1. Thus it is possible to define a conformally flat pseudosymmetric metric on $S^1 \times S^{n-1}$.

Example 4.2. Let $\overline{g}$ and $\widetilde{g}$ be metrics on $\mathbb{R}$ and $\mathbb{R}^{n-1}$ respectively, where $\overline{g}_{11} = \varepsilon$, $\widetilde{g}_{\alpha\beta} = \varepsilon_\alpha \delta_{\alpha\beta}$, $\varepsilon, \varepsilon_\alpha \in \{-1, 1\}$, $\alpha, \beta \in \{2, \ldots, n\}$, $n \geq 4$. The manifold $\mathbb{R} \times_F \mathbb{R}^{n-1}$, with $F(t) = \sin t + 2$, $t \in \mathbb{R}$, is non-semisymmetric, conformally flat and pseudosymmetric (cf. [9], Lemma 3.1). Let G be the group of translations generated by a suitable choice of a basis of $\mathbb{R}^n$ which leaves the metric $\overline{g} \times_F \widetilde{g}$ invariant. Thus the metric $\overline{g} \times_F \widetilde{g}$ determines a conformally flat pseudosymmetric metric on the n-torus $T^n = \mathbb{R}^n/G$.

REFERENCES

[1] A. Adamów and R. Deszcz, *On totally umbilical submanifolds of some class of Riemannian manifolds*, Demonstratio Math. 16 (1983), 39–59.

[2] A. L. Besse, *Einstein Manifolds*, Springer, Berlin 1987.

[3] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. 145 (1969), 1–49.

[4] F. Defever and R. Deszcz, *On semi-Riemannian manifolds satisfying the condition* $R \cdot R = Q(S, R)$, in: Geometry and Topology of Submanifolds, III, Leeds, May 1990, World Sci., Singapore 1991, 108–130.

[5] —,—, *A note on geodesic mappings of pseudosymmetric Riemannian manifolds*, Colloq. Math. 62 (1991), 313–319.

[6] —,—, *On warped product manifolds satisfying a certain curvature condition*, Atti Acad. Peloritana Cl. Sci. Fis. Mat. Natur., in print.

[7] —,—, *On Riemannian manifolds satisfying a certain curvature condition imposed on the Weyl curvature tensor*, Acta Univ. Palackiensis, in print.

[8] J. Deprez, R. Deszcz and L. Verstraelen, *Pseudosymmetry curvature conditions on hypersurfaces of Euclidean spaces and on Kählerian manifolds*, Ann. Fac. Sci. Toulouse 9 (1988), 183–192.

[9] —,—,—, *Examples of pseudosymmetric conformally flat warped products*, Chinese J. Math. 17 (1989), 51–65.

[10] R. Deszcz, *Notes on totally umbilical submanifolds*, in: Geometry and Topology of Submanifolds, Luminy, May 1987, World Sci., Singapore 1989, 89–97.

[11] —, *Examples of four-dimensional Riemannian manifolds satisfying some pseudosymmetry curvature conditions*, in: Geometry and Topology of Submanifolds, II, Avignon, May 1988, World Sci., Singapore 1990, 134–143.

[12] —, *On conformally flat Riemannian manifold satisfying certain curvature conditions*, Tensor (N.S.) 49 (1990), 134–145.

[13] —, *On four-dimensional Riemannian warped product manifolds satisfying certain pseudo-symmetry curvature conditions*, Colloq. Math. 62 (1991), 103–120.

[14] —, *Pseudosymmetry curvature conditions imposed on the shape operators of hypersurfaces in the affine space*, Results in Math. 20 (1991), 600–621.

[15] R. Deszcz and W. Grycak, *On some class of warped product manifolds*, Bull. Inst. Math. Acad. Sinica 15 (1987), 311–322.

[16] —,—, *On manifolds satisfying some curvature conditions*, Colloq. Math. 57 (1989), 89–92.

[17] —,—, *On certain curvature conditions on Riemannian manifolds*, ibid. 58 (1990), 259–268.

[18] R. Deszcz and M. Hotloś, *On geodesic mappings in pseudosymmetric manifolds*, Bull. Inst. Math. Acad. Sinica 16 (1988), 251–262.

[19] R. Deszcz and L. Verstraelen, *Hypersurfaces of semi-Riemannian conformally flat manifolds*, in: Geometry and Topology of Submanifolds, III, Leeds, May 1990, World Sci., Singapore 1991, 131–147.

[20] R. Deszcz, L. Verstraelen and L. Vrancken, *On the symmetry of warped product spacetimes*, Gen. Rel. Grav. 23 (1991), 671–681.

[21] G. I. Kruchkovich, *On semi-reducible Riemannian spaces*, Dokl. Akad. Nauk SSSR 115 (1957), 862–865 (in Russian).

[22] J. Mikesh, *Geodesic mappings of special Riemannian spaces*, in: Topics in Differential Geometry (Hajdoszoboszló 1984), Colloq. Math. Soc. János Bolyai 46, Vol. II, North-Holland, Amsterdam 1988, 793–813.

[23] M. Obata, *Certain conditions for a Riemannian manifold to be isometric with a sphere*, J. Math. Soc. Japan 14 (1962), 333–340.

[24] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y)\cdot R=0$. *I. The local version*, J. Differential Geom. 17 (1982), 531–582.

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