CONVERGENCE OF ITERATES OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS IN BANACH SPACES WITH THE UNIFORM OPIAL PROPERTY

BY

RONALD BRUCK (LOS ANGELES, CALIFORNIA),
TADEUSZ KUCZUMOW (LUBLIN)
AND SIMEON REICH (LOS ANGELES, CALIFORNIA)

Introduction. Throughout this paper X denotes a Banach space, C a subset of X (*not* necessarily convex), and $T : C \to C$ a self-mapping of C. There appear in the literature two definitions of asymptotically nonexpansive mapping. The weaker definition (cf. Kirk [14]) requires that

$$\limsup_{n \to \infty} \sup_{y \in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0$$

for every $x \in C$, and that T^N be continuous for some $N \ge 1$. The stronger definition (cf. Goebel and Kirk [8]) requires that each iterate T^n be Lipschitzian with Lipschitz constants $L_n \to 1$ as $n \to \infty$. For our iteration method we find it convenient to introduce a definition somewhere between these two: T is *asymptotically nonexpansive in the intermediate sense* provided T is uniformly continuous and

$$\limsup_{n \to \infty} \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|) \le 0 \,.$$

Many papers on the weak convergence of iterates of asymptotically nonexpansive mappings have appeared recently; their setting is either a uniformly convex space with a Fréchet-differentiable norm or a uniformly convex space with the Opial property. In this paper we are primarily interested in a generalization of the second case. Our proofs are not only simpler, they are more general: when τ is a Hausdorff linear topology and X satisfies the uniform τ-Opial property, we prove that $\{T^n x\}$ is τ-convergent if and only if $\{T^n x\}$ is τ-asymptotically regular, i.e.

$$T^{n+1} x - T^n x \xrightarrow{\tau} 0 \,.$$

The τ-limit is a fixed point of T.

In the second part of the paper we show how to construct (in uniformly convex Banach spaces) a fixed point of a mapping which is asymptotically nonexpansive in the intermediate sense as the τ-limit of a sequence $\{x_i\}$ defined by an iteration of the form

$$x_{i+1} = \alpha_i T^{n_i} x_i + (1 - \alpha_i) x_i \,,$$

where $\{\alpha_i\}$ is a sequence in $(0, 1)$ bounded away from 0 and 1 and $\{n_i\}$ is a sequence of nonnegative integers. Schu [25] has considered this iteration for $n_i \equiv i$, under the assumptions that X is Hilbert, C is compact, and T^n has Lipschitz constant $L_n \geq 1$ such that $\sum_n (L_n^2 - 1) < +\infty$; our results considerably generalize this result.

Recall the classical definition of the Opial property: whenever $x_n \rightharpoonup x$, then

$$\limsup_n \|x_n - x\| < \limsup_n \|x_n - y\|$$

for all $y \neq x$, where $\rightharpoonup$ denotes weak convergence. Henceforth we shall denote by τ a Hausdorff linear topology on X. The τ-Opial property is defined analogously to the classical Opial property, replacing weak convergence by τ-sequential convergence. We say that X has the *uniform τ-Opial property* if for each $c > 0$ there exists $r > 0$ with the property that for each $x \in X$ and each sequence $\{x_n\}$ the conditions

$$x_n \xrightarrow{\tau} 0, \quad 1 \leq \limsup_n \|x_n\| < +\infty, \quad \|x\| \geq c$$

imply that $\limsup_n \|x_n - x\| \geq 1 + r$ (cf. Prus [21]). Note that a uniformly convex space which has the τ-Opial property necessarily has the uniform τ-Opial property.

τ-Convergence of iterates. A common thread in each of our theorems is the convergence of a sequence of real numbers. We separate out the principle, but it is too trivial to offer a proof:

LEMMA 1. *Suppose $\{r_k\}$ is a bounded sequence of real numbers and $\{a_{k,m}\}$ is a doubly-indexed sequence of real numbers which satisfy*

$$\limsup_k \limsup_m a_{k,m} \leq 0, \quad r_{k+m} \leq r_k + a_{k,m} \quad \text{for each } k, m \geq 1 \,.$$

Then $\{r_k\}$ converges to an $r \in \mathbb{R}$; if $a_{k,m}$ can be taken to be independent of k, $a_{k,m} \equiv a_m$, then $r \leq r_k$ for each k.

THEOREM 1. *Suppose X has the uniform τ-Opial property, C is a norm-bounded, sequentially τ-compact subset of X, and $T : C \to C$ is asymptotically nonexpansive in the weak sense. If $\{y_n\}$ is a sequence in C such that $\lim_n \|y_n - w\|$ exists for each fixed point w of T, and if $\{y_n - T^k y_n\}$ is*

τ-*convergent to* 0 *for each* $k \geq 1$, *then* $\{y_n\}$ *is* τ-*convergent to a fixed point of* T.

P r o o f. We shall begin by proving that if $\{y_{n_i}\}$ is a subsequence such that $y_{n_i} \xrightarrow{\tau} z$, then $z = Tz$. Define

$$r_k = \limsup_i \|T^k y_{n_i} - z\|, \quad a_m = \sup_{y \in C}(\|T^m y - T^m z\| - \|y - z\|).$$

By the Opial property

(1) $\quad r_{k+m} = \limsup_i \|T^{k+m} y_{n_i} - z\| \leq \limsup_i \|T^{k+m} y_{n_i} - T^m z\| \leq r_k + a_m$,

where $\limsup_m a_m \leq 0$ by the weak definition of asymptotically nonexpansive. By Lemma 1, therefore, $\lim_k r_k = r$ exists and $r \leq r_k$ for each $k \geq 1$. Thus, given $\varepsilon > 0$, (1) implies that for sufficiently large k and m,

$$r \leq \limsup_i \|T^{k+m} y_{n_i} - T^m z\| < r + \varepsilon.$$

By the uniform τ-Opial property, $\lim_m T^m z = z$. Since T^N is continuous, z is therefore a fixed point of T^N, and since

$$z = \lim_j T^{jN+1} z = \lim_j T T^{jN} z = Tz,$$

z is also a fixed point of T.

We have proved that τ-subsequential limits of $\{y_n\}$ must be fixed points of T. Opial's classical argument [20] can now be followed to deduce that $\{y_n\}$ is τ-convergent to a fixed point of T; for otherwise, by the sequential τ-compactness of C, there must exist $z_1 \neq z_2$ and subsequences $\{y_{n_i}\}$ and $\{y_{m_i}\}$ such that $y_{n_i} \xrightarrow{\tau} z_1$ and $y_{m_i} \xrightarrow{\tau} z_2$. By the Opial property,

$$\limsup_i \|y_{n_i} - z_1\| < \limsup_i \|y_{n_i} - z_2\|$$

and

$$\limsup_i \|y_{m_i} - z_2\| < \limsup_i \|y_{m_i} - z_1\|.$$

But this is impossible; the sequences $\{\|y_n - z_1\|\}$ and $\{\|y_n - z_2\|\}$ both converge, so the limsup's over subsequences are actually limits over the full sequence. ∎

THEOREM 2. *Suppose the Banach space* X *has the uniform* τ-*Opial property, and let* C *be a nonempty, norm-bounded, sequentially* τ-*compact subset of* X. *If* $T : C \to C$ *is asymptotically nonexpansive in the weak sense and* $x \in C$, *then* $\{T^n x\}$ *is* τ-*convergent if and only if it is* τ-*asymptotically regular. The* τ-*limit of* $\{T^n x\}$ *is a fixed point of* T.

P r o o f. It is obvious that if $\{T^n x\}$ is τ-convergent, then $T^{n+1} x - T^n x \xrightarrow{\tau} 0$. Conversely, suppose that $T^{n+1} x - T^n x \xrightarrow{\tau} 0$.

If w is a fixed point of T, define

$$r_k = \|T^k x - w\|, \quad a_m = \sup_{y \in C}(\|T^m y - w\| - \|y - w\|),$$

so that $r_{k+m} \leq r_k + a_m$. By the asymptotic nonexpansiveness of T, $\limsup_m a_m \leq 0$, hence by Lemma 1, $\{r_n\}$ converges. We have proved that $\{\|T^n x - w\|\}$ converges for each fixed point w of T. By the τ-asymptotic regularity of T,

$$T^n x - T^k T^n x \xrightarrow{\tau} 0 \quad \text{as } n \to \infty$$

for each integer $k \geq 1$. Theorem 1 now shows that $\{T^n x\}$ is τ-convergent to a fixed point of T. (In particular, this proves that T *has* a fixed point.) ∎

R e m a r k 1. In the case X is a Hilbert space and τ is its weak topology, Theorem 1 was proved by Bruck in [4]. In this case the result also follows from the nonlinear mean ergodic theorem [1, 22, 23]. See [2, 5, 10, 11, 19, 24, 26, 28] for more recent results and a comprehensive and updated bibliography.

R e m a r k 2. There is still another definition of "asymptotically nonexpansive" mapping which appears in the literature:

$$\limsup_n \|T^n x - T^n y\| \leq \|x - y\| \quad \text{for each } x, y \in C.$$

However, this is unsatisfactory from the point of view of fixed point theory: Tingley [27] has constructed an example of a bounded closed convex C in Hilbert space and a continuous but fixed-point-free $T : C \to C$ which actually satisfies

$$\lim_n \|T^n x - T^n y\| = 0 \quad \text{for each } x, y \in C.$$

In his example it is even true that $\{T^n e_1\}$ is weakly convergent to 0, but of course 0 is not a fixed point.

The proof of Theorem 1 can also be applied to asymptotically nonexpansive commutative semigroups. Let C be a nonempty subset of a Banach space X. Let $\mathcal{T} = \{T(t) : t \geq 0\}$ be a family of mappings from C into itself. $\mathcal{T}$ is called an *asymptotically nonexpansive semigroup on C* if $T(t+s) = T(t)T(s)$ for all $s, t \geq 0$, $T(t_0)$ is continuous for some $t_0 > 0$, and for each $x \in C$,

$$\limsup_{t \to +\infty} \sup_{y \in C} (\|T(t)x - T(t)y\| - \|x - y\|) \leq 0.$$

THEOREM 3. *In the setting of Theorem 1, a trajectory $\{T(t)x\}$ of an asymptotically nonexpansive semigroup $\mathcal{T}$ on C is τ-convergent as $t \to +\infty$ iff $T(t+s)x - T(t)x \xrightarrow{\tau} 0$ as $t \to +\infty$ for each $s \geq 0$. The limit is a common fixed point of $\mathcal{T}$.*

R e m a r k 3. Theorems 2 and 3 can be easily generalized to metric spaces (X, d).

R e m a r k 4. Theorems 2 and 3 can be proved in the nonexpansive case under the weaker assumption that X has the Opial property and τ is "locally metrizable" (see Dye, Kuczumow, Lin and Reich [6] and Kuczumow [15]).

An averaging iteration of Schu. J. Schu [25] considered the averaging iteration

$$x_{i+1} = \alpha_i T^i x_i + (1 - \alpha_i)x_i$$

when $T : C \to C$ is asymptotically nonexpansive in the stronger, Lip-schitzian sense. Here $\{\alpha_i\}$ is a sequence in $(0, 1)$ which is bounded away from 0 and 1. We shall consider, instead, the more general iteration

$$(2) \qquad x_{i+1} = \alpha_i T^{n_i} x_i + (1 - \alpha_i)x_i \,,$$

where $\{n_i\}$ is a sequence of nonnegative integers (which need *not* be increasing). A strictly increasing sequence $\{m_i\}$ of positive integers will be called *quasi-periodic* if the sequence $\{m_{i+1} - m_i\}$ is bounded (equivalently, if there exists $b > 0$ so that any block of b consecutive positive integers must contain a term of the sequence).

THEOREM 4. *Suppose X is a uniformly convex Banach space, C is a bounded convex subset of X, and $T : C \to C$ is asymptotically nonexpansive in the intermediate sense. Put*

$$c_n = \max(0, \sup_{x,y \in C} (\|T^n x - T^n y\| - \|x - y\|)) \,,$$

so that $\lim_n c_n = 0$. Suppose $\{n_i\}$ is a sequence of nonnegative integers such that

$$\sum_i c_{n_i} < +\infty$$

and such that

$$\mathcal{O} = \{i : n_{i+1} = 1 + n_i\}$$

is quasi-periodic. Then for any $x_1 \in C$ and $\{x_i\}$ generated by (2) for $i \geq 1$, we have $\lim_i \|x_i - Tx_i\| = 0$. If, in addition, τ is a Hausdorff linear topology such that C is sequentially τ-compact and X has the τ-Opial property, then $\{x_i\}$ is τ-convergent to a fixed point of T.

P r o o f. We have not assumed C is closed, but since T is uniformly continuous it (and its iterates) can be extended to the (norm) closure $\overline{C}$ with the same modulus of uniform continuity and the same constants c_n, so it does no harm to assume C itself is closed. By a theorem of Kirk [14], T has at least one fixed point w in C.

We begin by showing that for a fixed point w, the limits $\lim_i \|x_i - w\|$ and $\lim_i \|T^{n_i} x_i - w\|$ exist and are equal. From (2) we have

$$
\begin{aligned}
\|x_{k+1} - w\| &\leq \alpha_k \|T^{n_k} x_k - w\| + (1 - \alpha_k)\|x_k - w\| \\
&= \alpha_k \|T^{n_k} x_k - T^{n_k} w\| + (1 - \alpha_k)\|x_k - w\| \\
&\leq \alpha_k(\|x_k - w\| + c_{n_k}) + (1 - \alpha_k)\|x_k - w\| \leq \|x_k - w\| + c_{n_k},
\end{aligned}
$$

and hence that

$$
(3) \qquad \|x_{k+m} - w\| \leq \|x_k - w\| + \sum_{i=k}^{k+m-1} c_{n_i}.
$$

Applying Lemma 1 with $r_k = \|x_k - w\|$ and $a_{k,m} = \sum_{i=k}^{k+m-1} c_{n_i}$, we see that $\lim_i \|x_i - w\| = r$ exists for each fixed point w of T.

If $r = 0$ then we immediately obtain

$$
\|Tx_i - x_i\| \leq \|Tx_i - w\| + \|w - x_i\| = \|Tx_i - Tw\| + \|w - x_i\|,
$$

and hence by the uniform continuity of T, that $\lim_i \|x_i - Tx_i\| = 0$. Therefore we must also have

$$
\|T^{n_i} x_i - w\| = \|T^{n_i} x_i - T^{n_i} w\| \leq c_{n_i} + \|x_i - w\| \to 0
$$

as $i \to \infty$.

If $r > 0$, we shall prove that $\lim_i \|T^{n_i} x_i - w\| = r$ by showing that for any increasing sequence $\{i_j\}$ of positive integers for which $\lim_j \|T^{n_{i_j}} x_{i_j} - w\|$ exists, it follows that the limit is r. Without loss of generality we may assume that the corresponding subsequence $\{\alpha_{i_j}\}$ converges to some α; we shall have $\alpha > 0$ because $\{\alpha_i\}$ is assumed to be bounded away from 0.

Thus we have

$$
\begin{aligned}
r = \lim_i \|x_i - w\| &= \lim_j \|x_{i_j+1} - w\| \\
&= \lim_j \|\alpha_{i_j} T^{n_{i_j}} x_{i_j} + (1 - \alpha_{i_j}) x_{i_j} - w\| \\
&\leq \alpha \liminf_j \|T^{n_{i_j}} x_{i_j} - w\| + (1 - \alpha) r \\
&\leq \alpha \limsup_j \|T^{n_{i_j}} x_{i_j} - w\| + (1 - \alpha) r \\
&\leq \alpha \limsup_j (\|x_{i_j} - w\| + c_{n_{i_j}}) + (1 - \alpha) r \\
&\leq \alpha \limsup_j \|x_{i_j} - w\| + (1 - \alpha) r = r.
\end{aligned}
$$

This completes the proof that

$$
\lim_i \|x_i - w\| = r = \lim_i \|T^{n_i} x_i - w\|.
$$

Let $\delta : [0, 2] \to [0, 1]$ be the modulus of uniform convexity of X, so that

whenever $0 < \alpha < 1$ and at least one of u, v is not zero, then

$$2\min(\alpha, 1 - \alpha)\delta\left(\frac{\|u - v\|}{\max(\|u\|, \|v\|)}\right) \le 1 - \frac{\|\alpha u + (1 - \alpha)v\|}{\max(\|u\|, \|v\|)}\,.$$

Take $u = T^{n_i}x_i - w$ and $v = x_i - w$; then

$$2\min(\alpha_i, 1 - \alpha_i)\delta\left(\frac{\|T^{n_i}x_i - x_i\|}{\max(\|u\|, \|v\|)}\right) \le 1 - \frac{\|\alpha_i u + (1 - \alpha_i)v\|}{\max(\|u\|, \|v\|)}$$

$$= 1 - \frac{\|x_{i+1} - w\|}{\max(\|T^{n_i}x_i - w\|, \|x_i - w\|)}\,.$$

Since $\|T^{n_i}x_i - w\|$, $\|x_i - w\|$ and $\|x_{i+1} - w\|$ all converge to $r > 0$ as $i \to \infty$, and since $\{\alpha_i\}$ remains bounded away from 0 and 1, we conclude that

$$\lim_i \delta(\|T^{n_i}x_i - x_i\|/r) = 0\,.$$

Therefore

(4) $$\lim_i \|T^{n_i}x_i - x_i\| = 0\,.$$

This is equivalent to

(5) $$\lim_i \|x_i - x_{i+1}\| = 0\,.$$

We claim that $x_j - Tx_j \to 0$ as $j \to \infty$ through $\mathcal{O}$. Indeed, since $n_{j+1} = 1 + n_j$ for such j, we have

(6) $$\|x_j - Tx_j\| \le \|x_j - x_{j+1}\| + \|x_{j+1} - T^{n_{j+1}}x_{j+1}\|$$
$$+ \|T^{n_{j+1}}x_{j+1} - T^{n_{j+1}}x_j\| + \|TT^{n_j}x_j - Tx_j\|$$
$$\le \|x_{j+1} - x_j\| + \|x_{j+1} - T^{n_{j+1}}x_{j+1}\|$$
$$+ \|x_{j+1} - x_j\| + c_{n_{j+1}} + \|TT^{n_j}x_j - Tx_j\|\,.$$

By (4)–(6) and the uniform continuity of T, we conclude that $\|x_j - Tx_j\| \to 0$ as $j \to \infty$ through $\mathcal{O}$.

But since $\mathcal{O}$ is quasi-periodic, there exists a constant $b > 0$ such that for each positive integer i we can find $j_i \in \mathcal{O}$ with $|j_i - i| \le b$. Thus (5) and the uniform continuity of $I - T$ imply $x_i - Tx_i \to 0$ as $i \to \infty$ through all of $\mathbb{N}$.

If X has the τ-Opial property and C is τ-sequentially compact, the strong convergence of $\|x_i - Tx_i\|$ to 0 implies $x_i - Tx_i \xrightarrow{\tau} 0$. Applying Theorem 1, we conclude that $\{x_i\}$ is τ-convergent to a fixed point of T. ∎

R e m a r k 5. Schu [25] assumed that X is Hilbert and that the iterates T^n have Lipschitz constants $L_n \ge 1$ such that $\sum_n(L_n^2 - 1)$ converges. Even for Schu's original iteration ($n_i \equiv i$), Theorem 4 is more general, since the convergence of $\sum_n(L_n^2 - 1)$ implies that of $\sum_n(L_n - 1)$, which in turn assures the convergence of our $\sum_n c_n$.

We can *always* choose a sequence $\{n_i\}$ satisfying the conditions of Theorem 4: since $\lim_n c_n = 0$, we can choose a subsequence $\{c_{m_i}\}$ such that $\sum_i c_{m_i} < +\infty$ and $\sum_i c_{1+m_i} < +\infty$, then put $n_{2i} = m_i$ and $n_{2i+1} = 1+m_i$.

If T is nonexpansive we can take $n_{2i} = 1$, $n_{2i+1} = 0$, recovering a well-known result on the iteration of averaged mappings (although it is not as general as the theorems of Ishikawa [13] and Edelstein and O'Brien [7] on asymptotic regularity).

Theorem 4 would be more satisfying if we had no condition of quasi-periodicity on $\{n_i\}$, but we do not know whether such a result is true.

The uniform Opial property. We conclude by recalling a few examples of spaces with the uniform Opial property.

EXAMPLE 1. If X is a Banach space with a weakly sequentially continuous duality map J_Φ associated with a gauge function Φ which is continuous, strictly increasing, with $\Phi(0) = 0$ and $\lim_{t \to +\infty} \Phi(t) = +\infty$, then X has the uniform Opial property with respect to the weak topology (cf. Gossez and Lami-Dozo [12]). In particular, ℓ^p has the uniform Opial property with respect to the weak topology for $1 < p < +\infty$.

EXAMPLE 2. $\ell^1 = c_0^*$ has the uniform Opial property with respect to the weak-$*$ topology (cf. Goebel and Kuczumow [9], Lim [18]).

EXAMPLE 3. The James Tree $JT = B^*$ (B is generated by the biorthogonal functionals $\{f_{n,i}\}$ corresponding to the basis $\{e_{n,i}\}$) has the uniform Opial property with respect to its weak-$*$ topology. This is also true for the James space $J = I^*$ (I is generated by the biorthogonal functionals $\{f_i\}$ corresponding to the basis $\{e_1 + \ldots + e_n\}$). See Kuczumow and Reich [16] for details.

EXAMPLE 4. It is known that $L^p[0,1]$ does not have the Opial property for $1 \le p \le +\infty$ and $p \ne 2$ (Opial [20]). Nevertheless, if (Ω, Σ, μ) is a positive σ-finite measure space, then for $1 \le p < +\infty$ the space $L^p(\mu)$ does have the uniform Opial property with respect to the topology of convergence locally in measure (cf. Brezis and Lieb [3], Lennard [17]).

REFERENCES

[1] J.-B. B a i l l o n, *Un théorème de type ergodique pour les contractions non linéaires dans un espace de Hilbert*, C. R. Acad. Sci. Paris Sér. A 280 (1975), 1511–1514.
[2] J.-B. B a i l l o n and R. E. B r u c k, *Ergodic theorems and the asymptotic behavior of contraction semigroups*, preprint.
[3] H. B r e z i s and E. L i e b, *A relation between pointwise convergence of functions and convergence of functionals*, Proc. Amer. Math. Soc. 88 (1983), 486–490.

[4] R. E. Bruck, *On the almost-convergence of iterates of a nonexpansive mapping in Hilbert space and the structure of the weak ω-limit set*, Israel J. Math. 29 (1978), 1–16.

[5] —, *Asymptotic behavior of nonexpansive mappings*, in: Contemp. Math. 18, Amer. Math. Soc., 1983, 1–47.

[6] J. M. Dye, T. Kuczumow, P.-K. Lin and S. Reich, *Random products of nonexpansive mappings in spaces with the Opial property*, ibid. 144, 1993, to appear.

[7] M. Edelstein and R. C. O'Brien, *Nonexpansive mappings, asymptotic regularity, and successive approximations*, J. London Math. Soc. (2) 1 (1978), 547–554.

[8] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 35 (1972), 171–174.

[9] K. Goebel and T. Kuczumow, *Irregular convex sets with the fixed point property for nonexpansive mappings*, Colloq. Math. 40 (1978), 259–264.

[10] J. Górnicki, *Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces*, Comment. Math. Univ. Carolinae 30 (1989), 249–252.

[11] —, *Nonlinear ergodic theorems for asymptotically nonexpansive mappings in Banach spaces satisfying Opial's condition*, J. Math. Anal. Appl. 161 (1991), 440–446.

[12] J. P. Gossez and E. Lami-Dozo, *Some geometric properties related to the fixed point theory for nonexpansive mappings*, Pacific J. Math. 40 (1972), 565–575.

[13] S. Ishikawa, *Fixed points and iterations of nonexpansive mappings in Banach space*, Proc. Amer. Math. Soc. 5 (1976), 65–71.

[14] W. A. Kirk, *Fixed point theorems for non-Lipschitzian mappings of asymptotically nonexpansive type*, Israel J. Math. 17 (1974), 339–346.

[15] T. Kuczumow, *Weak convergence theorems for nonexpansive mappings and semigroups in Banach spaces with Opial's property*, Proc. Amer. Math. Soc. 93 (1985), 430–432.

[16] T. Kuczumow and S. Reich, *Opial's property and James' quasi-reflexive spaces*, preprint.

[17] C. Lennard, *A new convexity property that implies a fixed point property for L_1*, Studia Math. 100 (1991), 95–108.

[18] T. C. Lim, *Asymptotic center and nonexpansive mappings in conjugate Banach spaces*, Pacific J. Math. 90 (1980), 135–143.

[19] H. Oka, *Nonlinear ergodic theorems for commutative semigroups of asymptotically nonexpansive mappings*, Nonlinear Anal. 18 (1992), 619–635.

[20] Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967), 591–597.

[21] S. Prus, *Banach spaces with the uniform Opial property*, Nonlinear Anal. 18 (1992), 697–704.

[22] S. Reich, *Nonlinear evolution equations and nonlinear ergodic theorems*, ibid. 1 (1976/77), 319–330.

[23] —, *Almost convergence and nonlinear ergodic theorems*, J. Approx. Theory 24 (1978), 269–272.

[24] —, *A note on the mean ergodic theorem for nonlinear semigroups*, J. Math. Anal. Appl. 91 (1983), 547–551.

[25] J. Schu, *Iterative construction of fixed points of asymptotically nonexpansive mappings*, ibid. 158 (1991), 407–413.

[26] K.-K. Tan and H.-K. Xu, *A nonlinear ergodic theorem for asymptotically nonexpansive mappings*, Bull. Austral. Math. Soc. 45 (1992), 25–36.

[27]   D. T i n g l e y, *An asymptotically nonexpansive commutative semigroup with no fixed points*, Proc. Amer. Math. Soc. 97 (1986), 107–113.

[28]   H.-K. X u, *Existence and convergence for fixed points of mappings of asymptotically nonexpansive type*, Nonlinear Anal. 16 (1991), 1139–1146.

Added in proof. It seems worthwhile to point out that Schu's iteration is valid in the class of spaces in which the nonlinear mean ergodic theorem is usually set:

THEOREM 5. *If, in Theorem* 4, τ *is the weak topology, then the conclusion remains valid if the hypothesis that* X *has the* τ-*Opial property is replaced by the hypothesis that* X *has Fréchet differentiable norm, and the assumption that* T *is asymptotically nonexpansive in the intermediate sense is strengthened to the strong* (*Lipschitzian*) *asymptotic nonexpansiveness of* T.

We sketch the proof: first, as in Theorem 4 we have $\lim_i \|x_i - Tx_i\| = 0$. Xu [28] has proved that $I - T$ is demiclosed, which in our context means:

(7)   All weak subsequential limits of $\{x_i\}$ are fixed points of T.

To prove the uniqueness of the weak subsequential limit we use an "orthogonality" relationship between fixed points, as in the proof of the nonlinear mean ergodic theorem. The idea is adapted from S. Reich [*Weak convergence theorems for nonexpansive mappings in Banach spaces*, J. Math. Anal. Appl. 67 (1979), 274–276].

Put $S_i = \alpha_i T^{n_i} + (1 - \alpha_i)I$ and, for $k \geq j$, $S(k, j) = S_{k-1}S_{k-2} \ldots S_j$, so that $x_k = S(k, j)x_j$. Let L_{kj} denote the Lipschitz constant of $S(k, j)$. The condition of Theorem 4 that $\sum c_{n_i} < +\infty$ implies that

(8) $$\lim_{j \to \infty} \sup_{k \geq j} L_{kj} = 1.$$

The proof of Theorem 4 that $\{\|x_i - w\|\}$ converges for each fixed point w of T is still valid, but we need a stronger result:

(9) $\{\|tx_i + (1 - t)w_1 - w_2\|\}$ converges for all fixed points w_1, w_2 and all $0 < t < 1$.

It follows from R. E. Bruck [*A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. 32 (1979), 107–116] that there exists a strictly increasing, continuous convex function $\gamma : [0, +\infty) \to [0, +\infty)$ with $\gamma(0) = 0$ such that for each $S : C \to C$ with Lipschitz constant L,

$$\|S(tu_1 + (1 - t)u_2) - tSu_1 - (1 - t)Su_2\| \leq L\gamma^{-1}\left(\|u_1 - u_2\| - \frac{1}{L}\|Su_1 - Su_2\|\right)$$

for all $u_1, u_2 \in C$ and $0 < t < 1$. Applying this to $u_1 = x_j$, $u_2 = w_1$, a fixed point of T, and $S = S(k, j)$ for $k \geq j$, we see by virtue of (8) and the convergence of $\{\|x_i - w_1\|\}$ that

(10) $$\lim_{j \to \infty} \sup_{k \geq j} \|S(k, j)(tx_j + (1 - t)w_1) - tx_k - (1 - t)w_1\| = 0.$$

Since

$$\|tx_k + (1 - t)w_1 - w_2\| \leq \|tx_k + (1 - t)w_1 - S(k, j)(tx_j + (1 - t)w_1)\|$$
$$+ \|S(k, j)(tx_j + (1 - t)w_1) - w_2\|$$
$$\leq \|tx_k + (1 - t)w_1 - S(k, j)(tx_j + (1 - t)w_1)\|$$
$$+ L_{kj}\|tx_j + (1 - t)w_1 - w_2\|,$$

(9) follows from (8) and (10) by first taking the lim sup as $k \to \infty$ and then taking the lim inf as $j \to \infty$.

Put $g_i(t) = (1/2)\|tx_i + (1-t)w_1 - w_2\|^2$. We have proved that $\lim_i g_i(t)$ exists. By the hypothesis of Fréchet differentiability of the norm,

$$\lim_{t \to 0+} \frac{g_i(t) - g_i(0)}{t} = \langle x_i - w_1, J(w_1 - w_2) \rangle$$

exists *uniformly in i*, where J is the normalized duality map of X (the gradient of $(1/2)\| \cdot \|^2$). It is an elementary exercise in analysis that if a sequence $\{g_i\}$ of functions is pointwise convergent and equidifferentiable from the right at a point, then the sequence of derivatives converges at the point; thus

(11) $\quad \lim_{i \to \infty} \langle x_i - w_1, J(w_1 - w_2) \rangle$ exists for any fixed points w_1, w_2 of T.

In particular, if w_1 and w_2 are weak subsequential limits of $\{x_i\}$, then when we first let $i \to \infty$ through a subsequence so $x_i \rightharpoonup w_1$, then through a subsequence such that $x_i \rightharpoonup w_2$, the resulting subsequential limits in (11) must be equal, i.e.

$$0 = \langle w_1 - w_1, J(w_1 - w_2) \rangle = \langle w_2 - w_1, J(w_1 - w_2) \rangle = -\|w_1 - w_2\|^2.$$

This proves the uniqueness of weak subsequential limits of $\{x_i\}$ and completes the proof that $\{x_i\}$ converges weakly.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CALIFORNIA 90089-1113
U.S.A.
E-mail: BRUCK@MTHA.USC.EDU and
      SREICH@MTHA.USC.EDU

DEPARTMENT OF MATHEMATICS
LUBLIN TECHNICAL UNIVERSITY
20-618 LUBLIN
POLAND
E-mail: CIESLAK@PLUMCS11.BITNET