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LIPSCHITZ CONTINUITY OF DENSITIES OF STABLE SEMIGROUPS OF MEASURES

BY

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In this paper we raise the question of regularity of the densities h_t of a symmetric stable semigroup $\{\mu_t\}$ of measures on the homogeneous group $\mathcal{N}$ under the mere assumption that the densities exist. (For a criterion of the existence of the densities of such semigroups see [11].)

If $\mathcal{N}$ is abelian, that is, isomorphic to $\mathbb{R}^n$ with a possibly non-isotropic family of dilations, the densities, if they exist, are always C^∞ functions. In addition they are square-integrable together with all their derivatives.

This need not be the case in general. An example of A. Hulanicki and the author (see [13]) shows that the derivatives of higher orders of the densities of the semigroup generated by the operator

$$X^2 - |Y|$$

on the three-dimensional Heisenberg group fail to be in $L^2(\mathcal{N})$. As a matter of fact, this fails even locally, as was shown by W. Hebisch (personal communication). The question whether the densities h_t are always C^1 remains open but the answer is likely to be in the negative.

In this article we show that, first of all, the densities, once they are known to exist, are automatically square-integrable, hence continuous. The result is not new. In [11] it is obtained as a corollary to a much more complex theorem (Corollary (4.25)). Here we give a direct and simple proof. The property is closely related to the maximal estimate

$$\|Sf\|_2 \le C\|Tf\|_2$$

valid for every convolution operator S on $\mathcal{N}$ which is homogeneous of the same order as the infinitesimal generator T of the semigroup. Recall that the order of homogeneity of T is equal to the characteristic exponent $0 < \theta \le 2$ of the semigroup.

However, the main results of this paper are the Lipschitz continuity of the densities and the pointwise estimate

$$h_t(x) \le t^{a/\theta}\Omega(\overline{x})|x|^{-Q-a},$$

valid for every $0 < a < \theta$, where Ω is an integrable function on the unit sphere satisfying an integral Lipschitz condition. Ω depends on a. The significance of this estimate is that it assures the weak $(1,1)$ type of the maximal operator

$$\mathcal{M}f(x) = \sup_{t>0} |f * h_t(x)|$$

associated with the semigroup (see Stein [18]). Let us remark that the estimate is sharp in the sense that the critical value $a = \theta$ implies that the Lévy measure of the generating functional T is absolutely continuous, which, of course, is not always the case. (See [12] for a detailed discussion of this case.)

The case $\theta = 2$ is very special. This may happen only if the group $\mathcal{N}$ is stratified in the sense of Folland [7] and the generator of such a semigroup coincides with a sublaplacian

$$\mathcal{L} = -\sum_{j=1}^{m} X_j^2$$

so that the semigroup is in fact Gaussian. (See e.g. Folland [6] for description of these semigroups.) In this case our theory is of no interest at all, even though the results we obtain remain valid.

The proof of the maximal estimate is based on the techniques of [11] depending on those of Helffer and Nourrigat [14]. In the proof of the Lipschitz continuity of the densities we imitate the classical method based on convolutions with the Gaussian semigroup. In that we follow pretty closely Folland [7] who adapted the method for homogeneous groups. The last part of the paper was strongly motivated and influenced by Stein [18].

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1. Preliminaries. Let $\mathcal{N}$ be an n-dimensional homogeneous group endowed with a family of dilations $\{\delta_t\}$ and a homogeneous norm $x \to |x|$ which is smooth away from the identity. Let dx denote Haar measure on $\mathcal{N}$ and Q the homogeneous dimension of $\mathcal{N}$. Let

$$\Sigma = \{x \in \mathcal{N} : |x| = 1\}$$

be the unit sphere relative to the homogeneous norm. For nonzero $x \in \mathcal{N}$, let

$$\overline{x} = \delta_{|x|^{-1}} x\,.$$

There exists a unique Radon measure $d\overline{x}$ on Σ such that for all continuous functions f on $\mathcal{N}$ with compact support

$$\int_{\mathcal{N}} f(x)\,dx = \int_0^\infty r^{Q-1} \int_{\Sigma} f(\delta_r \overline{x})\,d\overline{x}\,dr\,.$$

Since $\mathcal{N}$ is a connected and simply connected nilpotent Lie group it may be identified *via* the exponential map with its Lie algebra. We shall stick to this identification throughout the paper and think of $\mathcal{N}$ as being a nilpotent Lie algebra with the Campbell–Hausdorff multiplication. Let us remark that our convention implies that the origin 0 plays the rôle of the group identity and $-x$ is the inverse of $x \in \mathcal{N}$. Moreover, the dilations δ_t are also automorphisms of the Lie algebra structure of $\mathcal{N}$. It will be convenient to introduce an Euclidean norm $\|\cdot\|$ on $\mathcal{N}$ and adjust the homogeneous norm so that

$$\Sigma = \{x \in \mathcal{N} :\ \|x\| = 1\}\,.$$

This identification will allow the orthogonal group $\mathcal{O}(n)$ act in a natural way on Σ. We shall take advantage of it in Sections 5 and 6.

Let $e_1, \ldots, e_n \in \mathcal{N}$ be a linear basis consisting of eigenvectors of the dilations so that

$$\delta_t e_j = t^{d_j}\,, \qquad 1 \le j \le n,\ \ t > 0\,,$$

where the smallest of the exponents of homogeneity d_j is assumed to be 1. For $x \in \mathcal{N}$, let x_j denote the jth coordinate of x with respect to this basis. We may assume that

$$\sum_{j=1}^{n} |x_j|^{1/d_j} \le |x| \tag{1.1}$$

for $x = \sum_{j=1}^n x_j e_j \in \mathcal{N}$. Let also

$$X_j f(x) = \frac{d}{dt}\bigg|_{t=0} f(x \cdot te_j), \qquad 1 \le j \le n\,,$$

for $f \in C^1(\mathcal{N})$. If $I = (i_1, \ldots, i_n) \in \mathbb{Z}_+^n$, we set

$$x^I = x_1^{i_1} \ldots x_n^{i_n}\,.$$

A tempered distribution T on $\mathcal{N}$ is said to be a *kernel of order* $\theta \in \mathbb{R}$ if it coincides with a Radon measure away from the origin and satisfies

$$\langle f \circ \delta_t, T\rangle = t^\theta \langle f, T\rangle \qquad \text{for } f \in C_{\mathrm{c}}^\infty(\mathcal{N}),\ t > 0\,.$$

Note that, by homogeneity, any kernel of order $\theta > 0$ coincides with a bounded measure outside any neighbourhood of the origin and thus extends to a continuous linear form on the space $C_{\mathrm{b}}^\infty(\mathcal{N})$ of bounded smooth functions on $\mathcal{N}$ with natural topology.

A real distribution T on $\mathcal{N}$ is said to be *accretive* if

$$\langle f, T\rangle \geq 0$$

for all real $f \in C_{\mathrm{c}}^{\infty}(\mathcal{N})$ that take on their maximal value at the identity. It follows directly from the definition that such a T coincides with a (negative) Radon measure away from the origin. This measure, called the *Lévy measure* of T and denoted by $T(dx)$, is bounded on the complement of any neighbourhood of the origin so every accretive T extends by continuity to a linear form on $C_{\mathrm{b}}^{\infty}(\mathcal{N})$.

A distribution T is accretive if and only if there exists a unique continuous semigroup of subprobability measures $\{\mu_t\}$ for which T is the *generating functional*, that is,

$$\langle f, T\rangle = -\frac{d}{dt}\bigg|_{t=0} \langle f, \mu_t\rangle \qquad \text{for } f \in C_{\mathrm{c}}^{\infty}(\mathcal{N})\,.$$

If, in addition, T is a kernel of order θ, then $0 < \theta \leq 2$ and μ_t are probability measures.

For a strongly continuous representation π on a Banach space $\mathcal{H}$ and a kernel T of order $\theta > 0$, the operator π_T is defined on the space $C^{\infty}(\pi)$ of smooth vectors for π by

$$\langle \pi_T f, g^*\rangle = \langle \varphi_{f,g^*}, T\rangle, \qquad f \in \mathcal{C}^{\infty}(\pi),\ g^* \in \mathcal{H}^*\,,$$

where $\varphi_{f,g^*}(x) = \langle \pi_x f, g^*\rangle$ is in $C_{\mathrm{b}}^{\infty}(\mathcal{N})$. It is easily seen that π_T thus defined is a closable operator.

If T is a symmetric accretive kernel of order θ such that

$$-\int_{|x|\geq 1} \|\pi_x\|\, T(dx) < \infty\,,$$

then it follows from the theorem of Duflo–Hulanicki (see Duflo [4]) that $\{\pi_{\mu_t}\}$ is a strongly continuous semigroup of operators on $\mathcal{H}$ whose infinitesimal generator is the closure $\overline{\pi}_T$ of π_T. If $\mathcal{H}$ is a Hilbert space, then π_T is essentially selfadjoint.

Denote the centre of $\mathcal{N}$ by V, and choose once for all a linear complement $\widetilde{\mathcal{N}}$ to $\mathcal{N}$ invariant under the action of the dilations. Then the corresponding projections

$$\sigma : \mathcal{N} \to \widetilde{\mathcal{N}}, \qquad \upsilon : \mathcal{N} \to V$$

commute with the dilations, and every $x \in \mathcal{N}$ can be represented as

$$x = \upsilon(x) + \sigma(x) = \upsilon(x)\sigma(x) = \sigma(x)\upsilon(x) \tag{1.2}$$

in a unique way. It is fairly evident that the multiplication $x \circ y = \sigma(xy)$ makes $\widetilde{\mathcal{N}}$ into a group isomorphic to $\mathcal{N}/V$ with σ being the canonical homomorphism. Of course, $\widetilde{\mathcal{N}}$ is a homogeneous group if endowed with the dilations $x \to \delta_t x$, and $x \to |x|$ is a homogeneous norm on $\widetilde{\mathcal{N}}$. For a functional $\lambda \in V^*$ we shall denote by π^λ the unitary representation of $\mathcal{N}$ induced by the character $z \to e^{i\langle z,\lambda\rangle}$ from the centre V of $\mathcal{N}$. Note that π^λ is a representation on the Hilbert space $L^2(\widetilde{\mathcal{N}})$.

For a bounded measure μ on $\mathcal{N}$,

$$C_c^\infty(\widetilde{\mathcal{N}}) \ni f \to \pi_\mu^\lambda f(0) \in \mathbb{C}$$

defines, as is easily seen, a bounded measure μ^λ on $\widetilde{\mathcal{N}}$. We also have

$$f(a) = \int_{V^*} e^{i\langle v(a),\lambda\rangle} f^\lambda(\sigma(a))\,d\lambda \qquad \text{for } f \in C_c^\infty(\mathcal{N}),\ a \in \mathcal{N}\,,$$

where

$$f^\lambda(x) = \int_V f(yx)e^{-i\langle y,\lambda\rangle}\,dy \qquad \text{for } x \in \widetilde{\mathcal{N}}\,.$$

If R is a kernel of order $\theta > 0$ on $\mathcal{N}$, then

$$\langle f, R^0\rangle = \langle f\circ\sigma, R\rangle, \qquad f \in C_c^\infty(\widetilde{\mathcal{N}})\,,$$

defines a kernel of order θ on $\widetilde{\mathcal{N}}$. In addition, $\pi_R^0 f = f * R^0$ for $f \in C_c^\infty(\widetilde{\mathcal{N}})$.

If T is a symmetric accretive distribution on $\mathcal{N}$, then for every $\lambda \in V^*$, π_T^λ is an essentially selfadjoint operator with domain $C_c^\infty(\widetilde{\mathcal{N}})$. We denote by $\overline{\pi}_T^\lambda$ its closure. If $\{\mu_t\}$ is the semigroup generated by T, then $\{\mu_t^0\}$ is that generated by T^0.

(1.3) Lemma. *Let T be a symmetric accretive distribution on $\mathcal{N}$ and let $\{\mu_t\}$ be the corresponding semigroup of measures. Let $\lambda \in V^*$ be fixed. If the measures μ_t^λ have square-integrable densities h_t^λ on $\widetilde{\mathcal{N}}$, then h_t^λ are smooth vectors for $\overline{\pi}_T^\lambda$ and for every positive integer k and every $t > 0$, there exists a constant $C_k(t)$ such that*

$$\|(\overline{\pi}_T^\lambda)^k h_t^\lambda\|_2 \le C_k(t)\|h_{t/2}^\lambda\|_2\,.$$

Proof. This is Proposition (2.19) of [11]. ∎

For a general account of homogeneous groups, see Folland and Stein [8]. As a reference to the theory of continuous semigroups of measures we recommend Hulanicki [15] and Duflo [4]. See also Pazy [17] and Yosida [20] for the general theory of strongly continuous semigroups of operators on a Banach space.

2. Maximal estimates. Assume for the moment that the homogeneous group $\mathcal{N}$ is abelian, that is, isomorphic to $\mathbb{R}^n$. Let T be a symmetric accretive kernel of order θ on $\mathcal{N}$ such that the measures μ_t in the semigroup generated by T are absolutely continuous relative to Haar measure on $\mathcal{N}$. Then the Fourier transform $\widehat{T}$ is a positive continuous function on $\widehat{\mathcal{N}}$. Moreover,

$$\widehat{\mu_1}(\xi) = e^{-\widehat{T}(\xi)} \quad \text{for } \xi \in \widehat{\mathcal{N}}\,.$$

Since μ_1 has a density, $\lim_{\xi\to\infty} \widehat{\mu_1}(\xi) = 0$, which implies

$$\widehat{T}(\xi) > 0 \tag{2.1}$$

for $|\xi| = 1$, hence, by homogeneity, for $\xi \neq 0$. Consequently, for any kernel R of order θ on $\mathcal{N}$, there is a constant $C > 0$ such that $|\widehat{R}(\xi)| \leq C\widehat{T}(\xi)$ for $\xi \in \widehat{\mathcal{N}}$, which, by Plancherel theorem, implies $\|f * R\|_2 \leq C\|f * T\|_2$ for $f \in C_c^\infty(\mathcal{N})$. Another immediate consequence of (2.1) is that the densities of the measures μ_t are square-integrable.

The main purpose of this section is to extend this observation to the general case of a homogeneous group. Namely, we are going to prove

(2.2) THEOREM. *Let T be a symmetric accretive kernel of order $\theta \in (0,2)$ such that the measures μ_t in the semigroup it generates are absolutely continuous with respect to Haar measure. Then, for every kernel R of order θ, there exists a constant C such that*

$$\|f * R\|_2 \leq C\|f * T\|_2 \quad \textit{for } f \in C_c^\infty(\mathcal{N})\,. \tag{2.3}$$

Moreover, the densities of the measures μ_t are square-integrable with respect to Haar measure on $\mathcal{N}$.

The following lemma is well-known and easy to prove.

(2.4) LEMMA. *For every $F \in L^1(\mathcal{N})$,*

$$\lim_{|\lambda|\to\infty} \|\pi_F^\lambda\|_{\mathrm{op}} = 0\,,$$

where $\|A\|_{\mathrm{op}}$ stands for the operator norm of $A \in \mathcal{L}(L^2(\widetilde{\mathcal{N}}))$.

We shall need one more lemma which is a simple consequence of M. Christ's theorem on the boundedness of the Hilbert transform along a homogeneous curve (see Christ [2]).

(2.5) LEMMA. *Let S be an odd kernel of order 0. Then the singular integral operator defined by convolution by S on the right for $f \in C_c^\infty(\mathcal{N})$ extends to a bounded operator on $L^2(\mathcal{N})$.*

Proof. This is derived from Christ's theorem by the classical rotation method of Calderón and Zygmund [1]. ∎

(2.6) PROPOSITION. *Let R and T be symmetric kernels of order $\theta \in (0,2)$. Assume that there exists a constant $C_0 > 0$ such that*

$$\|\pi_R^0 f\|_2 \leq C_0\|\pi_T^0 f\|_2 \quad \textit{for } f \in C_c^\infty(\widetilde{\mathcal{N}})\,.$$

Then there exists another constant C_1 *such that*

$$\|\pi_R^\lambda f\|_2 \le C_1(\|\pi_T^\lambda f\|_2 + \|f\|_2) \quad \textit{for } f \in C_c^\infty(\widetilde{\mathcal{N}}) \textit{ and } |\lambda| \le 1\,.$$

P r o o f. If the kernels under consideration are smooth, this is a consequence of Theorem (3.19) of [10]. An analysis of the proof of that theorem shows that the smoothness is only essential to handle the case where for a kernel under consideration, say R, $x^I R$ happens to be of order 0 for some multi-index I. Then $x^I R$ is a smooth singular integral kernel so it gives rise to a bounded operator on $L^2(\widetilde{\mathcal{N}})$ and that is all which is needed in the proof.

Now, in our situation the critical order may occur only if $d_j = \theta$ for some $1 \le j \le n$ and so all we have to know is that the operator of convolution by $x_j R$ is bounded on $L^2(\widetilde{\mathcal{N}})$. (Of course, this may be the case only if $1 \le \theta < 2$.) Note, however, that, by hypothesis, the kernel $x_j R$ is odd so Lemma (2.5) applies and we are done. ∎

(2.7) LEMMA. *Under the assumptions of Theorem* (2.2), *there exists a constant* $C > 0$ *such that*

$$\|f\|_2 \le C\|\pi_T^\lambda f\|_2 \quad \textit{for } f \in C_c^\infty(\widetilde{\mathcal{N}}),\ |\lambda| \ge 1\,.$$

P r o o f. Denote the density of μ_t by h_t for $t > 0$. For $x \in \mathcal{N}$, let

$$F(x) = -\int_0^\infty e^{-t} h_t(x)\, dt\,.$$

Since $h_t \in L^1(\mathcal{N})$, F is integrable as well and $\|F\|_1 \le 1$. It is directly checked that

$$F * T = \delta - F\,,$$

where δ stands for the Dirac point mass located at the origin. Consequently,

$$\|f\|_2 \le \|\pi_F^\lambda \pi_T^\lambda f\|_2 + \|\pi_F^\lambda f\|_2 \le \|\pi_F^\lambda\|_{\mathrm{op}}(\|\pi_T^\lambda f\|_2 + \|f\|_2) \quad \text{for } \lambda \in V^*\,.$$

By Lemma (2.4), $\|\pi_F^\lambda\|_{\mathrm{op}} \le 1/2$ for $|\lambda| \ge M$; therefore

$$\|f\|_2 \le \frac{\|\pi_F^\lambda\|_{\mathrm{op}}}{1 - \|\pi_F^\lambda\|_{\mathrm{op}}}\|\pi_T^\lambda f\|_2 \le \|\pi_T^\lambda f\|_2 \quad \text{for } |\lambda| \ge M\,.$$

By homogeneity, $\|f\|_2 \le C\|\pi_T^\lambda f\|_2$ for $|\lambda| \ge 1$, where $C = M^\theta$. ∎

P r o o f o f T h e o r e m (2.2). As was observed at the beginning of this section, our theorem is true for abelian homogeneous groups. Now, since μ_t are absolutely continuous on $\mathcal{N}$ so are μ_t^0 on $\widetilde{\mathcal{N}}$. We may, therefore, assume that, by induction, (2.3) holds for R^0 and T^0. Then, by Proposition (2.6), there exists a constant $C_1 > 0$ such that

$$\|\pi_R^\lambda f\|_2 \le C_1(\|\pi_T^\lambda f\|_2 + \|f\|_2) \quad \text{for } f \in C_c(\widetilde{\mathcal{N}}),\ |\lambda| \le 1\,,$$

which combined with Lemma (2.7) shows that

$$\|\pi_R^\lambda f\|_2 \le C_2 \|\pi_T^\lambda f\|_2 \tag{2.8}$$

for $|\lambda| = 1$. By homogeneity, the estimate (2.8) must hold for all $\lambda \in V^*$.

Now the proof can be completed by decomposing the right-regular representation π into a direct integral of π^λ over V^*. In fact, for every $\lambda \in V^*$, we have $f^\lambda \in C_c^\infty(\widetilde{\mathcal{N}})$ and

$$\|\pi_a f\|_2^2 = \int_{V^*} \|\pi_a^\lambda f^\lambda\|_2^2 \, d\lambda \qquad \text{for } a \in \mathcal{N}.$$

Consequently,

$$\|\pi_R f\|_2^2 = \int_{V^*} \|\pi_R^\lambda f^\lambda\|_2^2 \, d\lambda \le C_2 \int_{V^*} \|\pi_T^\lambda f^\lambda\|_2^2 \, d\lambda = C_2 \|\pi_T f\|_2^2$$

for $f \in C_c^\infty(\widetilde{\mathcal{N}})$, which establishes the estimate (2.3).

It remains to show that the functions h_t are square-integrable. Again we shall proceed by induction. Assume, therefore, that our assertion is true for the quotient group $\widetilde{\mathcal{N}}$. It is easily seen that for every $t > 0$ the density of μ_t^0 is equal to h_t^0 and, by our inductive hypothesis, is square-integrable. Since

$$|\langle f, \mu_t^\lambda \rangle| \le \langle |f|, \mu_t^0 \rangle \le \|f\|_2 \|h_t^0\|_2 \qquad \text{for } \lambda \in V^*,\ f \in C_c^\infty(\widetilde{\mathcal{N}}), \tag{2.9}$$

it follows that $d\mu_t^\lambda = h_t^\lambda \, dx$, where $h_t^\lambda \in L^2(\widetilde{\mathcal{N}})$ and

$$\|h_t^\lambda\|_2 \le \|h_t^0\|_2 \tag{2.10}$$

independently of λ. Hence,

$$\int h_t(x)^2 \, dx = \int_{V^*} \|h_t^\lambda\|_2 \, d\lambda = \int_{|\lambda| \le 1} \|h_t^\lambda\|_2 \, d\lambda + \int_{|\lambda| > 1} \|h_t^\lambda\|_2 \, d\lambda .$$

By (2.10) the first integral is finite. To estimate the second, it is sufficient to observe that, by Lemma (2.7), Lemma (1.3), and (2.10),

$$\begin{aligned} \|h_t^\lambda\|_2 &\le C^k |\lambda|^{-k\theta} \|(\overline{\pi}_T^\lambda)^k h_t^\lambda\|_2 \\ &\le C^k C_k(t) |\lambda|^{-k\theta} \|h_{t/2}^\lambda\|_2 \le C_{k,t} |\lambda|^{-k\theta} \|h_{t/2}^0\|_2 \end{aligned}$$

for $|\lambda| \ge 1$ and positive integers k. This establishes our theorem. ∎

(2.11) COROLLARY. *For every kernel R of order $0 < \eta \le \theta$, there exists a constant $C > 0$ such that*

$$\|f * R\|_2 \le C(\|f * T\|_2 + \|f\|_2) \qquad \textit{for } f \in C_c^\infty(\mathcal{N}).$$

Since T can be represented as the sum of a compactly supported distribution and a bounded measure (cf. Section 1), the convolution $f * T$ is well defined for every $f \in L^p(\mathcal{N})$, where $1 \le p \le \infty$. Let

$$\operatorname{dom} T = \{ f \in L^2(\mathcal{N}) : f * T \in L^2(\mathcal{N}) \}.$$

Note that $\operatorname{dom} T$ coincides with the domain of the closure of the operator $C_c^\infty(\mathcal{N}) \ni f \to f * T \in L^2(\mathcal{N})$. We shall also write $Tf = f * T$. Let

$$L^2(\mathcal{N}) * \operatorname{dom} T = \{f * g : f \in L^2(\mathcal{N}),\ g \in \operatorname{dom} T\}.$$

(2.12) COROLLARY. *Let T be a symmetric accretive kernel of order θ satisfying the hypothesis of Theorem* (2.2). *Then for every kernel R of order θ and every $\varphi \in L^2(\mathcal{N}) * \operatorname{dom} T$, $\varphi * R \in L^\infty(\mathcal{N})$.*

Proof. Let $\varphi = f * g$, where $f \in L^2(\mathcal{N})$ and $g \in \operatorname{dom} T$. By Corollary (2.11), g is in the domain of the closure of $C_c^\infty \ni k \to k * R \in L^2(\mathcal{N})$, which implies $g * R \in L^2(\mathcal{N})$ and, hence, $\varphi * R = f * (g * R) \in L^\infty(\mathcal{N})$. ∎

3. Regularity of densities. From now on T denotes an accretive kernel of order $0 < \theta < 2$ such that the measures μ_t in the semigroup generated by it are absolutely continuous with respect to Haar measure on $\mathcal{N}$. We denote by h_t the corresponding densities. We also choose and fix an auxiliary accretive kernel P of order θ which is *smooth* away from the origin. One may take, for instance,

$$\langle f, P\rangle = \lim_{\varepsilon\to 0} \int_{|x|>\varepsilon} \frac{[f(0) - f(x)]}{|x|^{Q+\theta}}\,dx\,, \qquad f \in C_c^\infty(\mathcal{N})\,.$$

The semigroup of measures associated with P consists of measures with smooth densities p_t. Recall from [10] that for all $m \in \mathbb{Z}_+$, $I \in \mathbb{Z}_+^n$, $t > 0$,

$$P^m X^I p_t \in L^\infty(\mathcal{N}) \cap L^1(\mathcal{N})\,. \tag{3.1}$$

We shall make use of the following crude Taylor estimates for $f \in C^2(\mathcal{N})$:

$$|f(xy) - f(x)| \le \sum_{j=1}^n \|X_j f\|_\infty |y|^{d_j} \tag{3.2}$$

and

$$\Big|f(xy) - f(x) - \sum_{j=1}^n X_j f(x) y_j\Big| \le \sum_{i,j=1}^n \|X_i X_j f\|_\infty |y|^{d_i + d_j} \tag{3.3}$$

for $x, y \in \mathcal{N}$ (cf. Folland and Stein [8], Theorem 1.37). Note that by (1.1) the constants on the right may be taken to be 1.

(3.4) LEMMA. *Let $\varphi \in L^2(\mathcal{N}) * \operatorname{dom} T$. There exists a constant C_1 such that*

$$|\varphi * p_t(z) - \varphi(z)| \le C_1 t \quad \textit{for } z \in \mathcal{N},\ t > 0\,.$$

Proof. We have

$$\begin{aligned}\frac{d}{dt}(\varphi * p_t)(x) &= \varphi * \frac{d}{dt} p_t(x)\\ &= \varphi * P p_t(x) = P\varphi * p_t(x) \quad \text{for } x \in \mathcal{N},\ t > 0\,.\end{aligned}$$

Therefore, by Corollary (2.12),

$$\left|\frac{d}{dt}(\varphi * p_t)(x)\right| \le \|P\varphi\|_\infty \|p_t\|_1 = \|P\varphi\|_\infty = C_1 \qquad \text{for } x \in \mathcal{N},\ t > 0\,.$$

Consequently,

$$|\varphi * p_t(z) - \varphi(z)| \le \int\limits_0^t \left|\frac{d}{ds}\varphi * p_s(z)\right| ds \le C_1 t\,. \ \blacksquare$$

(3.5) THEOREM. *Let* $\varphi \in L^2(\mathcal{N}) * \operatorname{dom} T$. *There exists a constant* C *such that for every* $x, y \in \mathcal{N}$,

$$|\varphi(xy) - \varphi(x)| \le C|y|^\theta \qquad \text{if } 0 < \theta < 1\,, \tag{3.6}$$

$$|\varphi(xy) + \varphi(xy^{-1}) - 2\varphi(x)| \le C|y|^\theta \qquad \text{if } 1 \le \theta < 2\,. \tag{3.7}$$

Proof. In fact, we have

$$\begin{aligned} |\varphi(xy) - \varphi(x)| &\le |\varphi(xy) - \varphi * p_t(xy)| \\ &\quad + |\varphi * p_t(xy) - \varphi * p_t(x)| + |\varphi * p_t(x) - \varphi(x)|\,, \end{aligned} \tag{3.8}$$

where the value of t is to be chosen later on. Note that Lemma (3.4) provides an estimate for the first and last term of (3.8). Moreover, by the Taylor inequality (3.2),

$$|\varphi * p_t(xy) - \varphi * p_t(x)| \le \sum_{j=1}^n \|X_j(\varphi * p_t)\|_\infty |y|^{d_j}$$

and

$$\begin{aligned} |X_j(\varphi * p_t)(x)| &= \left|\int\limits_t^\infty \frac{d}{ds} X_j(\varphi * p_s)(x)\,ds\right| = \left|\int\limits_t^\infty P\varphi * X_j p_s(x)\,ds\right| \\ &\le \int\limits_t^\infty \|P\varphi\|_\infty \|X_j p_s\|_1\,ds \le C_1 \|X_j p_1\|_1 \int\limits_t^\infty s^{-d_j/\theta} ds = C_2 t^{1-d_j/\theta} \end{aligned}$$

so that

$$|\varphi * p_t(xy) - \varphi * p_t(x)| \le C_2 \sum_{j=1}^n t^{1-d_j/\theta} |y|^{d_j} \tag{3.9}$$

for $x, y \in \mathcal{N}$. Finally, letting $t = |y|^\theta$ and using Lemma (3.4), (3.8), and (3.9), we get

$$|\varphi(xy) - \varphi(x)| \le (2C_1 + C_2 n)|y|^\theta = C|y|^\theta\,,$$

which completes the proof of (3.6).

The proof of (3.7) is quite parallel to that of (3.6). In fact, we have

$$\begin{aligned}(3.10)\qquad &|\varphi(xy)+\varphi(xy^{-1})-2\varphi(x)| \\ &\le |\varphi(xy)-\varphi*p_t(xy)| \\ &\quad + |\varphi(xy^{-1})-\varphi*p_t(xy^{-1})| + 2|\varphi(x)-\varphi*p_t(x)| \\ &\quad + |\varphi*p_t(xy)+\varphi*p_t(xy^{-1})-2\varphi*p_t(x)|\,.\end{aligned}$$

Note that all the terms of (3.10) except the last one are estimated by Lemma (3.4). Moreover,

$$\begin{aligned}|X_iX_j(\varphi*p_t)(x)| &= \Big|\int_t^\infty \frac{d}{ds}X_iX_j(\varphi*p_s)(x)\,ds\Big| \\ &\le \int_t^\infty |P\varphi*X_iX_jp_s(x)|\,ds \le \|P\varphi\|_\infty \int_t^\infty \|X_iX_jp_s\|_1\,ds \\ &\le C_1\|X_iX_jp_1\|_1 \int_t^\infty s^{-(d_i+d_j)/\theta}\,ds = C_3t^{1-(d_i+d_j)/\theta}\end{aligned}$$

so that, by (3.3),

$$\begin{aligned}(3.11)\qquad &|\varphi*p_t(xy)+\varphi*p_t(xy^{-1})-2\varphi*p_t(x)| \\ &\le C_3\sum_{i,j=1}^n t^{1-(d_i+d_j)/\theta}|y|^{d_i+d_j} \qquad \text{for } x,y\in\mathcal{N}\,.\end{aligned}$$

Finally, letting $t=|y|^\theta$ and using Lemma (3.4), (3.10), and (3.11) we get

$$|\varphi(xy)+\varphi(xy^{-1})-2\varphi(x)| \le (4C_1+C_3n^2)|y|^\theta = C|y|^\theta\,. \ \blacksquare$$

We conclude this section with the following simple corollaries.

(3.12) Corollary. *For every* $0<\tau<\min(\theta,1)$, *there exists a constant* C *such that*

$$|\varphi(xy)-\varphi(x)|\le C|y|^\tau \qquad \textit{for } x,y\in\mathcal{N}\,.$$

(3.13) Corollary. *If, moreover,* φ *is symmetric, then there exists a constant* $C>0$ *such that*

$$|\varphi(x)-\varphi(0)|\le C|x|^\theta \qquad \textit{for } x\in\mathcal{N}, t>0\,.$$

(3.14) Corollary. *Theorem* (3.5) *and Corollaries* (3.12) *and* (3.13) *apply to the functions* T^kh_t, *where* k *is a nonnegative integer. In particular, they apply to the densities* h_t.

Proof. In fact, by [9], $T^kh_t\in L^2(\mathcal{N})$ for every $k\in\mathbb{Z}_+$ and every $t>0$. Therefore, $T^kh_t=T^kh_{t/2}*h_{t/2}\in L^2(\mathcal{N})*\operatorname{dom}T$. Since T^kh_t are symmetric, this proves our claim. $\blacksquare$

4. Holomorphic semigroups. Recall that T is an accretive kernel of order $0 < \theta < 2$ and the measures μ_t in the semigroup generated by T have square-integrable densities h_t.

Let

$$\langle x\rangle = \gamma(1 + |x|)\,,$$

where the constant $\gamma \ge 1$ is chosen so that $\langle x\rangle$ is a submultiplicative function, that is, $\langle xy\rangle \le \langle x\rangle\langle y\rangle$ for $x, y \in \mathcal{N}$. For $1 \le p \le \infty$ and $0 < a < \theta$, let $L^p(a) = L^p(\mathcal{N}, \langle x\rangle^a dx)$. In particular, $L^p(0) = L^p(\mathcal{N})$.

Let us consider the right-regular representation of $\mathcal{N}$ on the Banach space $L^p(a)$. As is easily seen,

$$-\int\limits_{|x|\ge 1} \langle x\rangle^a\, T(dx) < \infty \qquad \text{for } 0 < a < \theta\,,$$

so, by Hulanicki [15],

$$\int\limits_{\mathcal{N}} \langle x\rangle^a h_t(x)\, dx < \infty \qquad \text{for } t > 0\,.$$

Moreover, by the theorem of Duflo–Hulanicki (cf. Section 1),

$$H_t f = f * \mu_t$$

is a strongly continuous semigroup of operators acting on $L^p(a)$ for every $1 \le p < \infty$ and the operator

$$T : \operatorname{dom}_{L^p(a)} T \ni f \to f * T \in L^p(a)\,, \tag{4.1}$$

where $\operatorname{dom}_{L^p(a)} T = \{f \in L^p(a) : f * T \in L^p(a)\}$, is the infinitesimal generator of the semigroup (cf. remarks following Corollary (2.11)).

We recall the definition of a holomorphic semigroup of operators on a Banach space. Let $\{P_t\}$ be a strongly continuous semigroup of bounded operators on a Banach space $\mathbf{B}$. $\{P_t\}$ is called *holomorphic* in a sector $S_\vartheta = \{z \in \mathbb{C} : |\arg z| < \vartheta\}$ for a $0 < \vartheta < \pi/2$ if $t \to P_t$ admits a holomorphic extension to $S_\vartheta \ni z \to P_z \in \mathcal{L}(\mathbf{B})$ such that for every $0 < \vartheta' < \vartheta$, there exists a constant $\omega \ge 0$ such that

$$\sup_{z\in S_{\vartheta'}} e^{-\omega|z|}\|P_z\|_{\mathcal{L}(\mathbf{B})} < \infty\,.$$

Let A be the infinitesimal generator of $\{P_t\}$. The semigroup $\{P_t\}$ is holomorphic if and only if for every $t > 0$, the operator AP_t has a bounded extension to $\mathbf{B}$ and

$$\sup_{0<t\le 1} t\|AP_t\|_{\mathcal{L}(\mathbf{B})} < \infty\,.$$

(See Pazy [17] or Yosida [20].)

(4.2) Theorem. *For every* $0 < a < \theta$ *and every* $t > 0$,
$$\int |h_t * T(x)|\langle x\rangle^a\,dx < \infty\,.$$

As an immediate corollary we obtain

(4.3) Corollary. *The semigroup* $\{H_t\}$ *is holomorphic on* $L^p(a)$ *for every* $1 \le p < \infty$ *and every* $0 < a < \theta$.

Proof. In fact, for every $t > 0$ and every $f \in C_c^\infty$,
$$\|TH_tf\|_{L^p(a)} \le \|h_t * T\|_{L^1(a)}\|f\|_{L^p(a)}$$
and, by homogeneity,
$$\|TH_t\|_{\mathcal{L}(L^p(a))} \le \frac{1}{t}\|h_1 * T\|_{L^1(a)} \qquad \text{for } 0 < t \le 1\,. \ \blacksquare$$

In the proof of Theorem (4.2) we shall need the following lemma:

(4.4) Lemma. *For every* $0 < a < b < 1$, *there exists* $1 < p_0 < \infty$ *such that for every* $1 < p < p_0$,
$$\int_{\mathcal{N}} |f(x)|\langle x\rangle^a\,dx \le C_0\Big(\int_{\mathcal{N}} |f(x)|^p\langle x\rangle^b\,dx\Big)^{1/p} \qquad \text{for } f \in \mathrm{Mes}(\mathcal{N})\,,$$
where $C_0 = \int_{\mathcal{N}}\langle x\rangle^{-Q-1}dx$.

Proof. Let $q_0 = \frac{Q+b+1}{b-a}$ and let $1/p_0 + 1/q_0 = 1$. Then, by the Hölder inequality,
$$\begin{aligned}\int_{\mathcal{N}} |f(x)|\langle x\rangle^a\,dx &\le \Big(\int_{\mathcal{N}} \langle x\rangle^{q(a-b)+b}\,dx\Big)^{1/q}\Big(\int_{\mathcal{N}} |f(x)|^p\langle x\rangle^b\,dx\Big)^{1/p}\\ &\le C_0\Big(\int_{\mathcal{N}} |f(x)|^p\langle x\rangle^b\,dx\Big)^{1/p},\end{aligned}$$
for $1 < p < p_0$ and $1/p + 1/q = 1$. $\blacksquare$

Proof of Theorem (4.2). Let $a < b < c < \theta$. Since T is selfadjoint on $L^2(\mathcal{N}) = L^2(0)$, $\{H_t\}$ is holomorphic on $L^2(0)$ in every sector S_ϑ, where $0 < \vartheta < \pi/2$ with the norms $\|H_z\|_{\mathcal{L}(L^2(0))}$ being bounded by 1. On the other hand, the operators H_t are bounded on $L^1(c)$ with $\|H_t\|_{\mathcal{L}(L^1(c))} \le 1+t$. We are going to interpolate between these two extreme cases. To this end, fix $p = 2c/(c+b)$ and let $1/p + 1/q = 1$. Let $f, g \in C_c(\mathcal{N})$ be such that $\|f\|_{L^p(b)} = 1$ and $\|g\|_{L^q(b)} = 1$. For $\zeta \in \mathcal{D} = \{w \in \mathbb{C} : 0 \le \mathrm{Re}\, w \le 1\}$, let
$$\begin{aligned}f_\zeta(x) &= \mathrm{sgn}\, f(x)|f(x)|^{(2-\zeta)p/2}\langle x\rangle^{(1-\zeta/2)(b-c+\zeta c)}\,,\\ g_\zeta(x) &= \mathrm{sgn}\, g(x)|g(x)|^{\zeta q/2}\langle x\rangle^{(1-\zeta/2)(b-c+\zeta c)}\,.\end{aligned}$$

We have
$$|f_\zeta(x)| \le F(x)\,,$$

where $F(x)=A(|f(x)|^p+|f(x)|^{p/2})$ and $A=\sup_{x\in\operatorname{supp} f}\langle x\rangle^\theta$. Similarly,

$$|g_\zeta(x)|\le G(x)\,,$$

where $G(x)=B(|g(x)|^{q/2}+\chi(x))$. Here χ is the characteristic function of the support of g and $B=\sup_{x\in\operatorname{supp} g}\langle x\rangle^\theta$. Note that both F and G are bounded with compact support.

For fixed $\eta>0$ and $0<\vartheta<\vartheta_0<\pi/2$, let

$$H(\zeta)=\exp\{-\eta e^{i\vartheta\zeta}\}H_{\eta\exp(i\vartheta\zeta)},\qquad \zeta\in\mathcal{D}\,.$$

The operator-valued function $\mathcal{D}\ni\zeta\to H(\zeta)\in\mathcal{L}(L^2(\mathcal{N}))$ is continuous on $\mathcal{D}$ and holomorphic in the interior of $\mathcal{D}$.

We define the function

$$\Phi(\zeta)=\int_{\mathcal{N}} H(\zeta)f_\zeta(x)g_\zeta(x)\langle x\rangle^{(1-\zeta)c}\,dx\,,\qquad \zeta\in\mathcal{D}\,,$$

and observe that it is continuous on $\mathcal{D}$ and holomorphic in the interior of $\mathcal{D}$. Note that

$$|\Phi(\zeta)|\le\int_{\operatorname{supp} G}|H(\zeta)f_\zeta(x)|G(x)\,dx\le\|F\|_2\|G\|_2$$

so Φ is also bounded. In addition, $|\Phi(\zeta)|\le1$ for $\operatorname{Re}\zeta=1$ and $|\Phi(\zeta)|\le1$ for $\operatorname{Re}\zeta=0$. Therefore, by the three lines theorem (see, e.g., Stein and Weiss [19]),

$$\Big|\int_{\mathcal{N}} H_zf(x)g(x)\langle x\rangle^b\,dx\Big|=\left|e^z\Phi\left(1-\frac{b}{c}\right)\right|\le|e^z|$$

for every $z=\eta\exp i(1-b/c)\vartheta$. Since the bound depends neither on the choice of f,g in $C_{\mathrm{c}}(\mathcal{N})$ which is dense both in $L^p(b)$ and $L^q(b)$ nor that of $\eta>0$ and $0<\vartheta<\vartheta_0<\pi$, we conclude that

$$\|H_z\|_{\mathcal{L}(L^p(b))}\le|e^z|\le e^{|z|}\qquad\text{for } z\in S_{(1-b/c)\vartheta_0}\,.$$

Moreover, $z\to\int H_zf(x)g(x)\langle x\rangle^b\,dx$ is holomorphic in this sector for all f,g in a common dense subspace of $L^p(b)$ and its dual so the mapping

$$S_{a\vartheta_0/b}\ni z\to H_z\in\mathcal{L}(L^p(b))$$

is holomorphic.

Thus we have proved that $\{H_t\}$ is a holomorphic semigroup of operators on $L^p(b)$, where $p=2c/(c+b)$. This implies that for every $t>0$, TH_t is a bounded operator on $L^p(b)$. The last step in the proof will be the application of Lemma (4.4). Note that enlarging b, if necessary, at the expense of the size of the sector we can assume that $p<p_0$, where p_0 is as in the lemma. Consequently, by Lemma (4.4),

$$\|h_t*T\|_{L^1(a)}\le C_0\|h_t*T\|_{L^p(b)}=\|TH_{t/2}h_{t/2}\|_{L^p(b)}\le\frac{C_1}{t}\|h_{t/2}\|_{L^p(b)}\,.$$

But

$$\|h_t\|^p_{L^p(b)} = \int h_t(x)^p \langle x\rangle^b\,dx = t^{(1-p)Q/\theta} \int h_1(x)^p \langle \delta_{t^{1/\theta}} x\rangle^b\,dx$$
$$\le (1+t)t^{(1-p)Q/\theta}\|h_1\|^{p-1}_\infty \int h_1(x)\langle x\rangle^b\,dx < \infty$$

so the proof is complete. ■

(4.5) COROLLARY. *For every $k \in \mathbb{Z}^+$ and every $0 < a < \theta$,*

$$\int |T^k h_t(x)|\langle x\rangle^a\,dx < \infty\,. \tag{4.6}$$

5. Pointwise estimates. As before T denotes an accretive symmetric kernel of order $0 < \theta < 2$ and $\{\mu_t\}$ the continuous semigroup of measures generated by T. We assume that the measures are absolutely continuous with square-integrable densities. The density of μ_t is denoted by h_t.

Here is the main result of this paper.

(5.1) THEOREM. *For every $k \in \mathbb{Z}^+$ and every $0 < a < \theta$, there exists a function $\Omega = \Omega_{a,k} \in L^1(\Sigma)$ such that*

$$|T^k h_t(x)| \le t^{a/\theta - k}\Omega(\overline{x})|x|^{-Q-a} \qquad \text{for } x \ne 0,\ t > 0\,.$$

Moreover, there exists $\varepsilon > 0$ and a constant $A > 0$, both depending on a and k, such that

$$\int_\Sigma |\Omega(\omega\overline{x}) - \Omega(\overline{x})|\,d\overline{x} \le A\|\omega - I\|^\varepsilon$$

for every $\omega \in \mathcal{O}(n)$.

Proof. Fix $k \in \mathbb{Z}^+$ and let

$$\varphi(x) = T^k h_1(x), \quad \Phi(x) = (k+1)|\varphi(x)| + |T\varphi(x)|\,, \quad x \in \mathcal{N}\,.$$

Note that Φ and φ are continuous and vanish at infinity. Fix $0 < a < \theta$. By Corollary (4.5),

$$\theta \int \Phi(x)\langle x\rangle^a\,dx \le C(a) < \infty\,,$$

whereas Corollaries (3.14) and (3.12) imply

$$|\Phi(x) - \Phi(y)| \le C_1|x^{-1}y|^\tau \tag{5.2}$$

for $x, y \in \mathcal{N}$ and some constants $C, \tau > 0$. For a fixed $\overline{x} \in \Sigma$, let

$$v(s) = s^{-a/\theta}\varphi_s(\overline{x}), \quad \text{where} \quad \varphi_s(x) = s^{-Q/\theta}\varphi(\delta_{s^{-1/\theta}}x)\,.$$

Therefore,

$$\begin{aligned} v'(s) &= -\frac{a}{\theta}s^{-a/\theta-1}\varphi_s(\overline{x}) + s^{-a/\theta}\frac{d}{ds}(s^k T^k h_s(\overline{x})) \\ &= -\frac{a}{\theta}s^{-a/\theta-1}\varphi_s(\overline{x}) + s^{-a/\theta}(ks^{k-1}T^k h_s(\overline{x}) + s^k T^{k+1}h_s(\overline{x})) \\ &= -\frac{a}{\theta}s^{-a/\theta-1}\varphi_s(\overline{x}) + s^{-a/\theta-1}(k(T^k h)_s(\overline{x}) + (T^{k+1}h)_s(\overline{x})) \end{aligned}$$

$$\begin{aligned}&= \Big(k - \frac{a}{\theta}\Big) s^{-a/\theta-1}\varphi_s(\overline{x}) + s^{-a/\theta-1}(T\varphi)_s(\overline{x})\\ &\le s^{-a/\theta-1}\Phi_s(\overline{x})\,.\end{aligned}$$

Consequently, for $s > 0$,

$$|v(s)| \le \int\limits_0^\infty |v'(s)|\,ds \le \int\limits_0^\infty s^{-a/\theta}\Phi_s(\overline{x})\,\frac{ds}{s} = \theta\int\limits_0^\infty t^{Q-1+a}\Phi(\delta_t\overline{x})\,dt\,.$$

Let

$$\Omega(\overline{x}) = \theta\int\limits_0^\infty t^{Q-1+a}\Phi(\delta_t\overline{x})\,dt, \qquad \overline{x}\in\Sigma\,.$$

Then, of course,

$$|(T^k h)_s(\overline{x})| \le s^{a/\theta}\Omega(\overline{x}) \qquad \text{for } s > 0\,,$$

which, by homogeneity, implies

$$|T^k h_t(x)| \le t^{a/\theta-k}\Omega(\overline{x})|x|^{-Q-a} \qquad \text{for } t > 0\,.$$

We also have

$$\int\limits_\Sigma \Omega(\overline{x})\,d\overline{x} = \theta\int\limits_{\mathcal{N}} \Phi(x)|x|^a\,dx \le C(a)\,.$$

It remains to show that Ω satisfies the integral Lipschitz condition. First let us remark that there exists a constant L such that

$$|x^{-1}y| \le L\|y-x\|$$

for $|x| = 1$, $|y| = 1$, where the norm on the right-hand side is the Euclidean norm on $\mathcal{N}$. For $a < b < \theta$ and $\omega\in\mathcal{O}(n)$, let

$$U_z(\overline{x}) = \int\limits_0^\infty t^{Q-1}(1+t)^z|\Phi(\delta_t\overline{x}) - \Phi(\delta_t(\omega\overline{x}))|\,dt\,.$$

Then the function

$$V(z) = \int\limits_\Sigma U_z(\overline{x})\,d\overline{x}\,,$$

is holomorphic in the strip $-(Q+1) < \operatorname{Re} z < b$ and continuous on its closure. Moreover,

$$|V(z)| \le 2\int \langle x\rangle^b\Phi(x)\,dx \le C(b) \qquad \text{for } -Q-1 \le \operatorname{Re} z \le b\,,$$

and, by (5.2),

$$\begin{aligned}|V(z)| &\le C\Big(\int\limits_0^\infty t^{Q-1}(1+t)^{-Q-1}t^\tau\,dt\Big)|\overline{x}^{-1}(\omega\overline{x})|^\tau\\ &\le C_2\|\omega - I\|^\tau \qquad \text{for } \operatorname{Re} z = -Q-1\,.\end{aligned}$$

Therefore, by the three lines theorem, $V(a) \le C\|\omega - I\|^\varepsilon$ for some C, $\varepsilon > 0$, which implies

$$\int_{\Sigma} |\Omega(\overline{x}) - \Omega(\omega\overline{x})|\, d\overline{x} \le V(a) \le C\|\omega - I\|^\varepsilon$$

and thus completes the proof. ■

6. Application. The *Zo norm* of an L^1 function φ on a homogeneous group $\mathcal{N}$ is defined by

$$\|\varphi\|_{\mathrm{Zo}} = \sup_{x\in\mathcal{N}} \int_{|y|\ge 2|x|} \sup_{t>0} |\delta_t\varphi(xy) - \delta_t\varphi(y)|\, dy\,,$$

where, by definition,

$$\delta_t\varphi(x) = t^{-Q}\varphi(\delta_{t^{-1}}x), \qquad x \in \mathcal{N},\ t > 0\,.$$

Let us recall the well-known lemma due to Zo (cf. Zo [21] as well as Stein [18], 71–73).

(6.1) Lemma. *Let $k \in L^1(\mathcal{N})$. If $\|k\|_{\mathrm{Zo}} < \infty$, then the maximal operator K defined by*

$$Kf(x) = \sup_{t>0} |f * k_t(x)|$$

is of weak type $(1,1)$.

The following theorem is due to E. M. Stein. The proof of it is implicitly contained in Stein [18].

(6.2) Theorem. *Let k be an integrable function on a homogeneous group $\mathcal{N}$ such that*

$$|k(x)| \le \Omega(\overline{x})|x|^{-Q-\varepsilon}\,,$$

where $0 \le \Omega \in L^1(\Sigma)$ and $\varepsilon > 0$. Let, moreover,

$$\int_{\Sigma} \Omega(\overline{x})\, d\overline{x} \le A\,,$$

and

$$\int_{\Sigma} |\Omega(\omega\overline{x}) - \Omega(\overline{x})|\, d\overline{x} \le A\|\omega - I\|^\varepsilon$$

for every $\omega \in \mathcal{O}(n)$ and some constant $A > 0$. Then the Zo norm of k is estimated by

$$\|k\|_{\mathrm{Zo}} \le C(\varepsilon)A\,,$$

where $C(\varepsilon)$ depends just on ε.

As an immediate corollary from Theorem (6.2), Theorem (5.1), and Lemma (6.1) we get

(6.3) COROLLARY. *The maximal function*

$$\mathcal{M}f(x) = \sup_{t>0} |f * h_t(x)|$$

is of weak type $(1, 1)$.

Finally, by the routine technique employing the Marcinkiewicz interpolation theorem (see, e.g., Stein and Weiss [19]), we obtain

(6.4) COROLLARY. *The maximal function*

$$\mathcal{M}f(x) = \sup_{t>0} |f * h_t(x)|$$

is of strong type (p, p) *for* $1 < p \leq \infty$.

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