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A contribution to the topological classification of the spaces $C_{\mathrm{p}}(X)$

by

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Abstract. We prove that for each countably infinite, regular space X such that $C_{\mathrm{p}}(X)$ is a Z_σ-space, the topology of $C_{\mathrm{p}}(X)$ is determined by the class $\mathcal{F}_0(C_{\mathrm{p}}(X))$ of spaces embeddable onto closed subsets of $C_{\mathrm{p}}(X)$. We show that $C_{\mathrm{p}}(X)$, whenever Borel, is of an exact multiplicative class; it is homeomorphic to the absorbing set Ω_α for the multiplicative Borel class $\mathcal{M}_\alpha$ if $\mathcal{F}_0(C_{\mathrm{p}}(X)) = \mathcal{M}_\alpha$. For each ordinal $\alpha \geq 2$, we provide an example X_α such that $C_{\mathrm{p}}(X_\alpha)$ is homeomorphic to Ω_α.

1. Introduction. For a countable, regular (T_3) space X, let $C_{\mathrm{p}}(X)$ be the space of all continuous real-valued functions on X with the topology of pointwise convergence. Thus $C_{\mathrm{p}}(X)$ is a dense linear subspace of $\mathbb{R}^X$, the latter space being identified with the countable product of lines.

In the paper we apply the method of absorbing sets [2] to the topological classification of $C_{\mathrm{p}}(X)$ spaces. This subject was previously treated in several papers (see [5], [12]–[14], [23] and references therein). The method applies to spaces $C_{\mathrm{p}}(X)$ which are of the first category in $\mathbb{R}^X$, more precisely, to those that are countable unions of Z-sets (briefly, Z_σ-spaces). Let us recall that the key notion of the absorbing set method is the strong $\mathcal{C}$-universality for a class $\mathcal{C}$ of spaces (see Section 2 for definitions). The uniqueness theorem for absorbing sets states that two such function spaces $C_{\mathrm{p}}(X)$ and $C_{\mathrm{p}}(Y)$ are homeomorphic provided they are strongly $\mathcal{C}$-universal and can be expressed as countable unions of closed sets that are elements of $\mathcal{C}$. In order to apply the method, for a given space X, one must identify the class $\mathcal{C}$ and then show the strong $\mathcal{C}$-universality of $C_{\mathrm{p}}(X)$.

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The question of the strong universality of $C_{\mathrm{p}}(X)$ will be entirely resolved, for we shall prove that every space $C_{\mathrm{p}}(X)$ (not necessarily of the first category in $\mathbb{R}^X$) is strongly universal for the class $\mathcal{F}_0(C_{\mathrm{p}}(X))$ of spaces homeomorphic to closed subsets of $C_{\mathrm{p}}(X)$. This is a consequence of the strong $\mathcal{F}_0(E)$-universality of an arbitrary metric linear space E which is an absolute retract and which admits $\mathbb{R}^\infty$ as a factor, and the fact that (for noncompact X) $C_{\mathrm{p}}(X)$ always has such a factor. Applying the uniqueness theorem for absorbing sets, we conclude that for a Z_σ-space $C_{\mathrm{p}}(X)$ its topology is entirely determined by the class $\mathcal{F}_0(C_{\mathrm{p}}(X))$. This means that two Z_σ-spaces $C_{\mathrm{p}}(X)$ and $C_{\mathrm{p}}(Y)$ are homeomorphic if and only if each is homeomorphic to a closed subset of the other. It is remarkable that this result can be applicable even if we are not able to explicitly determine the class $\mathcal{F}_0(C_{\mathrm{p}}(X))$. For example, we show that if X has exactly one accumulation point and $C_{\mathrm{p}}(X)$ is a Z_σ-space then $(C_{\mathrm{p}}(X))^\infty$ is homeomorphic to a closed subset of $C_{\mathrm{p}}(X)$, and therefore $C_{\mathrm{p}}(X)$ and $(C_{\mathrm{p}}(X))^\infty$ are homeomorphic.

Subsequently, we apply the above general results to Borelian spaces $C_{\mathrm{p}}(X)$. It has been proved in [14] that if X is nondiscrete and $C_{\mathrm{p}}(X)$ is an absolute $F_{\sigma\delta}$-set, then $C_{\mathrm{p}}(X)$ is homeomorphic to Ω_2 (where Ω_α is the absorbing set for the multiplicative Borel class $\mathcal{M}_\alpha$ [2]). In view of this result, a conjecture was posed in [14] that for all α, every space $C_{\mathrm{p}}(X)$ of exact multiplicative class α must be homeomorphic to Ω_α. Since every Borelian space $C_{\mathrm{p}}(X)$ is a Z_σ-space, by applying our general theorems, the above conjecture reduces to $\mathcal{F}_0(C_{\mathrm{p}}(X)) = \mathcal{M}_\alpha$. Until now, it was not known that for $\alpha \geq 3$ there are spaces X_α so that $C_{\mathrm{p}}(X_\alpha)$ is homeomorphic to Ω_α nor that $C_{\mathrm{p}}(X)$ must always be of an exact multiplicative Borel class. We prove these statements. In fact, for X with exactly one accumulation point, we present two methods of constructing spaces $C_{\mathrm{p}}(X)$ of arbitrarily high Borel complexity. Every such X can be regarded as a space $\mathbb{N}_F$ induced by a filter F on the set $\mathbb{N}$ of integers (cf. Section 2 for definition). The first method provides, by transfinite induction, the spaces $C_{\mathrm{p}}(X)$ that are of even multiplicative classes; the basic obstacle to carrying out this construction for odd ordinals is the nonexistence of filters of type G_δ. The second method, which is a variation of the construction in [2], assigns to every subset A of the Hilbert cube a filter F_A such that $C_{\mathrm{p}}(\mathbb{N}_{F_A})$ contains A as a closed subset. For $A = \Omega_\alpha$, $\alpha \geq 2$, the space $C_{\mathrm{p}}(\mathbb{N}_{F_A})$ is homeomorphic to Ω_α.

Actually, our techniques work for all pairs $(\mathbb{R}^X, C_{\mathrm{p}}(X))$ and triples $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X))$. In particular, this allows us to give a complete classification of the triples $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X))$ for which $C_{\mathrm{p}}(X)$ is an absolute $F_{\sigma\delta}$-set.

2. Notations, definitions and auxiliary results. The symbol $\cong$ means "homeomorphic to". Maps are always continuous, and A^n designates the product of n copies of A, whereas A^∞ is the product of a countably infinite set of copies of A. The set of positive integers and the set of reals are denoted by $\mathbb{N}$ and $\mathbb{R}$, respectively. We let $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. I^∞ is the Hilbert cube $[0,1]^\infty$ and $2^\infty = \{0,1\}^\infty$.

Let X be a countable regular space. By 2^X we denote the set of all subsets of X. Identifying each subset of X with its characteristic function, we consider 2^X as the subspace $\{0,1\}^X$ of $\mathbb{R}^X$. We denote by $C_{\mathrm{p}}^{\mathrm{loc}}(X)$ the subspace of $C_{\mathrm{p}}(X)$ consisting of all locally constant functions.

Filters on a countable infinite set X are always assumed to contain the Fréchet filter F_0 consisting of all cofinite sets of X. Given a filter F on $\mathbb{N}$, we denote by $\mathbb{N}_F$ the space $\mathbb{N} \cup \{\infty\}$ topologized by isolating the points of $\mathbb{N}$ and using the family $\{A \cup \{\infty\} \mid A \in F\}$ to be a neighborhood base at ∞. We write

$$c_F = \{(x_n) \in \mathbb{R}^\infty \mid \forall_{\varepsilon>0} \exists_{A \in F} \forall_{n \in A}\ |x_n| < \varepsilon\}\,,$$
$$s_F = \{(x_n) \in \mathbb{R}^\infty \mid \forall_{\varepsilon>0} \exists_{A \in F} \forall_{n \in A}\ x_n = 0\}\,.$$

It is known [23, Lemma 2.1] that $C_{\mathrm{p}}(\mathbb{N}_F)$ is (linearly) homeomorphic to c_F; one can easily adapt the proof of [23, Lemma 2.1] to show that $(\mathbb{R}^{\mathbb{N}_F}, C_{\mathrm{p}}(\mathbb{N}_F), C_{\mathrm{p}}^{\mathrm{loc}}(\mathbb{N}_F)) \cong (\mathbb{R}^\infty, c_F, s_F)$.

If Y is a separable metrizable space, and α is a countable ordinal, we write $\mathcal{A}_\alpha(Y)$ (resp., $\mathcal{M}_\alpha(Y)$) to denote the family of subsets of Y that are Borel of additive (resp., multiplicative) class α. By $\mathcal{A}_\alpha$ (resp., $\mathcal{M}_\alpha$) we denote the class of spaces that are absolute Borel of additive (resp., multiplicative) class α. If $A \in \mathcal{A}_\alpha \setminus \mathcal{M}_\alpha$ (resp., $\mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$ or $\mathcal{A}_\alpha \cap \mathcal{M}_\alpha \setminus \bigcup_{\beta<\alpha}(\mathcal{A}_\beta \cap \mathcal{M}_\beta)$), then we say that A is of *exact additive* (resp., *multiplicative* or *ambiguous*) *class* α. By $\mathcal{P}_n$, $n \geq 0$, we denote the nth projective class. Let $\mathcal{C}$ be a class of spaces. We say that a pair (X, X_0) is *Wadge* $(Y, \mathcal{C})$*-complete* if, for every $A \subseteq Y$, $A \in \mathcal{C}$, there exists a map $\varphi : Y \to X$ such that $\varphi^{-1}(X_0) = A$ (usually, $\mathcal{C} = A_\alpha$, $\mathcal{M}_\alpha$ or $\mathcal{P}_n$).

A subset A of a metric space X is said to be *locally homotopy negligible* in X if for every open subset U of X, the inclusion of $U \setminus A$ into U is a weak homotopy equivalence. A closed subset A of X is a *Z-set* (resp., *strong Z-set*) if, for every open cover $\mathcal{U}$ of X, there exists a map $f : X \to X$ that is $\mathcal{U}$-close to the identity and satisfies $f(X) \cap A = \emptyset$ (resp., $\overline{f(X)} \cap A = \emptyset$). It is known that the classes of Z-sets and strong Z-sets coincide in any X which admits a completion $\widehat{X}$ homeomorphic to either $\mathbb{R}^\infty$ or $\overline{\mathbb{R}}^\infty$ and such that $\widehat{X} \setminus X$ is locally homotopy negligible in $\widehat{X}$ (see [2]). For a space X that is an absolute neighborhood retract, a subset (resp., closed subset) A of X is locally homotopy negligible (resp., a Z-set) if every map of I^n into X is approximable by maps into X whose images miss A, $n = 1, 2, \ldots$ A space which is a countable union of Z-sets is called a *Z_σ-space* (see [25]). By a *Z-embedding* we mean an embedding $f : Y \to X$ such that $f(Y)$ is a Z-set in X.

Let $(K_1, \ldots, K_k)$, $k \geq 1$, be a topological k-tuple (briefly, a *k-tuple*), i.e., $K_1 \supseteq \ldots \supseteq K_k$. We say that a k-tuple $(X_1, \ldots, X_k)$ is *strongly* $(K_1, \ldots, K_k)$-*universal* if, for every closed subset D of K_1, every map $f : K_1 \to X_1$ whose restriction to D is a Z-embedding and for which $(f|D)^{-1}(X_i) = D \cap K_i$, $i = 1, \ldots, k$, and every open cover $\mathcal{U}$ of X_1, there exists a Z-embedding $g : K_1 \to X_1$ which is $\mathcal{U}$-close to f and satisfies $g|D = f|D$ and $g^{-1}(X_i) = K_i$ for $i = 1, \ldots, k$. If $\mathcal{K}$ is a class of k-tuples, then $(X_1, \ldots, X_k)$ is strongly $\mathcal{K}$-universal provided it is strongly $(K_1, \ldots, K_k)$-universal for each $(K_1, \ldots, K_k) \in \mathcal{K}$.

A class $\mathcal{K}$ of k-tuples is said to be *topological* if it contains every homeomorph of an element of $\mathcal{K}$. It is *additive* if, given a k-tuple $(K_1, \ldots, K_k)$ such that $K_1 = K_1^1 \cup K_1^2$, K_1^1 and K_1^2 closed in K_1, and such that each $(K_1^i, K_1^i \cap K_2, \ldots, K_1^i \cap K_k)$ belongs to $\mathcal{K}$, $(K_1, \ldots, K_k)$ belongs to $\mathcal{K}$. Finally, it is called *hereditary with respect to closed subsets* if, for every $(K_1, \ldots, K_k) \in \mathcal{K}$ and every closed $C \subseteq K_1$, $(C, C \cap K_2, \ldots, C \cap K_k) \in \mathcal{K}$. If $\mathcal{C}_1, \ldots, \mathcal{C}_k$ are classes of spaces, we denote by $(\mathcal{C}_1, \ldots, \mathcal{C}_k)$ the class consisting of all k-tuples $(K_1, \ldots, K_k)$ such that $K_i \in \mathcal{C}_i$ for $i = 1, \ldots, k$.

Let $\mathcal{C}$ be a class of (separable metrizable) spaces which is topological, additive and hereditary with respect to closed subsets. A subset X of $\mathbb{R}^\infty$ is called a $\mathcal{C}$-*absorbing set* [2] in $\mathbb{R}^\infty$ if it satisfies the following conditions:

(i) $\mathbb{R}^\infty \setminus X$ is locally homotopy negligible in $\mathbb{R}^\infty$,
(ii) $X = \bigcup_{n=1}^\infty Z_n$, where each Z_n is a Z-set in X and belongs to $\mathcal{C}$,
(iii) X is strongly $\mathcal{C}$-universal.

Here is a particular case of [2, Theorem 3.1] (which we will refer to as the *uniqueness theorem for absorbing sets*).

2.1. Theorem. *Any two $\mathcal{C}$-absorbing sets in $\mathbb{R}^\infty$ are homeomorphic.* ∎

The notion of $\mathcal{C}$-absorbing set has its origin in research done by Anderson, Bessaga and Pełczyński, Toruńczyk and West (see [1]). They mostly considered absorbing sets in complete metric spaces M with $\mathcal{C}$ being a subclass of the class of all Z-sets in M. Then any two $\mathcal{C}$-absorbing sets were ambiently homeomorphic in M. The same can be achieved by using the above strong universality for k-tuples; this concept was originated by Cauty in [6].

It is routine to check that whenever a k-tuple $(X_1, \ldots, X_k)$ is strongly $\mathcal{K}$-universal for some class $\mathcal{K}$ of k-tuples then, under some restrictions on X_1, it is also strongly universal with respect to the smallest class $\widetilde{\mathcal{K}}$ that is topological, additive, hereditary with respect to closed subsets and contains $\mathcal{K}$. Specifically, this is true if $X_1 \cong \mathbb{R}^\infty$ or $X_1 \cong \overline{\mathbb{R}}^\infty$ (or X_1 is a Z_σ-space that is an absolute retract; see [15, p. 412]).

2.2. THEOREM. *Let* $(A_1^i, A_2^i, \ldots, A_k^i)$, $i = 1, 2$, *be* k*-tuples in* $\mathbb{R}^\infty$. *Suppose* $A_1^i \subseteq \bigcup_{n=1}^\infty X_n^i$, $i = 1, 2$, *where* X_n^i *are* Z*-sets in* $\overline{\mathbb{R}}^\infty$.

(a) *If each* $(k+1)$*-tuple* $(\mathbb{R}^\infty, A_1^i, \ldots, A_k^i)$ *is strongly* $(X_n^j \cap \mathbb{R}^\infty, X_n^j \cap A_1^j, \ldots, X_n^j \cap A_1^j)$*-universal for* $j = 1, 2$ *and* $n \geq 1$, *then* $(\mathbb{R}^\infty, A_1^1, \ldots \ldots, A_k^1) \cong (\mathbb{R}^\infty, A_1^2, \ldots, A_k^2)$.

(b) *If each* $(k+2)$*-tuple* $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, A_1^i, \ldots, A_k^i)$ *is strongly* $(X_n^j \cap \overline{\mathbb{R}}^\infty, X_n^j \cap \mathbb{R}^\infty, X_n^j \cap A_1^j, \ldots, X_n^j \cap A_k^j)$*-universal for* $j = 1, 2$ *and* $n \geq 1$, *then* $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, A_1^1, \ldots, A_k^1) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, A_1^2, \ldots, A_k^2)$.

Proof. The proof employs a version of a standard back and forth argument. More specifically, to get (a) follow the version elaborated in [6, Theorem 2.1] and use the remark made before the statement of the theorem; part (b) needs some adjustments.

Since $\overline{\mathbb{R}}^\infty \setminus \mathbb{R}^\infty$ is strongly $\mathcal{M}_0$-universal [1], for every compactum Z, the $(k+2)$-tuples in question are strongly $(Z, \emptyset, \ldots, \emptyset)$-universal. Let $\mathcal{K}$ be the smallest topological additive class which is hereditary with respect to closed subsets and contains all $(k+2)$-tuples of the form $(X_n^j, X_n^j \cap \mathbb{R}^\infty, X_n^j \cap A_1^j, \ldots, X_n^j \cap A_k^j)$ and $(Z, \emptyset, \emptyset, \ldots, \emptyset)$, where Z is a compactum. It follows that the $(k+2)$-tuples in question are strongly $\mathcal{K}$-universal. To obtain (b) it suffices to repeat the proof of [6, Theorem 2.1] replacing the pairs $(\overline{Y}_n \cup f_{2n}(\overline{X}_{n+1}), Y_n \cup f_{2n}(X_{n+1}))$ and $(\overline{X}_{n+1} \cup f_{2n+1}^{-1}(\overline{Y}_{n+1}), X_{n+1} \cup f_{2n+1}^{-1}(Y_{n+1}))$ therein by the $(k+2)$-tuples $\mathcal{Z}_n^2 \cup f_{2n}(\mathcal{Z}_{n+1}^1)$ and $\mathcal{Z}_{n+1}^1 \cup f_{2n+1}^{-1}(\mathcal{Z}_{n+1}^2)$, respectively, where $\mathcal{Z}_n^i$ are defined below. Let $\overline{\mathbb{R}}^\infty \setminus \mathbb{R}^\infty = \bigcup_{n=0}^\infty B_n$, where $B_0 = \emptyset$ and B_n are compacta. Set

$$\mathcal{Z}_n^i = (X_n^i \cup B_n, X_n^i \cap \mathbb{R}^\infty, X_n^i \cap A_1^i, \ldots, X_n^i \cap A_k^i)$$

and observe that since $B_n \cap \mathbb{R}^\infty = \emptyset$, $\mathcal{Z}_n^i \in \mathcal{K}$. ∎

Let $(X_1, \ldots, X_k)$ be a k-tuple. We denote by $\mathcal{F}_0(X_1, \ldots, X_k)$ the class of all k-tuples homeomorphic to a k-tuple of the form $(C, C \cap X_2, \ldots, C \cap X_k)$, where C is a closed subset of X_1. In particular, $\mathcal{F}_0(X)$ is the class of spaces that are homeomorphic to closed subsets of X.

2.3. COROLLARY. *Let* $K^i = (\mathbb{R}^\infty, A_1^i, \ldots, A_k^i)$ *and* $L^i = (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, A_1^i, \ldots \ldots, A_k^i)$, $i = 1, 2$, *be tuples. Assume that*

(i) *each* A_1^i *is a* Z_σ*-space*,

(ii) *each* $\mathbb{R}^\infty \setminus A_1^i$ *is locally homotopy negligible in* $\mathbb{R}^\infty$,

(iii) *each* K^i (*resp.*, L^i) *is strongly* $\mathcal{F}_0(K^i)$*-universal* (*resp.*, $\mathcal{F}_0(L^i)$*-universal*).

If $\mathcal{F}_0(K^1) = \mathcal{F}_0(K^2)$ (*resp.*, $\mathcal{F}_0(L^1) = \mathcal{F}_0(L^2)$), *then* $K^1 \cong K^2$ (*resp.*, $L^1 \cong L^2$).

Proof. By the Z_σ-property, $A_1^i = \bigcup_{n=1}^{\infty} Z_n^i$, where Z_n^i are Z-sets in A_1^i. Since $\mathbb{R}^\infty \backslash A_1^i$ is locally homotopy negligible in $\mathbb{R}^\infty$, the closure X_n^i of Z_n^i in $\overline{\mathbb{R}}^\infty$ is a Z-set. Now, 2.2 is applicable. ■

Let Ω_α and Λ_α be the absorbing sets in $\mathbb{R}^\infty$ for the classes $\mathcal{M}_\alpha$ and $\mathcal{A}_\alpha$, respectively, constructed in [2]. The space Λ_1 is $\{(x_n) \in \mathbb{R}^\infty : (x_n)$ is bounded$\}$ and is commonly denoted by Σ. By [9, Proposition 4.1 and Remark 4.8], $(\mathbb{R}^\infty, \Omega_\alpha)$ (resp., $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha)$) is strongly $(\mathcal{M}_1, \mathcal{M}_\alpha)$-universal (resp., $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_\alpha)$-universal), $\alpha \geq 2$. From 2.3, it follows that the respective strong universality characterizes $(\mathbb{R}^\infty, X)$ or triples $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, X)$ such that $X \cong \Omega_\alpha$ (the fact that $\mathbb{R}^\infty \backslash X$ is locally homotopy negligible in $\mathbb{R}^\infty$ follows from the respective strong universality). Similarly, the strong $(\mathcal{M}_1, \mathcal{A}_\alpha)$-universality (resp., $(\mathcal{M}_0, \mathcal{M}_1, \mathcal{A}_\alpha)$-universality) characterizes the pairs $(\mathbb{R}^\infty, X)$ (resp., triples $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, X)$) such that $X \cong \Lambda_\alpha$.

Let us recall that in [8], there have been constructed absorbing sets Π_n and Π_n' for the class $\mathcal{P}_n$, $n \geq 1$, in a copy E of $\mathbb{R}^\infty$ and Q of the Hilbert cube, respectively. Moreover, the pairs (E, Π_n) and (Q, Π_n') are strongly $(\mathcal{M}_1, \mathcal{P}_n)$- and $(\mathcal{M}_0, \mathcal{P}_n)$-universal, respectively. Writing $(\mathbb{R}^\infty, \Pi_n)$ and $(\overline{\mathbb{R}}^\infty, \Pi_n)$ we will mean the above pairs (E, Π_n) and (Q, Π_n'), respectively.

In Section 4 we shall need a particular case of the following fact.

2.4. THEOREM. *Let Z_1 and Z_2 be subsets of Σ such that each (Σ, Z_i) is strongly $(\mathcal{M}_0, \mathcal{C})$-universal for some class $\mathcal{C}$. Assume each Z_i is a countable union of closed sets that are elements of $\mathcal{C}$. Then the quadruples $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Sigma, Z_1)$ and $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Sigma, Z_2)$ are homeomorphic.*

Proof. Let $Z_i = \bigcup_{n=1}^{\infty} Z_n^i$, where Z_n^i are closed in Z_i and $Z_n^i \in \mathcal{C}$. Write $\Sigma = \bigcup_{k=1}^{\infty} B_k$, where B_k are compacta. Put $X_{n,k}^i = B_k \cap \overline{Z}_n^i$, the closure taken in $\overline{\mathbb{R}}^\infty$. Clearly, each quadruple $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Sigma, Z_i)$ is strongly $(X_{n,k}^j, X_{n,k}^j, X_{n,k}^j, X_{n,k}^j \cap Z_j)$-universal. To get the result apply 2.2(b). ■

3. Criteria of strong universality. The following result on the strong universality of linear spaces $E \times \mathbb{R}^\infty$ plays a fundamental role in this paper.

3.1. THEOREM. *Let E be a separable metric linear space that is an absolute retract and let E_0 be a dense linear subspace of E. Then the pair $(E \times \mathbb{R}^\infty, E_0 \times \mathbb{R}^\infty)$ is strongly $\mathcal{F}_0(E \times \mathbb{R}^\infty, E_0 \times \mathbb{R}^\infty)$-universal.*

Proof. Since $\mathbb{R}^\infty$ is strongly $\mathcal{F}_0(\mathbb{R}^\infty)$-universal we can assume that E is infinite-dimensional. Let $\widehat{E}$ be the linear completion of E. Endow $\widehat{E}$ with an F-norm $|\cdot|_1$ that is increasing on each ray emanating from the origin (see [1, p. 285]). We consider $\mathbb{R}^\infty$ as the product $\prod_{n=1}^{\infty} R_n$, where $R_n = \mathbb{R}$

for all n, and endow it with the F-norm

$$|x|_2 = \sum_{n=1}^{\infty} 2^{-n} \frac{|x_n|}{1+|x_n|} \quad \text{for } x = (x_n) \in \mathbb{R}^\infty .$$

Note that

$$|x|_2 \le \frac{1}{n+1} \quad \text{if } x_k = 0 \text{ for } k \le n . \tag{1}$$

We define an F-norm on $\widehat{E} \times \mathbb{R}^\infty$ by letting for $y = (z, x) \in \widehat{E} \times \mathbb{R}^\infty$,

$$|y| = |z|_1 + |x|_2 .$$

We identify $\widehat{E}$ and $\mathbb{R}^\infty$ with $\widehat{E} \times \{0\}$ and $\{0\} \times \mathbb{R}^\infty$ in $\widehat{E} \times \mathbb{R}^\infty$, respectively. Let $\pi : \widehat{E} \times \mathbb{R}^\infty \to \widehat{E}$ be the projection. Each element $y \in \widehat{E} \times \mathbb{R}^\infty$ is of the form $(y_0, (y_n))$, where $y_0 = \pi(y)$ and y_n is the projection of y onto R_n.

Proposition 4.1 in [25] assures that there exists a set $\widetilde{E}$ such that $E \subseteq \widetilde{E} \subseteq \widehat{E}$, $\widetilde{E}$ is a G_δ-subset of $\widehat{E}$ (hence, topologically complete) and $\widetilde{E}$ is an absolute retract with $\widetilde{E} \backslash E$ locally homotopy negligible in $\widetilde{E}$. By [16] (see Lemma 1 and Sec. 2 therein), both $\widetilde{E}$ and $\widetilde{E} \times \mathbb{R}^\infty$ are copies of $\mathbb{R}^\infty$. It follows that every Z-set in $E \times \mathbb{R}^\infty$ is a strong Z-set.

Let $T = \{(x_n) \in \mathbb{R}^\infty \mid x_n \neq 0 \text{ for infinitely many } n\}$. It is easy to check that $E \times T$ and $E \times \mathbb{R}^\infty \backslash E_0 \times T$ are locally homotopy negligible in $E \times \mathbb{R}^\infty$. Then, by [9, Proposition 2.1] applied to $X = E \times \mathbb{R}^\infty$, $Y = E_0 \times T$, $Y' = E \times T$, $Z = E_0 \times \mathbb{R}^\infty$ and $(K, L) = (K, K \cap (E_0 \times \mathbb{R}^\infty)) \in \mathcal{F}_0(E \times \mathbb{R}^\infty, E_0 \times \mathbb{R}^\infty)$, it suffices to verify the following:

(∗) *Given a closed subset K of X, an open subset U of K, an open subset V of X, an open cover $\mathcal{V} = \{V_j \mid j \in \mathcal{J}\}$ of V and a map $f : K \to X$ satisfying $f(U) \subset V \cap Y$ and $f(K \backslash U) \subset X \backslash V$, there exists a closed embedding $g : U \to V$ that is $\mathcal{V}$-close to $f|U$ and satisfies $g(U) \subset Y'$ and $g^{-1}(V \cap Y) = U \cap (E_0 \times \mathbb{R}^\infty)$.*

For each $j \in \mathcal{J}$, find an open set $\widehat{V}_j \subset \widehat{E} \times \mathbb{R}^\infty$ such that $\widehat{V}_j \cap (E \times \mathbb{R}^\infty) = V_j$. Then $\widehat{V} = \bigcup_{j \in \mathcal{J}} \widehat{V}_j$ is open in $\widehat{E} \times \mathbb{R}^\infty$, $\widehat{V} \cap (E \times \mathbb{R}^\infty) = V$ and $\widehat{\mathcal{V}} = \{\widehat{V}_j \mid j \in \mathcal{J}\}$ is an open cover for $\widehat{V}$. Pick a map $\omega : \widehat{V} \to (0, 1]$ such that

(2) whenever $y \in \widehat{V}$, $y' \in \widehat{E} \times \mathbb{R}^\infty$ and $|y - y'| < 4\omega(y)$, then $y, y' \in \widehat{V}_j$ for some $j \in \mathcal{J}$.

Lavrent'ev's theorem guarantees the existence of a subset $\widetilde{K}$ of $\widetilde{E}$ that is a G_δ-subset of $\widetilde{E}$ (hence, $\widetilde{K}$ is topologically complete), $K \subseteq \widetilde{K}$, and such that f admits a continuous extension $\tilde{f} : \widetilde{K} \to \widetilde{E} \times \mathbb{R}^\infty$. We can assume that $\widetilde{K} \cap (E \times \mathbb{R}^\infty) = K$. Let $\widetilde{V} = \widehat{V} \cap \widetilde{E}$ and $\widetilde{U} = \tilde{f}^{-1}(\widetilde{V})$; hence $\widetilde{U} \cap K = U$.

We need the following lemma.

3.2. Lemma. *There exists a map* $\Psi = (\Psi_0, (\Psi_n)) : \widetilde{E} \times \mathbb{R}^\infty \times [1,\infty] \to \widetilde{E} \times \mathbb{R}^\infty$ *satisfying*

(i) $\Psi(\widetilde{E} \times \mathbb{R}^\infty \times [1,\infty)) \subset E_0 \times \mathbb{R}^\infty$,

(ii) $\Psi(y,\infty) = y$ *for all* $y \in \widetilde{E} \times \mathbb{R}^\infty$,

(iii) $\Psi_n(y,t) = 0$ *for* $t \le n-1$ *and* $y \in \widetilde{E} \times \mathbb{R}^\infty$,

(iv) *if* $\lim \Psi(y_i,t_i) = y \in \widehat{E} \times \mathbb{R}^\infty$, $(y_i,t_i) \in \widetilde{E} \times \mathbb{R}^\infty \times [1,\infty)$, *and* $\lim t_i = \infty$, *then* $\lim y_i = y$.

Proof. Since $\widetilde{E} \backslash E$ is locally homotopy negligible in $\widetilde{E}$ and $E \backslash E_0$ is locally homotopy negligible in E, $\widetilde{E} \backslash E_0$ is locally homotopy negligible in $\widetilde{E}$. Since $\widetilde{E}$ is an absolute retract, by [25, Theorem 2.4] there exists $\psi : \widetilde{E} \times [1,\infty] \to \widetilde{E}$ such that

(a) $\psi(\widetilde{E} \times [1,\infty)) \subset E_0$,

(b) $\psi(x,\infty) = x$ for every $x \in \widetilde{E}$,

(c) $|\psi(x,t) - x|_1 < 1/t$ for all $(x,t) \in \widetilde{E} \times [1,\infty)$.

Define Ψ as follows: $\Psi_0 = \psi \circ \pi$ and

$$\Psi_n(y,t) = \begin{cases} y_n & \text{if } n \le t, \\ s y_n & \text{if } t = n-1+s,\ 0 \le s \le 1, \\ 0 & \text{if } t \le n-1, \end{cases}$$

for $y = (y_0,(y_n)) \in \widetilde{E} \times \mathbb{R}^\infty$. It is clear that Ψ satisfies (i)–(iii). The condition (iv) is a consequence of (c) and the fact that $\Psi_n(y,t) = y_n$ for $n \le t$. ■

We go back to the proof of 3.1. Applying (ii) and the continuity of Ψ we can choose a map $\varepsilon : \widetilde{V} \to (0,1]$ with the properties

$$|\Psi(y,(\varepsilon(y))^{-1}) - y| < \omega(y)\,, \tag{3}$$

$$\varepsilon(y) < \omega(y)\,, \tag{4}$$

for all $y \in \widetilde{V}$.

Denote by τ_n the projection of $\mathbb{R}^\infty$ onto $\prod_{k=n+1}^\infty R_k$. Since $\widetilde{K}$ is topologically complete, so is $\widetilde{U} \subset \widetilde{K}$. Consequently, one can find a map $\chi = (\chi_n) : \widetilde{U} \to \mathbb{R}^\infty$ such that

(5) $\tau_n \circ \chi$ is a closed embedding for $n \ge 1$,

(6) for every $c \in \widetilde{U}$ there are infinitely many indices k such that $\chi_k(c) \ne 0$ (i.e., $\chi(c) \in T$).

Define $\Phi = (\Phi_k) : \widetilde{U} \times [1,\infty) \to \mathbb{R}^\infty$ by the formula

$$\Phi_k(c,t) = \begin{cases} \chi_k(c) & \text{if } t \le k-1, \\ (1-s)\chi_k(c) & \text{if } t = k-1+s,\ 0 \le s \le 1, \\ 0 & \text{if } t \ge k. \end{cases}$$

If $n \le t < n+1$, then $\Phi_k(c,t) = 0$ for $k \le n$; consequently, by (1), $|\Phi_k(c,t)|_2 \le 1/(n+1)$. It follows that

$$(7) \qquad |\Phi(c,t)|_2 < 1/t \quad \text{for } (c,t) \in \widetilde{U} \times [1,\infty)\,.$$

Write $\tilde{\varepsilon}(c) = \varepsilon(\tilde{f}(c))$ for $c \in \widetilde{U}$. Let

$$\alpha(c) = \sup\{t \in [0,1] \mid |t\pi(c)|_1 < \tilde{\varepsilon}(c)\}\,.$$

Then $\alpha(c) > 0$ and the continuity of $\tilde{\varepsilon}$ implies the lower semicontinuity of α. We can find [17, p. 428] a map $\lambda : \widetilde{U} \to [0,1]$ satisfying $0 < \lambda(c) < \alpha(c)$ for all $c \in \widetilde{U}$. Using the fact that $|\cdot|_1$ is monotone on each ray emanating from 0, we get

$$(8) \qquad |\lambda(c)\pi(c)|_1 < \tilde{\varepsilon}(c) \quad \text{for all } c \in \widetilde{U}\,.$$

Define $\widetilde{g} : \widetilde{U} \to \widehat{E} \times \mathbb{R}^\infty$ by

$$\widetilde{g}(c) = \Psi(\widetilde{f}(c), \varepsilon^{-1}) + \lambda(c)\pi(c) + \Phi(c, \varepsilon^{-1})\,,$$

where $\varepsilon = \widetilde{\varepsilon}(c)$. By (i), (iii) and (6), $\Psi(\widetilde{f}(c), \varepsilon^{-1}) + \Phi(c, \varepsilon^{-1}) \in E_0 \times T$. As a consequence, $\widetilde{g}(c)$ belongs to $\widehat{E} \times T$; moreover, $\widetilde{g}(c) \in E \times \mathbb{R}^\infty$ (resp., $\widetilde{g}(c) \in E_0 \times \mathbb{R}^\infty$) if and only if $\pi(c) \in E$ (resp., $\pi(c) \in E_0$). This yields $\widetilde{g}^{-1}(E \times \mathbb{R}^\infty) = \widetilde{g}^{-1}(E \times T) = \widetilde{U} \cap (E \times \mathbb{R}^\infty) = U$ and $\widetilde{g}^{-1}(E_0 \times T) = U \cap (E_0 \times \mathbb{R}^\infty)$. We claim that $g = \widetilde{g}|U$ is as required in $(*)$.

It follows from (7) and (8) that

$$(9) \qquad |g(c) - \Psi(f(c), (\varepsilon(f(c)))^{-1})| < 2\varepsilon(f(c)) \quad \text{for all } c \in U\,.$$

Consequently, by (3) and (4), we have

$$(10) \qquad |g(c) - f(c)| < 2\varepsilon(f(c)) + \omega(f(c)) < 3\omega(f(c))\,.$$

Using (2), we find $j \in \mathcal{J}$ such that $g(c)$ and $f(c)$ belong to $\widehat{V}_j \cap (E \times \mathbb{R}^\infty) = V_j$. This shows that g is $\mathcal{V}$-close to $f|U$ and, in particular, the range of g is V. Assume $g(c) = g(c')$ for some $c, c' \in U$. Write $\varepsilon = \varepsilon(f(c))$ and $\varepsilon' = \varepsilon(f(c'))$. Since for each k,

$$(11) \qquad \begin{aligned} \Psi_k(f(c), \varepsilon^{-1}) + \Phi_k(c, \varepsilon^{-1}) &= g_k(c) = g_k(c') \\ &= \Psi_k(f(c'), \varepsilon'^{-1}) + \Phi_k(c', \varepsilon'^{-1})\,, \end{aligned}$$

and for large k, $\Psi_k(f(c), \varepsilon^{-1}) = \Psi_k(f(c'), \varepsilon'^{-1}) = 0$, $\Phi_k(c, \varepsilon^{-1}) = \chi_k(c)$ and $\Phi_k(c', \varepsilon'^{-1}) = \chi_k(c')$, it follows that $\chi_k(c) = \chi_k(c')$ for large k. By (5), we infer that $c = c'$. Hence, g is injective. To prove that $g : U \to V$ is a closed embedding it suffices to show that whenever $\lim g(c_i) = y \in V$ for some $\{c_i\} \subset U$, then $\{c_i\}$ has a subsequence that converges in U. Set $\varepsilon_i = \varepsilon(f(c_i))$. We may assume that $\lim \varepsilon_i = \varepsilon_0 \in [0,1]$. We claim $\varepsilon_0 > 0$. In fact, if $\varepsilon_0 = 0$ then, by (9), $\lim \Psi(f(c_i), \varepsilon_i^{-1}) = y$. Then 3.2(iv) implies that $\lim f(c_i) = y$. By the continuity of ε, $\lim \varepsilon_i = \varepsilon(y) > 0$, a contradiction. Let N be so large that $\varepsilon_0^{-1} < N - 1$. We can assume that $\varepsilon_i^{-1} < N - 1$

for all i. Since $\Psi_k(f(c_i),\varepsilon_i^{-1}) = 0$ and $\Phi_k(c_i,\varepsilon_i^{-1}) = \chi_k(c_i)$ for $k \ge N$, we have

$$g_k(c_i) = \Psi_k(f(c_i),\varepsilon_i^{-1}) + \Phi_k(c_i,\varepsilon_i^{-1}) = \chi_k(c_i)$$

for $k \ge N$. As $\{g(c_i)\}$ is convergent, so is $\{\chi_k(c_i)\}$ for $k \ge N$. By (5), there exists $c \in \widetilde{U}$ such that $\lim c_i = c$. Since $\widetilde{g}(c) = y \in E \times \mathbb{R}^\infty$, we get $c \in C$. Verification of $(*)$ is now complete. ∎

Letting $E = E_0$ in 3.1 we get

3.3. Corollary. *For every separable metric linear space E that is an absolute retract, the space $E \times \mathbb{R}^\infty$ is strongly $\mathcal{F}_0(E \times \mathbb{R}^\infty)$-universal.* ∎

It would be interesting to extend the criterion of 3.1 to some spaces E that are not linear, e.g., to metric groups that are absolute retracts. Our proof actually works for some convex sets.

3.4. Remark. Let C be a convex subset of a separable metric linear space E and let E_0 be a linear subspace of E such that $E_0 \cap C = C_0$ is dense in C. Assume that C is a G_δ-subset of E and C is an absolute retract. Then the pair $(C \times \mathbb{R}^\infty, C_0 \times \mathbb{R}^\infty)$ is strongly $\mathcal{F}_0(C \times \mathbb{R}^\infty, C_0 \times \mathbb{R}^\infty)$-universal. In particular, for every convex subset C of E, $C \times \mathbb{R}^\infty$ is strongly $\mathcal{F}_0(C \times \mathbb{R}^\infty)$-universal provided C is absolute retract and C is a G_δ-subset of E. For a proof, follow that of 3.1. Replace $\widetilde{E}$ by $\widetilde{C}$ with the same properties. Since C is a G_δ-subset of E, we can additionally assume $\widetilde{C} \cap E = C$. Find λ satisfying $|\lambda(c)\Psi_0(\widetilde{f}(c),\varepsilon^{-1})|_1 < \widetilde{\varepsilon}(c)$. Define $\widetilde{g}(c) = (1-\lambda(c))\Psi_0(\widetilde{f}(c),\varepsilon^{-1}) + \lambda(c)\pi(c) + (\Psi_n)(\widetilde{f}(c),\varepsilon^{-1}) + \Phi(c,\varepsilon^{-1})$. Then the restriction $\widetilde{g}|U$ will work. ∎

Our next result concerns the strong universality of certain triples $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, E)$.

3.5. Proposition. *Let $\mathbb{R}^{\mathbb{N}} = \prod_{k=1}^\infty \mathbb{R}^{N_k}$, where $\{N_k\}_{k=1}^\infty$ is a partition of $\mathbb{N}$ into nonempty sets. Let E_k be a dense linear subspace of $\mathbb{R}^{N_k}$ and write $E = \prod_{k=1}^\infty E_k$. Assume there exist maps $\mu_k : \overline{\mathbb{R}}^{N_k} \to \mathbb{R}^{N_k}$ with $\mu_k^{-1}(E_k) \cap \mathbb{R}^{N_k} = E_k$ for $k \ge 1$. Then $(\overline{\mathbb{R}}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}}, E)$ is strongly $\mathcal{F}_0(\overline{\mathbb{R}}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}}, E)$-universal.*

Proof. The proof is parallel to that of 3.1 and employs some of its notations. Let $T = \{(x_n) \in \overline{\mathbb{R}}^{\mathbb{N}} \mid x_n \neq 0 \text{ for infinitely many } n \in \mathbb{N}\}$. (Writing $x = (x_n) \in \overline{\mathbb{R}}^{\mathbb{N}}$, x_n denotes the nth coordinate of x in $\overline{\mathbb{R}}^{\mathbb{N}}$, $n \in \mathbb{N}$; writing $x = (x_k) \in \prod_{k=1}^\infty \overline{\mathbb{R}}^{N_k}$, x_k denotes the $\overline{\mathbb{R}}^{N_k}$-coordinate of x in the product $\prod_{k=1}^\infty \overline{\mathbb{R}}^{N_k}$.) Define $\mu : \overline{\mathbb{R}}^{\mathbb{N}} \to \prod_{k=1}^\infty \mathbb{R}^{N_k}$ by letting $\mu = (\mu_k)$. Denote by $\mu^n(x)$, $x \in \overline{\mathbb{R}}^{\mathbb{N}}$, the nth coordinate of $\mu(x)$ in $\mathbb{R}^{\mathbb{N}}$, $n \in \mathbb{N}$. Since $\overline{\mathbb{R}}^{\mathbb{N}}$ is compact, there exists $p_n \in \mathbb{R}$, $p_n \ge 1$, with $\max\{|\mu^n(x)| \mid x \in \overline{\mathbb{R}}^{\mathbb{N}}\} \le p_n$. It follows that

$$\sum_{n=1}^\infty (2^{n+1}p_n)^{-1}|\mu^n(x)| \le \frac{1}{2} \quad \text{for all } x \in \overline{\mathbb{R}}^{\mathbb{N}}. \tag{12}$$

Endow $\overline{\mathbb{R}}^{\mathbb{N}}$ with the metric

$$d(x,y)=\sum_{n=1}^{\infty}(2^{n+1}p_n)^{-1}\frac{1}{\pi}|\arctan x_n-\arctan y_n|\,,$$

for $x=(x_n)$ and $y=(y_n)$ in $\overline{\mathbb{R}}^{\mathbb{N}}$. We have

$$(1)' \qquad d(x,y)\le\frac{1}{n+1}\qquad \text{if } x_p=y_p \text{ for all } p\le n\,.$$

We will assume that $\{1,\ldots,n\}\subset N_1\cup\ldots\cup N_n$ for every n.

Using [9, Proposition 2.1] and the compactness of $\overline{\mathbb{R}}^{\mathbb{N}}$ it suffices to verify the following modification of the condition $(*)$ from the proof of 3.1:

$(*)'$ *Given a closed subset K of $\overline{\mathbb{R}}^{\mathbb{N}}$, an open subset U of K, an open subset V of $\overline{\mathbb{R}}^{\mathbb{N}}$, an open cover $\mathcal{V}$ of V and a map $f:K\to\overline{\mathbb{R}}^{\mathbb{N}}$ satisfying $f(U)\subset V$ and $f(K\backslash U)\subset\overline{\mathbb{R}}^{\mathbb{N}}\backslash V$, there exists an injective map $g:U\to V$ that is $\mathcal{V}$-close to $f|U$ and satisfies $g(U)\subset T$, $g^{-1}(T\cap\mathbb{R}^{\mathbb{N}})=U\cap\mathbb{R}^{\mathbb{N}}$ and $g^{-1}(T\cap E)=U\cap E$.*

We will make use of an analogue of 3.2.

3.6. Lemma. *There exists a map $\Psi=(\Psi_k):\overline{\mathbb{R}}^{\mathbb{N}}\times[1,\infty]\to\prod_{k=1}^{\infty}\overline{\mathbb{R}}^{N_k}=\overline{\mathbb{R}}^{\mathbb{N}}$ satisfying*

(i) $\Psi(\overline{\mathbb{R}}^{\mathbb{N}}\times[1,\infty))\subset E$,
(ii) $\Psi(y,\infty)=y$ *for all* $y\in\overline{\mathbb{R}}^{\mathbb{N}}$,
(iii) $\Psi_k(y,t)=0$ *for all* $t\le k-1$ *and* $y\in\overline{\mathbb{R}}^{\mathbb{N}}$.

Proof. By [25, Theorem 2.4], for each k there exists $\psi_k:\overline{\mathbb{R}}^{N_k}\times[1,\infty]\to\overline{\mathbb{R}}^{N_k}$ satisfying

(a) $\psi_k(\overline{\mathbb{R}}^{N_k}\times[1,\infty))\subset E_k$,
(b) $\psi_k(x,\infty)=x$ for all $x\in\overline{\mathbb{R}}^{N_k}$.

Define Ψ by letting

$$\Psi_k(y,t)=\begin{cases}\psi_k(y_k,t) & \text{if } k\le t,\\ s\psi_k(y_k,t) & \text{if } t=k-1+s,\ 0\le s\le 1,\\ 0 & \text{if } t\le k-1,\end{cases}$$

for $y=(y_k)\in\prod_{k=1}^{\infty}\overline{\mathbb{R}}^{N_k}$. ∎

Since E_k is nontrivial, there exist $0\ne v_k\in E_k$ and $n_k\in N_k$ so that the n_kth coordinate of v_k is 1 (use linearity of E_k). Let $\mathbb{R}_0^{N_k}=\{(x_i)\in\mathbb{R}^{N_k}\mid x_{n_k}=0\}$. We will identify the pairs $(\mathbb{R}_0^{N_k}\times\mathbb{R}v_k,(\mathbb{R}^{N_k}\cap E_k)\times\mathbb{R}v_k)$ and $(\mathbb{R}^{N_k},E_k)$ via the isomorphism $T(y,tv_k)=y+tv_k$. Observe that T extends to an injective map $\widetilde{T}:\mathbb{R}_0^{N_k}\times\overline{\mathbb{R}}v_k\to\overline{\mathbb{R}}^{N_k}$. Identifying $\mathbb{R}_0^{N_k}$ with $\mathbb{R}_0^{N_k}\times\{0\}$ and $\overline{\mathbb{R}}v_k$ with $\{0\}\times\overline{\mathbb{R}}v_k$ in $\mathbb{R}_0^{N_k}\times\overline{\mathbb{R}}v_k$ we will write $y+tv_k$ instead of $\widetilde{T}(y, tv_k)$ for $y \in \mathbb{R}_0^{N_k}$ and $t \in \overline{\mathbb{R}}$. Note that for every $y \in \mathbb{R}_0^{N_k}$ the map

$$t \to y + tv_k\,, \ t \in \overline{\mathbb{R}}\,, \text{ is an embedding.} \tag{13}$$

Let $\mu_k^0 : \overline{\mathbb{R}}^{N_k} \to \mathbb{R}^{N_k} = \mathbb{R}_0^{N_k} \times \mathbb{R}v_k$ be the $\mathbb{R}_0^{N_k}$-component of μ_k, i.e., $(\mu_k^0(y))_i = (\mu_k(y))_i - (\mu_k(y))_{n_k}$, for every $i \in N_k$ and $y \in \overline{\mathbb{R}}^{N_k}$. For $y = (y_k) \in \prod_{k=1}^\infty \mathbb{R}^{N_k}$, define $\pi : \overline{\mathbb{R}}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by letting

$$\pi(y) = (\mu_k^0(y_k))\,.$$

Let $(h_s) : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a homotopy such that $h_0 = \mathrm{id}$, $h_1(t) = 0$ for $t \in \overline{\mathbb{R}}$ and

$$h_s(t) \in \mathbb{R} \qquad \text{for all } t \in \overline{\mathbb{R}} \text{ and } s > 0\,. \tag{14}$$

Let $\{P_n\}_{n=1}^\infty$ be a partition of the set of odd positive integers into infinite sets. For each $k \in \mathbb{N}$ and $x = (x_n) \in \overline{\mathbb{R}}^{\mathbb{N}}$, we let $\chi_k(x) = 1$ if k is even and $\chi_k(x) = x_n$ if $k \in P_n$, $n \geq 1$. Next, for $x \in \overline{\mathbb{R}}^{\mathbb{N}}$, $1 \leq t < \infty$ and $k \in \mathbb{N}$, let

$$\Phi_k(x,t) = \begin{cases} \chi_k(x) & \text{if } t \leq k-1, \\ h_s(\chi_k(x)) & \text{if } t = k-1+s,\ 0 \leq s \leq 1, \\ 0 & \text{if } t \geq k, \end{cases}$$

and put

$$\Phi(x,t) = (\Phi_k(x,t)v_k) \in \prod_{k=1}^\infty \overline{\mathbb{R}}v_k \subset \prod_{k=1}^\infty \overline{\mathbb{R}}^{N_k}\,.$$

Choose $\omega : V \to (0,1]$ and $\varepsilon : V \to (0,1]$ as in the proof of 3.1. We let

$$g(c) = \Psi(f(c), \varepsilon^{-1}) + \varepsilon \cdot \pi(c) + \Phi(c, \varepsilon^{-1})\,, \qquad \varepsilon = \varepsilon(f(c))\,,$$

for $c \in U$; addition is the coordinatewise addition in $\prod_{k=1}^\infty \overline{\mathbb{R}}^{N_k}$ defined above.

Fix $c \in U$ with $n \leq \varepsilon^{-1} < n+1$. Since $\Phi_k(c,t) = 0$, for $k \leq n$, by $(1)'$ and the choice of ε, we have

$$\begin{aligned}
&d(g(c), f(c)) \\
&\leq d(\Psi(f(c),\varepsilon^{-1}) + \varepsilon\cdot\pi(c) + \Phi(c,\varepsilon^{-1}), \Psi(f(c),\varepsilon^{-1}) + \Phi(c,\varepsilon^{-1})) \\
&\quad + d(\Psi(f(c),\varepsilon^{-1}) + \Phi(c,\varepsilon^{-1}), f(c)) \\
&\leq d(\Psi(f(c),\varepsilon^{-1}) + \varepsilon\cdot\pi(c) + \Phi(c,\varepsilon^{-1}), \Psi(f(c),\varepsilon^{-1}) + \Phi(c,\varepsilon^{-1})) + 2\varepsilon(f(c)) \\
&\leq d(\Psi(f(c),\varepsilon^{-1}) + \varepsilon\cdot\pi(c), \Psi(f(c),\varepsilon^{-1})) + \frac{1}{2^n p_n} + 2\varepsilon(f(c)) \\
&\leq d(\Psi(f(c),\varepsilon^{-1}) + \varepsilon\cdot\pi(c), \Psi(f(c),\varepsilon^{-1})) + 3\varepsilon(f(c))\,.
\end{aligned}$$

Using $|\arctan(x+y) - \arctan x| \leq |y|$ and (12) we see that

$$\begin{aligned}
&d(\Psi(f(c),\varepsilon^{-1}) + \varepsilon\cdot\pi(c), \Psi(f(c),\varepsilon^{-1})) \\
&\qquad\qquad \leq \sum_{n=1}^\infty (2^{n+1}p_n)^{-1}\varepsilon(f(c)) \cdot |\mu^n(c)| \leq \varepsilon(f(c))\,.
\end{aligned}$$

We conclude that $d(g(c), f(c)) < 4\varepsilon(f(c))$. As in 3.1, it follows that g is $\mathcal{V}$-close to $f|U$ and the range of g is V. By 3.6(iii), the n_kth coordinate of $g(c)$ equals $\chi_k(c)$ for $k \geq \varepsilon^{-1}+1$. Using the properties of χ and (13), we infer that g is injective, $g(U\backslash\overline{\mathbb{R}}^{\mathbb{N}}) \subset \overline{\mathbb{R}}^{\mathbb{N}}\backslash\mathbb{R}^{\mathbb{N}}$ and $g(U) \subset T$. By 3.6(i) and (14), if $c \in U \cap \mathbb{R}^{\mathbb{N}}$, then $g(c) \in \mathbb{R}^{\mathbb{N}}$. Since for $c \in U \cap \mathbb{R}^{\mathbb{N}}$, $\Psi(f(c), \varepsilon^{-1}) + \Phi(c, \varepsilon^{-1}) \in E$, we infer that $g(c) \in E$ if and only if $\pi(c) \in E$. The last happens exactly when $c \in E$. ∎

3.7. Remark. If, in 3.1, E' is any linear space such that $E_0 \subseteq E' \subseteq E$, then $g^{-1}(V \cap (E' \times \mathbb{R}^\infty)) = U \cap (E' \times \mathbb{R}^\infty)$. This permits one to generalize Theorem 3.1 to systems of the form $(E \times \mathbb{R}^\infty, E_k \times \mathbb{R}^\infty, \ldots, E_1 \times \mathbb{R}^\infty, E_0 \times \mathbb{R}^\infty)$, where $E_0 \subseteq E_1 \subseteq \ldots \subseteq E_k$ are linear spaces. Also Proposition 3.5 is true for $(\overline{\mathbb{R}}^{\mathbb{N}}, \mathbb{R}^{\mathbb{N}}, E^m, \ldots, E^1, E)$, where each $E^i = \prod_{k=1}^\infty E^i_k$ and $E_k \subseteq E^1_k \subseteq \ldots \subseteq E^m_k \subseteq \mathbb{R}^{N_k}$ are linear spaces such that $\mu_k^{-1}(E^i_k) \cap \mathbb{R}^{N_k} = E^i_k$. ∎

In particular, we have

3.8. Corollary. *Let E_n be a separable metric linear space which is an absolute retract and let $E^n_0 \subseteq E^n_1 \subseteq \ldots \subseteq E^n_k \subseteq E_n$ be linear spaces such that $\{0\} \neq E^n_0$ is dense in E_n, $n = 1, 2, \ldots$ Then $\prod_{n=1}^\infty (E_n, E^n_k, \ldots, E^n_1, E^n_0)$ is strongly $\mathcal{F}_0(\prod_{n=1}^\infty (E_n, E^n_k, \ldots, E^n_1, E^n_0))$-universal.*

Proof. Pick a nonzero vector $v_n \in E^n_0$. By a result of Michael (see [1, p. 87]), $(E_n, E^n_k, \ldots, E^n_1, E^n_0)$ is homeomorphic to $(F_n, F^n_k, \ldots, F^n_1, F^n_0) \times \mathbb{R}v_n$, where $F^n_i = E^n_i/\mathbb{R}v_k$ are linear subspaces of the quotient space $F_n = E_n/\mathbb{R}v_n$. Hence, the product $\prod_{n=1}^\infty (E_n, E^n_k, \ldots, E^n_1, E^n_0)$ is homeomorphic to $\prod_{n=1}^\infty (F_n, F^n_k, \ldots, F^n_1, F^n_0) \times \mathbb{R}^\infty$ and 3.7 is applicable. ∎

4. Application to $C_{\mathrm{p}}(X)$.

Here is our application of the results of Section 3 to $C_{\mathrm{p}}(X)$.

4.1. Proposition. *Let X be a countable regular noncompact space. Let S be one of the following k-tuples, $1 \leq k \leq 4$: $C_{\mathrm{p}}(X)$, $C^{\mathrm{loc}}_{\mathrm{p}}(X)$, $(C_{\mathrm{p}}(X), C^{\mathrm{loc}}_{\mathrm{p}}(X))$, $(\mathbb{R}^X, C_{\mathrm{p}}(X))$, $(\mathbb{R}^X, C^{\mathrm{loc}}_{\mathrm{p}}(X))$, $(\mathbb{R}^X, C_{\mathrm{p}}(X), C^{\mathrm{loc}}_{\mathrm{p}}(X))$, $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X))$, $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C^{\mathrm{loc}}_{\mathrm{p}}(X))$ and $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X), C^{\mathrm{loc}}_{\mathrm{p}}(X))$. Then S is strongly $\mathcal{F}_0(S)$-universal.*

4.2. Lemma. *We have $X = \bigcup_{k=1}^\infty V_k$, where each V_k is nonempty and clopen, and $V_i \cap V_j = \emptyset$ for $i \neq j$. In particular,*

$$(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X), C^{\mathrm{loc}}_{\mathrm{p}}(X)) = \prod_{k=1}^\infty (\overline{\mathbb{R}}^{V_k}, \mathbb{R}^{V_k}, C_{\mathrm{p}}(V_k), C^{\mathrm{loc}}_{\mathrm{p}}(V_k))\,.$$

Proof. Since X is countable, it is Lindelöf and hence normal. Being Lindelöf and noncompact, X is not countably compact; hence it contains a closed discrete infinite set A. Enumerate A as $\{a_k\}_{k=1}^\infty$. Since A is discrete and X is normal, there exists a map $\lambda : X \to \mathbb{R}$ such that $\lambda(a_k) = k$, $k \geq 1$. Using the fact that X is countable, we pick $\alpha_k \in (k, k+1)\setminus\lambda(X)$. Put $\alpha_0 = -\infty$ and set $V_k = \lambda_k^{-1}((\alpha_{k-1}, \alpha_k))$. ∎

Proof of 4.1. When S does not contain $\overline{\mathbb{R}}^X$ we apply 4.2 and 3.8. If S contains $\overline{\mathbb{R}}^X$, we apply 3.5 (see also 3.7) with $\mathbb{R}^{N_k} = \mathbb{R}^{V_k}$, $E_k = C_{\mathrm{p}}^{\mathrm{loc}}(V_k)$ and $E'_k = C_{\mathrm{p}}(V_k)$, where V_k are those of Lemma 4.2. The map $\mu_k : \overline{\mathbb{R}}^{V_k} \to \mathbb{R}^{V_k}$ is given by

$$\mu_k(f)(x) = \arctan(f(x)), \qquad x \in V_k.$$

It is easy to see that $\mu_k^{-1}(C_{\mathrm{p}}(V_k)) \cap \mathbb{R}^{V_k} = C_{\mathrm{p}}(V_k)$ and $\mu^{-1}(C_{\mathrm{p}}^{\mathrm{loc}}(V_k)) \cap \mathbb{R}^{V_k} = C_{\mathrm{p}}^{\mathrm{loc}}(V_k)$. ∎

If $C_{\mathrm{p}}(X)$ in Proposition 4.1 is a Z_σ-space (e.g., $C_{\mathrm{p}}(X)$ is analytic [14, Corollary 3.6]), then it is an $\mathcal{F}_0(C_{\mathrm{p}}(X))$-absorbing set in $\mathbb{R}^\infty$. Thus, in this case we can say that the topology of $C_{\mathrm{p}}(X)$ is completely determined by the class $\mathcal{F}_0(C_{\mathrm{p}}(X))$. Below we show that, in such a case, $\mathcal{F}_0(\mathbb{R}^X, C_{\mathrm{p}}(X))$ not only determines the topology of the pair $(\mathbb{R}^X, C_{\mathrm{p}}(X))$ but also that of the triple $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X))$.

4.3. Theorem. *Let X and Y be countable regular noncompact spaces such that $C_{\mathrm{p}}(X)$ and $C_{\mathrm{p}}(Y)$ are Z_σ-spaces. Then*

(a) $C_{\mathrm{p}}(X)$ *is homeomorphic to* $C_{\mathrm{p}}(Y)$ *iff* $\mathcal{F}_0(C_{\mathrm{p}}(X)) = \mathcal{F}_0(C_{\mathrm{p}}(Y))$,
(b) *the following conditions are equivalent*:
  (i) $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X)) \cong (\overline{\mathbb{R}}^Y, \mathbb{R}^Y, C_{\mathrm{p}}(Y))$,
  (ii) $(\mathbb{R}^X, C_{\mathrm{p}}(X)) \cong (\mathbb{R}^Y, C_{\mathrm{p}}(Y))$,
  (iii) $\mathcal{F}_0(\mathbb{R}^X, C_{\mathrm{p}}(X)) = \mathcal{F}_0(\mathbb{R}^Y, C_{\mathrm{p}}(Y))$.

4.4. Remark. Theorem 4.3 remains true for arbitrary countable regular spaces X and Y. In fact, if X is compact then it is metrizable (combine Theorems 3.1.21 and 4.2.8 in [17]). By results of [5], [13], $C_{\mathrm{p}}(X)$ is homeomorphic to Ω_2. Now, (a) follows from [14]. As observed in Remark 6.8, if X is compact and $\mathcal{F}_0(X, C_{\mathrm{p}}(X)) = \mathcal{F}_0(Y, C_{\mathrm{p}}(Y))$, then Y is also compact and $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X)) \cong (\overline{\mathbb{R}}^Y, \mathbb{R}^Y, C_{\mathrm{p}}(Y))$. ∎

The proof of Theorem 4.3 is a direct consequence of 4.1, 2.3 and the fact below.

4.5. Proposition. *Let X and Y be countable regular noncompact spaces. If $\mathcal{F}_0(\mathbb{R}^X, C_{\mathrm{p}}(X)) = \mathcal{F}_0(\mathbb{R}^Y, C_{\mathrm{p}}(Y))$ then $\mathcal{F}_0(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X)) = \mathcal{F}_0(\overline{\mathbb{R}}^Y, \mathbb{R}^Y, C_{\mathrm{p}}(Y))$.* ∎

Define $\mu : \overline{\mathbb{R}}^Y \to \mathbb{R}^Y$ by $\mu(f)(y) = \arctan(f(y))$. If h is an embedding of $(\mathbb{R}^Y, C_{\mathrm{p}}(Y))$ into $(\mathbb{R}^X, C_{\mathrm{p}}(X))$ then $\varphi = h \circ \mu$ satisfies $\varphi^{-1}(C_{\mathrm{p}}(X)) \cap \mathbb{R}^Y = C_{\mathrm{p}}(Y)$. Hence, our proposition is a consequence of the following lemma.

4.6. LEMMA. *Let X be a countable regular noncompact space and $A \subseteq \mathbb{R}^\infty$. If there exists $\varphi : \overline{\mathbb{R}}^\infty \to \mathbb{R}^X$ with $\varphi^{-1}(C_{\mathrm{p}}(X)) \cap \mathbb{R}^\infty = A$, then $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, A) \in \mathcal{F}_0(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X))$.*

Proof. Let $\{V_k\}_{k=1}^\infty$ be a decomposition of X into pairwise disjoint nonempty clopen sets (Lemma 4.2). Fix x_k in each V_k. For $q = (q_k) \in \overline{\mathbb{R}}^\infty$, define $g(q) \in \overline{\mathbb{R}}^X$ by letting $g(q)(x) = \varphi(q)(x) - \varphi(q)(x_k) + q_k$ for $x \in V_k$, $k \geq 1$. It is clear that g is an injective map of $\overline{\mathbb{R}}^\infty$ into $\overline{\mathbb{R}}^X$ such that $g^{-1}(\mathbb{R}^X) = \mathbb{R}^\infty$ and $g^{-1}(C_{\mathrm{p}}(X)) = A$. ∎

4.7. Remark. Proposition 4.5, Lemma 4.6 and Theorem 4.3 remain true for $C_{\mathrm{p}}^{\mathrm{loc}}(X)$. Analogous results apply to $(\mathbb{R}^X, C_{\mathrm{p}}(X), C_{\mathrm{p}}^{\mathrm{loc}}(X))$. ∎

Fix $x_0 \in X$. Let $\overline{\mathbb{R}}_0^X = \{f \in \overline{\mathbb{R}}^X \mid f(x_0) = 0\}$. If $S = (E^1, \ldots, E^k)$, $1 \leq k \leq 4$, is one of the k-tuples from 4.1, we denote by S_0 the k-tuple $(E^1, \ldots, E^k) \cap \overline{\mathbb{R}}_0^X = (E^1 \cap \overline{\mathbb{R}}_0^X, \ldots, E^k \cap \overline{\mathbb{R}}_0^X)$.

4.8. PROPOSITION. *Let X be a countable regular noncompact space. If S is one of the k-tuples from 4.1, $1 \leq k \leq 4$, then S_0 is strongly $\mathcal{F}_0(S)$-universal.*

Proof. Pick a decomposition $\{V_k\}_{k=1}^\infty$ of X given by 4.2 and assume $x_0 \in V_1$. We have

$$(*) \quad (\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X), C_{\mathrm{p}}^{\mathrm{loc}}(X)) \cap \overline{\mathbb{R}}_0^X$$
$$= (\overline{\mathbb{R}}^{V_1}, \mathbb{R}^{V_1}, C_{\mathrm{p}}(V_1), C_{\mathrm{p}}^{\mathrm{loc}}(V_1)) \cap \overline{\mathbb{R}}_0^{V_1} \times \prod_{k=2}^{\infty} (\overline{\mathbb{R}}^{V_k}, \mathbb{R}^{V_k}, C_{\mathrm{p}}(V_k), C_{\mathrm{p}}^{\mathrm{loc}}(V_k)).$$

Identifying $\mathbb{R}_0^{V_1}$ with $\mathbb{R}^{V_1 \setminus \{x_0\}}$, we see that the argument from the proof of 4.1 works. ∎

Let $X = \mathbb{N}_F$ and $x_0 = \infty$, where F is a filter on $\mathbb{N}$ different from the Fréchet filter. Proposition 4.8 implies that each of the following tuples T: c_F, s_F, (c_F, s_F), $(\mathbb{R}^\infty, c_F)$, $(\mathbb{R}^\infty, s_F)$, $(\mathbb{R}^\infty, c_F, s_F)$, $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_F)$, $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_F)$ and $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_F, s_F)$ is strongly $\mathcal{F}_0(T)$-universal. Since for the Fréchet filter F_0, c_{F_0} is homeomorphic to Ω_2 ([5], [13]), in particular, we conclude that c_F is strongly $\mathcal{F}_0(c_F)$-universal for arbitrary F. This provides an affirmative answer to the first part of question 6.2 in [15].

4.9. Remark. Let F be a filter on $\mathbb{N}$ different from the Fréchet filter. Then Lemma 4.6 applies to c_F, s_F and (c_F, s_F). ∎

Our second application of Section 3 concerns spaces $C_{\mathrm{p}}(X)$, where $X = \mathbb{N}_F$. We start with the following fact that allows us to replace $C_{\mathrm{p}}(\mathbb{N}_F)$ and $C_{\mathrm{p}}^{\mathrm{loc}}(\mathbb{N}_F)$ by c_F and s_F, respectively.

4.10. LEMMA. *For every filter F on $\mathbb{N}$ different from the Fréchet filter*, $\mathcal{F}_0(\overline{\mathbb{R}}^{\mathbb{N}_F}, \mathbb{R}^{\mathbb{N}_F}, C_{\mathrm{p}}(\mathbb{N}_F), C_{\mathrm{p}}^{\mathrm{loc}}(\mathbb{N}_F)) = \mathcal{F}_0(\overline{\mathbb{R}}^{\infty}, \mathbb{R}^{\infty}, c_F, s_F)$. *If*, *additionally*, *c_F is a Z_σ-space*, *then* $(\overline{\mathbb{R}}^{\mathbb{N}_F}, \mathbb{R}^{\mathbb{N}_F}, C_{\mathrm{p}}(\mathbb{N}_F), C_{\mathrm{p}}^{\mathrm{loc}}(\mathbb{N}_F)) \cong (\overline{\mathbb{R}}^{\infty}, \mathbb{R}^{\infty}, c_F, s_F)$.

Proof. Evidently, we have $(\overline{\mathbb{R}}^{\infty}, \mathbb{R}^{\infty}, c_F, s_F) \in \mathcal{F}_0(\overline{\mathbb{R}}^{\mathbb{N}_F}, \mathbb{R}^{\mathbb{N}_F}, C_{\mathrm{p}}(\mathbb{N}_F), C_{\mathrm{p}}^{\mathrm{loc}}(\mathbb{N}_F))$. To show that $(\overline{\mathbb{R}}^{\mathbb{N}_F}, \mathbb{R}^{\mathbb{N}_F}, C_{\mathrm{p}}(\mathbb{N}_F), C_{\mathrm{p}}^{\mathrm{loc}}(\mathbb{N}_F)) \in \mathcal{F}_0(\overline{\mathbb{R}}^{\infty}, \mathbb{R}^{\infty}, c_F, s_F)$ define $\varphi : \overline{\mathbb{R}}^{\mathbb{N}_F} \to \mathbb{R}^{\infty}$ by

$$\varphi(f)(n) = \arctan(f(n)) - \arctan(f(\infty))$$

for $f \in \overline{\mathbb{R}}^{\mathbb{N}_F}$ and apply 4.9 (note that $\varphi^{-1}(c_F) \cap \mathbb{R}^{\mathbb{N}_F} = C_{\mathrm{p}}(\mathbb{N}_F)$ and $\varphi^{-1}(s_F) \cap \mathbb{R}^{\mathbb{N}_F} = C_{\mathrm{p}}^{\mathrm{loc}}(\mathbb{N}_F)$). The first part of our lemma follows. If c_F is a Z_σ-space then $C_{\mathrm{p}}(\mathbb{N}_F) \cong c_F \times \mathbb{R}$ is also a Z_σ-space and the second part of 4.10 follows from 4.1, 4.8 and 2.3. ∎

4.11. LEMMA. *For every filter F on $\mathbb{N}$ different from the Fréchet filter*, *we have* $\mathcal{F}_0(\overline{\mathbb{R}}^{\infty}, \mathbb{R}^{\infty}, c_F) = \mathcal{F}_0((\overline{\mathbb{R}}^{\infty})^{\infty}, (\mathbb{R}^{\infty})^{\infty}, c_F^{\infty})$.

Proof. It is obvious that $(\overline{\mathbb{R}}^{\infty}, \mathbb{R}^{\infty}, c_F) \in \mathcal{F}_0((\overline{\mathbb{R}}^{\infty})^{\infty}, (\mathbb{R}^{\infty})^{\infty}, c_F^{\infty})$. Now, it suffices to show that $((\overline{\mathbb{R}}^{\infty})^{\infty}, (\mathbb{R}^{\infty})^{\infty}, c_F^{\infty}) \in \mathcal{F}_0(\overline{\mathbb{R}}^{\infty}, \mathbb{R}^{\infty}, c_F)$. For $x = (x(j))_{j=1}^{\infty} \in (\mathbb{R}^{\infty})^{\infty}$, we let

$$\zeta_n(x) = \sum_{j=1}^{\infty} 2^{-j} \frac{|x_n(j)|}{1 + |x_n(j)|},$$

where $x(j) = (x_n(j)) \in \mathbb{R}^{\infty}$. The map $\zeta = (\zeta_n) : (\mathbb{R}^{\infty})^{\infty} \to \mathbb{R}^{\infty}$ satisfies $\zeta^{-1}(c_F) = c_F^{\infty}$. Define a map $\chi : (\overline{\mathbb{R}}^{\infty})^{\infty} \to (\mathbb{R}^{\infty})^{\infty}$ by letting $\chi((x_n(j))) = (\arctan(x_n(j)))$. Finally, let $\varphi = \zeta \circ \chi$ and observe that $\varphi^{-1}(c_F) \cap (\mathbb{R}^{\infty})^{\infty} = c_F^{\infty}$. Now, 4.10 is applicable. ∎

The following result which, in particular, provides a partial answer to the second part of question 6.12 in [15] (and generalizes [14, Theorem 8.8]) follows directly from 3.5, 4.8 and 2.3.

4.12. THEOREM. *Let F be a filter on $\mathbb{N}$ which is different from the Fréchet filter and such that c_F is a Z_σ-space. Then $(\overline{\mathbb{R}}^{\infty}, \mathbb{R}^{\infty}, c_F)$ is homeomorphic to $((\overline{\mathbb{R}}^{\infty})^{\infty}, (\mathbb{R}^{\infty})^{\infty}, c_F^{\infty})$.* ∎

4.13. Remark. For a filter F on $\mathbb{N}$ the following conditions are equivalent:

(i) c_F is a Z_σ-space,

(ii) c_F is a first category subset of $\mathbb{R}^{\infty}$,

(iii) F is a first category subset of $2^{\mathbb{N}}$,

(iv) F belongs to the σ-algebra generated by the open subsets and the first category subsets of $2^{\mathbb{N}}$.

This follows from [22, Theorem 5.1] and [14, Lemmas 2.2, 2.3 and Proposition 3.3]. Note that if F is analytic or coanalytic, then (iv) holds. ∎

4.14. Remark. Theorem 4.12 is false for the Fréchet filter F_0; though c_{F_0} is homeomorphic to $c_{F_0}^\infty$. In fact, $(\mathbb{R}^\infty, c_{F_0}) = (\mathbb{R}^\infty, c_0)$ is not homeomorphic to $(\mathbb{R}^\infty, \Omega_2)$ ([9], [12]) but one can show that $((\mathbb{R}^\infty)^\infty, c_0^\infty)$ is homeomorphic to $(\mathbb{R}^\infty, \Omega_2)$. ■

5. Determining the Borel class of $C_{\mathrm p}(X)$**.** It is known that for every filter on $\mathbb{N}$ the space c_F (and hence, $C_{\mathrm p}(\mathbb{N}_F)$), if Borel, must be of an exact multiplicative class. In this section we extend this result to all $C_{\mathrm p}(X)$ spaces.

5.1. Theorem. *Let X be a countable infinite regular space such that $C_{\mathrm p}(X)$ is Borel. Then there exists a countable ordinal $\alpha \ge 1$ such that $C_{\mathrm p}(X) \in \mathcal{M}_\alpha \backslash \mathcal{A}_\alpha$.*

Our theorem will easily follow from the lemma below.

5.2. Lemma. *Let $P \subset \mathbb{R}$ be the set of irrationals, and let $C_{\mathrm p}(X, P)$ be the subspace of $C_{\mathrm p}(X)$ consisting of functions that take values in P. For every countable infinite regular space X, $C_{\mathrm p}(X)$ is Borel if and only if $C_{\mathrm p}(X, P)$ is Borel; moreover, the exact Borel classes of $C_{\mathrm p}(X)$ and $C_{\mathrm p}(X, P)$ coincide.*

Proof. For discrete X, $C_{\mathrm p}(X) = \mathbb{R}^X$ and $C_{\mathrm p}(X, P) = P^X$ and both belong to $\mathcal{M}_1 \backslash \mathcal{A}_1$. Suppose X is not discrete. Evidently, $C_{\mathrm p}(X, P)$ is a G_δ-subset of $C_{\mathrm p}(X)$; consequently, $C_{\mathrm p}(X, P)$ is Borel provided $C_{\mathrm p}(X)$ is. Moreover, using the fact [11] that $C_{\mathrm p}(X) \notin \mathcal{A}_2$, we have $C_{\mathrm p}(X, P) \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$) provided $C_{\mathrm p}(X) \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$).

Conversely, suppose $C_{\mathrm p}(X, P)$ is Borel. Put

$$S = \{(f, t) \in \mathbb{R}^X \times \mathbb{R} \mid \forall_{x \in X} \ f(x) + t \in P\}\,.$$

Since X is countable, S is a G_δ-subset of a complete metrizable space $\mathbb{R}^X \times \mathbb{R}$. Let d be a complete metric on S. For $f \in \mathbb{R}^X$, we define $S_f = \{t \in \mathbb{R} \mid (f, t) \in S\} = \{t \in \mathbb{R} \mid \forall_{x \in X} f(x) + t \in P\}$. Since X is countable, $\bigcap_{x \in X}(P - f(x))$ is dense in $\mathbb{R}$. Hence, S_f is dense in $\mathbb{R}$. Define $\varphi : S \to P^X$ by letting

$$\varphi(f, t)(x) = f(x) + t\,.$$

Then φ is continuous and satisfies

$$\varphi^{-1}(C_{\mathrm p}(X, P)) = \{(f, t) \mid f \in C_{\mathrm p}(X)\} = T\,.$$

Claim. *The restriction $\pi|T$ of the projection of $C_{\mathrm p}(X) \times \mathbb{R}$ onto $C_{\mathrm p}(X)$ is open.*

The above follows immediately from the fact that $\pi^{-1}(\{f\}) = S_f$ is dense in $\mathbb{R}$ for every $f \in C_{\mathrm p}(X)$. Since S_f is closed in S, $(\pi^{-1}(\{f\}), d) = (S_f, d)$ is complete. It follows from [18, Theorem 5.9.16, p. 156] (see also [10]) that if $T \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$), then $\pi(T) = C_{\mathrm p}(X) \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$). Our lemma follows. ■

Proof of 5.1. Observe that

$$C_{\mathrm{p}}(X,P) \cong C_{\mathrm{p}}(X,P^{\infty}) = (C_{\mathrm{p}}(X,P))^{\infty}\,.$$

Now, it is enough to combine 5.2 with 8.3 which, in particular, states that a countable product of a Borel set is of an exact multiplicative class. ■

The proof of 5.2 works also for the projective classes $\mathcal{P}_n$. Using embeddings of 2^{∞} into P and of P into 2^{∞} one can easily verify that for nondiscrete X, the exact Borel classes of $C_{\mathrm{p}}(X,P)$ and $C_{\mathrm{p}}(X,2^{\infty}) = (C_{\mathrm{p}}(X)\cap 2^X)^{\infty}$ coincide. Consequently, applying 5.2 and 8.3, we obtain the following generalization of [14, Lemma 4.2].

5.3. Corollary. *Let X be a countable regular space. Then, for a countable ordinal $\alpha \ge 1$ and $n \ge 1$, we have*

(a) *if $C_{\mathrm{p}}(X)\cap 2^X \in \mathcal{A}_\alpha \backslash \mathcal{M}_\alpha$, then $C_{\mathrm{p}}(X) \in \mathcal{M}_{\alpha+1}\backslash\mathcal{A}_{\alpha+1}$,*
(b) *if $C_{\mathrm{p}}(X)\cap 2^X \in \mathcal{M}_\alpha \backslash \mathcal{A}_\alpha$, then $C_{\mathrm{p}}(X) \in \mathcal{M}_{\alpha}\backslash\mathcal{A}_{\alpha}$,*
(c) *if $C_{\mathrm{p}}(X)\cap 2^X \in \mathcal{A}_\alpha \cap \mathcal{M}_\alpha \backslash \bigcup_{\beta<\alpha}(\mathcal{A}_\beta\cup\mathcal{M}_\beta)$, then $C_{\mathrm{p}}(X) \in \mathcal{M}_{\alpha}\backslash\mathcal{A}_{\alpha}$.*
(d) *$C_{\mathrm{p}}(X)\cap 2^X \in \mathcal{P}_n$ iff $C_{\mathrm{p}}(X) \in \mathcal{P}_n$.* ■

Let us notice that $C_{\mathrm{p}}(X)\cap 2^X$ is the subspace of 2^X consisting of the characteristic functions of clopen subsets of X. For $X = \mathbb{N}_F$, $C_{\mathrm{p}}(X)\cap 2^X$ can be identified with $F\times\{0,1\}$. Let us also point out that all Borel and projective classes of the spaces $C_{\mathrm{p}}(X)\cap 2^X$ mentioned in the above corollary (except for $\alpha = 1$ in (b) and (c)) do occur (see 9.2).

6. Classification of Borel spaces $C_{\mathrm{p}}(X)$. In this section we discuss the following question.

6.1. Problem. Let X be a countable regular nondiscrete space such that $C_{\mathrm{p}}(X)$ is Borel. Does the Borel class of $C_{\mathrm{p}}(X)$ determine its topological type?

A particular case of this question has been treated in [14], where it was shown that $C_{\mathrm{p}}(X)$ is homeomorphic to Ω_2 provided it belongs to the class $\mathcal{M}_2$. By 5.1, if X is a countable nondiscrete regular space such that $C_{\mathrm{p}}(X)$ is Borel, then $C_{\mathrm{p}}(X) \in \mathcal{M}_\alpha\backslash\mathcal{A}_\alpha$ for some $\alpha\ge 2$. We shall later show that for every countable ordinal $\alpha \ge 2$, there exists a space X such that $C_{\mathrm{p}}(X)$ is homeomorphic to Ω_α. Since every $C_{\mathrm{p}}(X)$ which is Borel is a Z_σ-space [14], by 4.1, it is an absorbing set for the class $\mathcal{F}_0(C_{\mathrm{p}}(X))$. The uniqueness theorem for absorbing sets (Theorem 2.1) implies that 6.1 is equivalent to the following question.

6.2. Problem. Let X be a countable regular space such that $C_{\mathrm{p}}(X) \in \mathcal{M}_\alpha\backslash\mathcal{A}_\alpha$, $\alpha\ge 2$. Is $\mathcal{F}_0(C_{\mathrm{p}}(X))$ equal to $\mathcal{M}_\alpha$?

All proofs of the fact that $C_{\mathrm{p}}(X) \cong \Omega_\alpha$ we are aware of are based on the Wadge $(I^\infty, \mathcal{M}_\alpha)$-completeness of the pair $(\mathbb{R}^X, C_{\mathrm{p}}(X))$. Therefore it is reasonable to ask:

6.3. Problem. Let X be a countable regular space such that $C_{\mathrm{p}}(X) \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$, $\alpha \ge 2$. Is the pair $(\mathbb{R}^X, C_{\mathrm{p}}(X))$ Wadge $(I^\infty, \mathcal{M}_\alpha)$-complete?

An affirmative answer to this problem would not only resolve 6.1 but also provide a complete topological classification of the triples $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X))$ for Borel $C_{\mathrm{p}}(X)$.

6.4. Theorem. *Let X be a countable regular noncompact space such that $C_{\mathrm{p}}(X) \in \mathcal{M}_\alpha$, $\alpha \ge 2$. The following conditions are equivalent*:

(1) $(\mathbb{R}^X, C_{\mathrm{p}}(X))$ *is Wadge* $(I^\infty, \mathcal{M}_\alpha)$*-complete*,
(2) $\mathcal{F}_0(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X)) = (\mathcal{M}_0, \mathcal{M}_1, \mathcal{M}_\alpha)$,
(3) $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X)) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha)$.

Proof. (1)⇒(2). (1) and Lemma 4.6 show that $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha) \in \mathcal{F}_0(\overline{\mathbb{R}}^X,$
$\mathbb{R}^X, C_{\mathrm{p}}(X))$. Now, (2) follows.

(2)⇒(3). Apply 4.1, 2.3 and the fact that $C_{\mathrm{p}}(X)$ is a Z_σ-space (see [14]).

(3)⇒(1) is obvious. ∎

6.5. Remark. Here are two more conditions equivalent to those of 6.4:

(4) $(\mathbb{R}^X, C_{\mathrm{p}}(X))$ is Wadge $(\mathbb{R}^\infty, \mathcal{M}_\alpha)$-complete,
(5) $(\mathbb{R}^X, C_{\mathrm{p}}(X)) \cong (\mathbb{R}^\infty, \Omega_\alpha)$. ∎

Theorem 6.4 holds also for the spaces $C_{\mathrm{p}}^{\mathrm{loc}}(X)$. However, in this case, $C_{\mathrm{p}}^{\mathrm{loc}}(X)$ can also be of an exact additive class α. Then we must replace $\mathcal{M}_\alpha$ by $\mathcal{A}_\alpha$ and Ω_α by Λ_α. Also 6.4 can be extended to the classes $\mathcal{P}_n$, $n \ge 1$, for both $C_{\mathrm{p}}(X)$ and $C_{\mathrm{p}}^{\mathrm{loc}}(X)$ spaces that are Z_σ-spaces. Below we give a detailed statement of these facts for $X = \mathbb{N}_F$.

6.6. Remark. Let F be a filter on $\mathbb{N}$ that is not a Fréchet filter. Then

(a) if $c_F \in \mathcal{M}_\alpha$ (resp., $s_F \in \mathcal{M}_\alpha$), then (1)–(5) formulated for c_F (resp., s_F) are equivalent,

(b) if $s_F \in \mathcal{A}_\alpha$, then (1)–(5) formulated for s_F, with $\mathcal{M}_\alpha$ replaced by A_α and Ω_α by Λ_α, are equivalent,

(c) if $c_F \in \mathcal{P}_n$ (resp., $s_F \in \mathcal{P}_n$) and c_F is a Z_σ-space, then (1), (2), (4), and (5) formulated for c_F (resp., s_F), with $\mathcal{M}_\alpha$ replaced by $\mathcal{P}_n$ and Ω_α by Π_n, are equivalent. We also have $(\overline{\mathbb{R}}^\infty, c_F) \cong (\overline{\mathbb{R}}^\infty, \Pi_n)$ (resp., $(\overline{\mathbb{R}}^\infty, s_F) \cong (\overline{\mathbb{R}}^\infty, \Pi_n)$).

A proof of 6.6 can be obtained in the same way as that of 6.4 (use 4.9 and the fact that $C_{\mathrm{p}}(\mathbb{N}_F)$ (resp., $C_{\mathrm{p}}^{\mathrm{loc}}(\mathbb{N}_F)$) is Wadge complete if and only if c_F (resp., s_F) are). ∎

Since $C_{\mathrm{p}}(X) \in \mathcal{M}_2$ for compact X, Theorem 6.4 shows that an affirmative answer to 6.3 would determine the topological type of $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X))$ for $C_{\mathrm{p}}(X) \in \mathcal{M}_\alpha \backslash \mathcal{A}_\alpha$, $\alpha \geq 3$. Let us discuss the case where $\alpha = 2$. Let $B(X) = \{f \in \mathbb{R}^X \mid f \text{ is bounded}\}$. For countable infinite X, we have $B(X) \cong \Sigma$ (see [1]). If X is compact then $C_{\mathrm{p}}(X) \subset B(X)$.

6.7. Theorem. *Let X be a countable regular nondiscrete space such that $C_{\mathrm{p}}(X) \in \mathcal{M}_2$. Then*:

(a) *If X is not compact, then* $(\overline{\mathbb{R}}^X, \mathbb{R}^X, C_{\mathrm{p}}(X)) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_2)$.

(b) *If X is compact, then* $(\overline{\mathbb{R}}^X, \mathbb{R}^X, B(X), C_{\mathrm{p}}(X)) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Sigma, c_0)$.

Proof. (a) This follows from 6.4 and the fact ([5], [14, Remark 5.6]) that $(\mathbb{R}^X, C_{\mathrm{p}}(X))$ is Wadge $(I^\infty, \mathcal{M}_2)$-complete.

(b) It was implicitly shown in [5], [13] that (Σ, c_0) is strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal. To get the result we will check that also $(B(X), C_{\mathrm{p}}(X))$ is strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal, and then apply 2.4. Since X is metrizable, a standard argument ([5], [13]) yields a (linear) factorization $(B(X), C_{\mathrm{p}}(X)) \cong (E \times \Sigma, E \times c_0)$ for some linear space E. Now, since (Σ, c_0) is strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal, the argument of [2, Proposition 2.6] shows that $(B(X), C_{\mathrm{p}}(X))$ is also strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal. ∎

6.8. Remark. It has been observed ([9], [12]) that $(\mathbb{R}^\infty, \Omega_2)$ and $(\mathbb{R}^\infty, c_0)$ are not homeomorphic though the pairs $(\overline{\mathbb{R}}^\infty, \Omega_2)$ and $(\overline{\mathbb{R}}^\infty, c_0)$ are strongly $(\mathcal{M}_0, \mathcal{M}_2)$-universal. Consequently, if X is compact and Y is not compact such that $C_{\mathrm{p}}(Y) \in \mathcal{M}_2 \backslash \mathcal{A}_2$ then $(\mathbb{R}^X, C_{\mathrm{p}}(X)) \not\cong (\mathbb{R}^Y, C_{\mathrm{p}}(Y))$ (however, $(\overline{\mathbb{R}}^X, C_{\mathrm{p}}(X)) \cong (\overline{\mathbb{R}}^Y, C_{\mathrm{p}}(Y))$). ∎

7. Some observations on c_F and s_F. To attack Problems 6.2 and 6.3 it is tempting to utilize the strategy of [14]: first treat spaces X with exactly one nonisolated point and then deal with the general case. Having this in mind, we consider in this section a few specific aspects of spaces c_F and s_F.

First, using Theorem 4.12, we reduce 6.2 and 6.3 to spaces c_F by proving the following result.

7.1. Proposition. *Let F be a filter on $\mathbb{N}$ that is not the Fréchet filter and such that $c_F \in \mathcal{M}_\alpha$. Then*

(a) *$c_F \cong \Omega_\alpha$ iff $\mathcal{F}_0(c_F)$ contains all $\mathcal{A}_\beta$ for $\beta < \alpha$,*

(b) *$(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_F) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha)$ iff $(\mathbb{R}^\infty, c_F)$ is Wadge $(I^\infty, \mathcal{A}_\beta)$-universal for all $\beta < \alpha$.*

Proof. Since c_F is Borel, by [14], c_F is a Z_σ-space. By 4.8, c_F is then an $\mathcal{F}_0(c_F)$-absorbing set; hence it is homeomorphic to Ω_α iff $\mathcal{F}_0(c_F) = \mathcal{M}_\alpha$. According to the construction of [2], $(\mathbb{R}^\infty, \Omega_\alpha) = ((\mathbb{R}^\infty)^\infty, \prod_{n=1}^\infty F_n)$, where each $F_n \subset \mathbb{R}^\infty$ belongs to $\bigcup_{\beta<\alpha} \mathcal{A}_\beta$. Now, (a) follows from the fact (Theorem 4.12) that $c_F \cong c_F^\infty$. To get (b), assume $(\mathbb{R}^\infty, c_F)$ is Wadge $(I^\infty, \mathcal{A}_\beta)$-universal for all $\beta < \alpha$. Hence there exists a map $\varphi : (\overline{\mathbb{R}}^\infty)^\infty \to (\mathbb{R}^\infty)^\infty$ such that $\varphi^{-1}(c_F^\infty) = \prod_{n=1}^\infty F_n$. Since, by 4.12, $((\mathbb{R}^\infty)^\infty, c_F^\infty) \cong (\mathbb{R}^\infty, c_F)$, applying 4.7 we infer that $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha) \in \mathcal{F}_0(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_F)$. Now 6.6(a) is applicable to conclude that $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_F) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha)$. ∎

The remaining part of this section is devoted to the spaces s_F and their application to the study of c_F.

7.2. Lemma. *For $f \in \mathbb{R}^{\mathbb{N}}$, let $\varrho(f) = f^{-1}(\{0\}) \in 2^{\mathbb{N}}$. Then*

(a) *ϱ is a transformation of $\mathbb{R}^{\mathbb{N}}$ into $2^{\mathbb{N}}$ of the first Baire class,*

(b) *for every closed set $H \subset 2^{\mathbb{N}}$ the set*

$$V(H) = \{f \in \mathbb{R}^{\mathbb{N}} \mid \exists_{A\in H}\ A \subseteq \varrho(f)\}$$

is closed in $\mathbb{R}^{\mathbb{N}}$.

Proof. (a) This is folklore (cf. [22, Lemma 3.2]); we include its proof for the reader's convenience. Pick $x_1, \ldots, x_n$, $y_1, \ldots, y_m \in \mathbb{N}$ and write

$$\begin{aligned} U &= U(x_1, \ldots, x_n\,; y_1, \ldots, y_m) \\ &= \{A \in 2^{\mathbb{N}} \mid \{x_1, \ldots, x_n\} \subset A \subset \mathbb{N}\setminus\{y_1, \ldots, y_m\}\}\,. \end{aligned}$$

Then $\varrho^{-1}(U)$ is the intersection of a closed set $\{f \mid f(x_1) = \ldots = f(x_n) = 0\}$ and an open set $\{f \mid f(y_j) \neq 0,\ 1 \le j \le m\}$; hence $\varrho^{-1}(U)$ is an F_σ-set. Since the sets $U(x_1, \ldots, x_n\,; y_1, \ldots, y_m)$ form a basis in $2^{\mathbb{N}}$, (a) is shown.

To see (b), let (f_n) be a sequence in $V(H)$ which converges to $f \in \mathbb{R}^{\mathbb{N}}$. By definition of $V(H)$, there are $A_n \in H$ with $A_n \subseteq \varrho(f_n)$. Since H is compact we can assume that (A_n) converges to some $A \in H$. If we had $A \not\subseteq \varrho(f)$, there would exist $x \in A$ with $f(x) \neq 0$. Then, if n is sufficiently large, $x \in A_n$ and $f_n(x) \neq 0$, a contradiction. ∎

It turns out that the topological identification of s_F is easy for σ-compact filters F. For the Fréchet filter F_0, s_{F_0} is the space $\sigma = \{(x_i) \in \mathbb{R}^\infty \mid x_i = 0$ a.e.$\}$. The case of all remaining σ-compact filters is described below.

7.3. Proposition. *Let F be a σ-compact filter on $\mathbb{N}$ that is not the Fréchet filter. Then $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_F) \cong (\overline{\mathbb{R}}^\infty \times \overline{\mathbb{R}}^\infty, \mathbb{R}^\infty \times \mathbb{R}^\infty, \mathbb{R}^\infty \times \sigma)$.*

Proof. First we shall show that, for every σ-compact filter F, s_F is a countable union of Z-sets in $\mathbb{R}^\infty$. To this end, observe that s_F, as a linear subspace of a Z_σ-space c_F (see [14]), is itself a Z_σ-space. Let $F = \bigcup_{n=1}^\infty H_n$, where $H_n \subset 2^{\mathbb{N}}$ are compacta. Since $s_F = \bigcup_{n=1}^\infty V(H_n)$ and each $V(H_n)$ is closed in $\mathbb{R}^\infty$ (use 7.2(b)), s_F is an F_σ-subset of $\mathbb{R}^\infty$, and consequently is a countable union of Z-sets in $\mathbb{R}^\infty$.

Let F_0 be the Fréchet filter on $\mathbb{N}$. Consider the σ-compact filter $F_1 = 2^{\mathbb{N}} \times F_0$ on $\mathbb{N} \times \{0,1\}$. Identifying $\mathbb{N} \times \mathbb{N}$ with $\mathbb{N}$, we have

$$(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_{F_1}) \cong (\overline{\mathbb{R}}^\infty \times \overline{\mathbb{R}}^\infty, \mathbb{R}^\infty \times \mathbb{R}^\infty, \mathbb{R}^\infty \times \sigma)\,.$$

Since for every filter F, $\mathbb{R}^\infty \backslash s_F$ is locally homotopy negligible, our result will follow from 2.3 and 4.8 if we show that $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, X) \in \mathcal{F}_0(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_F)$ for every X that is a countable union of Z-sets in $\mathbb{R}^\infty$. However, if X is such a set, then there exists Y which is an F_σ-subset of $\overline{\mathbb{R}}^\infty$ and such that $Y \cap \mathbb{R}^\infty = X$. Hence, by [14, Lemma 5.4] (see also our Lemma 8.9), there exists a map $\varphi : \overline{\mathbb{R}}^\infty \to \mathbb{R}^\infty$ with $\varphi^{-1}(s_F) = Y$. Finally, 4.9 is applicable. ∎

Let us ask

7.4. Question. Let F be a σ-compact filter on $\mathbb{N}$ that is not the Fréchet filter. Is $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_F, s_F)$ homeomorphic to $(\overline{\mathbb{R}}^\infty \times \overline{\mathbb{R}}^\infty, \mathbb{R}^\infty \times \mathbb{R}^\infty, \mathbb{R}^\infty \times c_0, \mathbb{R}^\infty \times \sigma)$, where $c_0 = \{(x_i) \in \mathbb{R}^\infty \mid x_i \to 0\}$?

To answer this question in the affirmative it is enough to show that there are maps $\varphi : \mathbb{R}^\infty \to \mathbb{R}^\infty \times \mathbb{R}^\infty$ and $\psi : \mathbb{R}^\infty \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ with

$$\varphi^{-1}(\mathbb{R}^\infty \times c_0) = c_F \quad \text{and} \quad \varphi^{-1}(\mathbb{R}^\infty \times \sigma) = s_F\,, \quad \text{and}$$
$$\psi^{-1}(c_F) = \mathbb{R}^\infty \times c_0 \quad \text{and} \quad \psi^{-1}(s_F) = \mathbb{R}^\infty \times \sigma\,.$$

In contrast to the case of c_F, the relationship between the Borel complexity of F and that of s_F seems to be difficult to determine. We have the following partial result; part (a) is a consequence of the fact that F embeds onto a closed subset of s_F, and (b) and (c) follow from 7.2(a) and the observation that $s_F = \varrho^{-1}(F)$.

7.5. Corollary. *For a filter F on $\mathbb{N}$, we have*:

(a) *If $s_F \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$), then $F \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$).*

(b) *If $F \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$), then $s_F \in \mathcal{M}_{1+\alpha}$ (resp., $\mathcal{A}_{1+\alpha}$). In particular, for infinite α, $F \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$) iff $s_F \in \mathcal{M}_\alpha$ (resp., $\mathcal{A}_\alpha$).*

(c) *For every n, $s_F \in \mathcal{P}_n$ iff $F \in \mathcal{P}_n$.* ∎

The following general fact can be helpful in studying c_F.

7.6. Proposition. *For every filter F on $\mathbb{N}$, we have $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_F) \in \mathcal{F}_0((\overline{\mathbb{R}}^\infty)^\infty, (\mathbb{R}^\infty)^\infty, s_F^\infty)$.*

Proof. For every k, let $\psi_k : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ be a map such that $\psi_k(t) = t$ for $|t| \ge 1/k$, $\psi_k(t) = 0$ for $|t| < 1/(k+1)$ and ψ_k is linear on $[-1/k, -1/(k+1)]$ and on $[1/(k+1), 1/k]$. For $f \in \overline{\mathbb{R}}^\infty$, we let $\zeta_k(f) = \psi_k \circ f$. Then $\zeta = (\zeta_k) : \overline{\mathbb{R}}^\infty \to (\overline{\mathbb{R}}^\infty)^\infty$ is an embedding such that $\zeta^{-1}((\mathbb{R}^\infty)^\infty) = \mathbb{R}^\infty$. Moreover, for $f \in \mathbb{R}^\infty$, $\zeta_k(f) \in s_F$ if and only if $\{n \mid |f(n)| \le 1/(k+1)\} \in F$. It follows that $\zeta^{-1}(s_F^\infty) = c_F$. ∎

Note that for the Fréchet filter F_0, $c_{F_0} \cong s^\infty_{F_0}$. By 7.3, for every σ-compact filter F on $\mathbb{N}$, we have $c_F \cong \Omega_2 \cong (\mathbb{R}^\infty \times \sigma)^\infty \cong s^\infty_F$. Therefore, it is reasonable to ask:

7.7. Problem. Is c_F homeomorphic to s^∞_F?

In view of 7.6, 4.8 and 4.11, this problem for spaces c_F that are Z_σ-spaces is equivalent to the question of whether $s_F \in \mathcal{F}_0(c_F)$.

8. Construction of spaces c_F homeomorphic to Ω_α for even α. In this section, for every countable odd (resp., even) ordinal α, we inductively construct a filter $F_\alpha \in \mathcal{A}_\alpha \backslash \mathcal{M}_\alpha$ (resp., $G_\alpha \in \mathcal{M}_\alpha \backslash \mathcal{A}_\alpha$) such that c_{F_α} (resp., c_{G_α}) is homeomorphic to $\Omega_{\alpha+1}$ (resp., Ω_α). We start with some auxiliary constructions.

For a sequence $\{(X_n, A_n)\}_{n=1}^\infty$ of pairs of metrizable spaces, we define the Fréchet product of A_n with respect to X_n to be the subset

$$\mathbb{FP}(X_n, A_n) = \Big\{(x_n) \in \prod_{n=1}^\infty X_n : x_n \in A_n \text{ for almost all } n\Big\}$$

of the product $\prod_{n=1}^\infty X_n$. If $X_n = X$ (resp., $X_n = X$ and $A_n = A$) for $n \ge 1$, we abbreviate $\mathbb{FP}(X_n, A_n) = \mathbb{FP}(X, A_n)$ (resp., $\mathbb{FP}(X_n, A_n) = \mathbb{FP}(X, A)$).

The following fact is the key to this section.

8.1. Proposition. *Let Y be a separable metrizable space and let $\{(X_n, A_n)\}_{n=1}^\infty$ be a sequence of pairs of separable metrizable spaces.*

(a) *Let $\alpha \ge 1$ (or $\alpha \ge 0$ if Y is zero-dimensional) be a countable ordinal. If each (X_n, A_n) is Wadge $(Y, \mathcal{A}_\alpha(Y))$-complete then $(\prod_{n=1}^\infty X_n, \mathbb{FP}(X_n, A_n))$ (resp., $(\prod_{n=1}^\infty X_n, \prod_{n=1}^\infty A_n)$) is Wadge $(Y, \mathcal{A}_{\alpha+2}(Y))$-complete (resp., $(Y, \mathcal{M}_{\alpha+1}(Y))$-complete).*

(b) *Let α be a limit ordinal, and let (α_n) be a sequence of ordinals such that $\alpha_n < \alpha$ and $\sup \alpha_n = \alpha$. If each (X_n, A_n) is Wadge $(Y, \mathcal{A}_{\alpha_n}(Y))$-complete, then $(\prod_{n=1}^\infty X_n,\ \mathbb{FP}(X_n, A_n))$ (resp., $(\prod_{n=1}^\infty X_n, \prod_{n=1}^\infty A_n)$) is Wadge $(Y, \mathcal{A}_{\alpha+1}(Y))$-complete (resp., $(Y, \mathcal{M}_\alpha(Y))$-complete).*

The proof makes use of the lemma below which is essentially due to Calbrix [4].

8.2. Lemma. *Let Y be a separable metrizable space and let $\alpha \ge 2$ (or $\alpha \ge 1$ if Y is zero-dimensional) be a countable ordinal.*

(a) *If $A \in \mathcal{A}_{\alpha+1}(Y)$, then there exists $\{B_n\}_{n=1}^\infty$ with $B_n \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta(Y)$, $n \ge 1$, such that $A = \bigcup_{m=1}^\infty \bigcap_{n \ge m} B_n$.*

(b) *If $A \in \mathcal{M}_{\alpha+1}(Y)$, then there exists $\{B_n\}_{n=1}^\infty$ with $B_n \in \bigcup_{\beta<\alpha} \mathcal{M}_\beta(Y)$, $n \ge 1$, such that $A = \bigcap_{m=1}^\infty \bigcup_{n \ge m} B_n$.*

Proof. (b) There exist $C_i \in \mathcal{A}_\alpha(Y)$, $i \geq 1$, such that $A = \bigcap_{i=1}^\infty C_i$ and $C_{i+1} \subseteq C_i$ for all i. By [20, Theorem 2 and Remarks, p. 348], there exist sets $D_{ij} \in \bigcup_{\beta<\alpha} \mathcal{M}_\beta(Y)$ such that $C_i = \bigcup_{j=1}^\infty D_{ij}$ and $D_{ij} \cap D_{ik} = \emptyset$ for $j \neq k$. Let $\{B_n\}_{n=1}^\infty$ be an enumeration of $\{D_{ij}\}_{i,j=1}^\infty$. We claim that $A = \bigcap_{m=1}^\infty \bigcup_{n\geq m} B_n$. In fact, if $x \in A$ then x belongs to each C_i; and hence it belongs to some element of $\{D_{ij}\}_{j=1}^\infty$. Consequently, x belongs to infinitely many B_n. Conversely, if $x \notin A$ then there exists an integer i_0 such that $x \notin C_i$ for $i \geq i_0$. Since, for fixed i, x belongs to at most one of the sets $\{D_{ij}\}_{j=1}^\infty$, x cannot belong to infinitely many B_n.

(a) If $A \in \mathcal{A}_{\alpha+1}(Y)$, then $A' = Y\backslash A \in \mathcal{M}_{\alpha+1}(Y)$. By (b), $A' = \bigcap_{m=1}^\infty \bigcup_{n\geq m} B'_n$, where $B'_n \in \bigcup_{\beta<\alpha} \mathcal{M}_\beta(Y)$. Then

$$A = Y\backslash \bigcap_{m=1}^\infty \bigcup_{n\geq m} B'_n = \bigcup_{m=1}^\infty \Big(Y\backslash \bigcup_{n\geq m} B'_n\Big) = \bigcup_{m=1}^\infty \bigcap_{n\geq m} (Y\backslash B'_n)\,,$$

and $B_n = Y\backslash B'_n \in \bigcup_{\beta<\alpha} \mathcal{A}_\beta(Y)$. ■

Proof of 8.1. We only show (a); the proof of (b) is similar. Let $T \in \mathcal{A}_{\alpha+2}(Y)$ (resp., $T \in \mathcal{M}_{\alpha+1}(Y)$). It follows from 8.2(a) (resp., it is elementary) that there exist $S_n \in \mathcal{A}_\alpha(Y)$, $n \geq 1$, such that $T = \bigcup_{m=1}^\infty \bigcap_{n\geq m} S_n$ (resp., $T = \bigcap_{n=1}^\infty S_n$). For every n, let $\varphi_n : Y \to X_n$ be a map such that $\varphi_n^{-1}(A_n) = S_n$. Let $\varphi = (\varphi_n) : Y \to \prod_{n=1}^\infty X_n$. One can check that $\varphi^{-1}(\mathbb{FP}(X_n, A_n)) = T$ (resp., $\varphi^{-1}(\prod_{n=1}^\infty A_n) = T$). ■

Proposition 8.1 allows us to determine the Borel class of $\mathbb{FP}(X_n, A_n)$ (as well as that of $\prod_{n=1}^\infty A_n$, which we include herein though, presumably, this belongs to mathematical folklore).

8.3. Proposition. *Let $\{(X_n, A_n)\}_{n=1}^\infty$ be a sequence of pairs of separable metrizable spaces, and let X_n be complete metrizable, $n \geq 1$. Let $\alpha \geq 1$ be a countable ordinal.*

(a) *If $A_n \in \mathcal{A}_\alpha\backslash\mathcal{M}_\alpha$ (resp., $\mathcal{A}_1(X_n)\backslash\mathcal{M}_1$ for $\alpha = 1$), $n \geq 1$, then $\mathbb{FP}(X_n, A_n) \in \mathcal{A}_{\alpha+2}\backslash\mathcal{M}_{\alpha+2}$ and $\prod_{n=1}^\infty A_n \in \mathcal{M}_{\alpha+1}\backslash\mathcal{A}_{\alpha+1}$.*

(b) *If $A_n \in \mathcal{M}_\alpha\backslash\bigcup_{\beta<\alpha}\mathcal{M}_\beta$ (resp., $\mathcal{M}_1\backslash\mathcal{M}_0(X)$ for $\alpha = 1$), $n \geq 1$, then $\mathbb{FP}(X_n, A_n) \in \mathcal{A}_{\alpha+1}\backslash\mathcal{M}_{\alpha+1}$ and $\prod_{n=1}^\infty A_n \in \mathcal{M}_\alpha\backslash\mathcal{A}_\alpha$.*

(c) *Let α be a limit countable ordinal and (α_n) be a sequence of ordinals such that $\alpha_n < \alpha$ and $\sup \alpha_n = \alpha$. If $A_n \in \mathcal{A}_{\alpha_n} \cup \mathcal{M}_{\alpha_n} \backslash \bigcup_{\beta<\alpha_n}(\mathcal{A}_\beta \cup \mathcal{M}_\beta)$, $n \geq 1$, then $\mathbb{FP}(X_n, A_n) \in \mathcal{A}_{\alpha+1}\backslash\mathcal{M}_{\alpha+1}$ and $\prod_{n=1}^\infty A_n \in \mathcal{M}_\alpha\backslash\mathcal{A}_\alpha$.*

We need the following lemma.

8.4. Lemma. *Let A be a Borel subset of a separable complete metrizable space X and let α be a countable ordinal. If $A \notin \mathcal{M}_\alpha(X)$ then the pair (X, A) is Wadge $(2^\infty, \mathcal{A}_\alpha(2^\infty))$-complete.*

P r o o f. According to [19, Theorem 4], X contains a zero-dimensional compactum P such that $P \cap A \notin \mathcal{M}_\alpha$. We can consider P as a subset of the Cantor set 2^∞. Let $r : 2^\infty \to P$ be a retraction (see [17, Ex. 4.5.10, p. 363]) and let $B = r^{-1}(A \cap P)$. Obviously B is Borel and $B \notin \mathcal{M}_\alpha$ ($A \cap P$ is a closed subset of B). The Wadge Lemma (see [24]) shows that $(2^\infty, B)$ is Wadge $(2^\infty, \mathcal{A}_\alpha)$-complete. Using the retraction r we infer that $(P, P \cap A)$, and therefore (X, A), is Wadge $(2^\infty, \mathcal{A}_\alpha(2^\infty))$-complete. ∎

P r o o f o f 8.3. It is elementary that if $A_n \in \mathcal{A}_\alpha$ (resp., $A_n \in \mathcal{M}_\alpha$) then $\mathbb{FP}(X_n, A_n)$ belongs to $\mathcal{A}_{\alpha+2}$ and $\prod_{n=1}^\infty A_n \in \mathcal{M}_{\alpha+1}$ (resp., $\mathbb{FP}(X_n, A_n)$ belongs to $\mathcal{A}_{\alpha+1}$ and $\prod_{n=1}^\infty A_n$ belongs to $\mathcal{M}_\alpha$). Now, to evaluate the exact Borel classes of the spaces $\mathbb{FP}(X_n, A_n)$ and $\prod_{n=1}^\infty A_n$ apply 8.1 together with the suitable Wadge completeness of these spaces given by 8.4. ∎.

Let F be a filter on $\mathbb{N}$. We write

$$u_F = \{(x_n) \in \mathbb{R}^\infty \mid \forall_{A \in F} \sup_{i \in A} |x_i| = \infty\}$$

and set

$$X_F = c_F \cup u_F\,.$$

8.5. PROPOSITION. *Let $\{F_n\}_{n=1}^\infty$ be a sequence of filters on $\mathbb{N}$. Consider the filters $F = \mathbb{FP}(2^{\mathbb{N}}, F_n)$ and $P = \prod_{n=1}^\infty F_n$ on $\mathbb{N} \times \mathbb{N}$.*

(a) *If α is a countable ordinal such that (X_{F_n}, c_{F_n}) are Wadge $(I^\infty, \mathcal{A}_\alpha)$-complete for all n then*

(i) *the pair (X_F, c_F) is Wadge $(I^\infty, \mathcal{A}_{\alpha+2})$-complete,*

(ii) *the pair (X_P, c_P) is Wadge $(I^\infty, \mathcal{M}_{\alpha+1})$-complete.*

(b) *If α is a countable limit ordinal and $(\alpha_n)_{n=1}^\infty$ a sequence of ordinals satisfying $\alpha_n < \alpha$ and $\sup \alpha_n = \alpha$ such that (X_{F_n}, c_{F_n}) is Wadge $(I^\infty, \mathcal{A}_{\alpha_n})$-complete for all n, then*

(i) *the pair (X_F, c_F) is Wadge $(I^\infty, \mathcal{A}_{\alpha+1})$-complete,*

(ii) *the pair (X_P, c_P) is Wadge $(I^\infty, \mathcal{M}_\alpha)$-complete.*

P r o o f. (a) Let us note that

(1) $\mathbb{FP}(X_{F_n}, c_{F_n}) \subset c_F$,
(2) $(\prod_{n=1}^\infty X_{F_n}) \setminus \mathbb{FP}(X_{F_n}, c_{F_n}) \subset u_F$,
(3) $c_P = \prod_{n=1}^\infty c_{F_n}$,
(4) $\prod_{n=1}^\infty X_{F_n} \subset X_P$.

The assertion (i) (resp., (ii)) follows from (1), (2) and 8.1 (resp., (2), (3) and 8.2).

The proof of (b) is the same as that of (a). ∎

For odd countable ordinals α, define filters F_α inductively as follows. Let F_1 be any filter that belongs to $\mathcal{A}_1$. Suppose filters F_β have been defined for all odd $\beta < \alpha$. If $\alpha - 1$ is not a limit ordinal, put $\beta_n = \alpha - 2$, $n = 1, 2, \ldots$; if

$\alpha-1$ is a limit ordinal, pick (β_n) to be a sequence of odd ordinals satisfying $\beta_n < \alpha-1$ and $\sup\beta_n = \alpha-1$. Then let

$$F_\alpha = \mathbb{FP}(2^{\mathbb{N}}, F_{\beta_n}).$$

Let us also define, for all even ordinals $\alpha > 0$, filters G_α as follows. If α is not a limit ordinal, let $G_\alpha = F^\infty_{\alpha-1}$. If α is a limit ordinal, pick (β_n) a sequence of odd ordinals satisfying $\beta_n < \alpha$ and $\sup\beta_n = \alpha$, and let $G_\alpha = \prod_{n=1}^\infty F_{\beta_n}$.

8.6. Lemma. (a) *For every odd* α, $F_\alpha \in \mathcal{A}_\alpha \setminus \mathcal{M}_\alpha$.

(b) *For every even* $\alpha > 0$, $G_\alpha \in \mathcal{M}_\alpha \setminus \mathcal{A}_\alpha$.

Proof. Since no filter belongs to $\mathcal{M}_1$ (see [4]), $F_1 \in \mathcal{A}_1 \setminus \mathcal{M}_1$. Now, the assertions (a) and (b) follow inductively from 8.3. ∎

8.7. Theorem. (a) *For every odd* $\alpha > 1$, *we have* $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_{F_\alpha}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_{\alpha+1})$ *and* $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_{F_\alpha}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Lambda_{\alpha+1})$.

(b) *For every even* $\alpha > 0$, *we have* $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_{G_\alpha}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_{G_\alpha})$.

We will employ the following auxiliary result.

8.8. Lemma. *For every odd* $\alpha \geq 1$, $(X_{F_\alpha}, c_{F_\alpha})$ *is Wadge* $(I^\infty, \mathcal{A}_\alpha)$-*complete.*

Proof. It follows from 8.5 that whenever our lemma holds for $\alpha = 1$, then it holds for arbitrary odd α. Verification of the assertion for $\alpha = 1$ is a particular case of the following fact.

8.9. Lemma. *Let F be a filter on $\mathbb{N}$ that is an element of the σ-algebra generated by the open subsets and the first category subsets of* $2^{\mathbb{N}}$. *Then the pair* (X_F, c_F) *is Wadge* $(I^\infty, \mathcal{A}_1)$-*complete.*

We will make use of the fact below whose proof is implicitly contained in the proof of [14, Lemma 5.4].

8.10. Lemma. *Let X be a complete absolute retract and $Z \subseteq Y$ be subsets of X that satisfy*

(i) *Y is a countable union of Z-sets in X,*

(ii) *$X \setminus Z$ is locally homotopy negligible in X.*

Then for every σ-compact subset A of I^∞ there exists a map $\varphi : I^\infty \to X$ such that $\varphi(A) \subseteq Z$ and $\varphi(I^\infty \setminus A) \subseteq X \setminus Y$. ∎

Proof of 8.9. Applying [14, Lemmas 2.2 and 2.3], we can find a matrix $\{A(n,m)\}_{n,m=1}^\infty$ of pairwise disjoint finite subsets of $\mathbb{N}$ such that, for every $A \in F$, there exists $n \in \mathbb{N}$ with $A \cap A(n,m) \neq \emptyset$ for all m. Put

$$X(n,k) = \{(x_i) \in \mathbb{R}^\infty \mid \forall_m \exists_{i \in A(n,m)}\ |x_i| \leq k\}.$$

One can easily check that

(1) each $X(n,k)$ is a Z-set in $\mathbb{R}^\infty$,
(2) $c_F \subseteq \bigcup_{n,k=1}^\infty X(n,k) = Y$,
(3) $\mathbb{R}^\infty \backslash Y \subset u_F$.

It suffices to apply 8.10 with $X = \mathbb{R}^\infty$, Y and $Z = c_F$. ■

Proof of 8.7. Combining 8.6 and [14, Lemma 4.2], we infer that $c_{F_\alpha} \in \mathcal{M}_{\alpha+1}$. This together with 8.8 and 7.1 yields that $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_{F_\alpha}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_{\alpha+1})$.

Similarly, we deduce that c_{G_α} belongs to $\mathcal{M}_\alpha$. By 8.8 and 8.5, $(X_{G_\alpha}, c_{G_\alpha})$ is Wadge $(I^\infty, \mathcal{M}_\alpha)$-complete (hence, $(\mathbb{R}^\infty, c_{G_\alpha})$ is also Wadge $(I^\infty, \mathcal{M}_\alpha)$-complete). Now, 7.1 shows that $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_{G_\alpha}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha)$.

Observe that for every sequence $\{F_n\}_{n=1}^\infty$ of filters on $\mathbb{N}$, writing $F = \mathbb{FP}(2^{\mathbb{N}}, F_n)$ and $P = \prod_{n=1}^\infty F_n$, we have

(1) $s_F = \mathbb{FP}(\mathbb{R}^\infty, s_{F_n})$,
(2) $s_P = \prod_{n=1}^\infty s_{F_n}$.

Note that $s_{F_1} \in \mathcal{A}_1(\mathbb{R}^\infty) \backslash \mathcal{M}_1(\mathbb{R}^\infty)$ (for a filter F_1 that is not the Fréchet filter, apply 7.3; if F_1 is the Fréchet filter then $s_{F_1} = \sigma$). Using 8.3 and (1), we inductively deduce that $s_{F_\alpha} \in \mathcal{A}_\alpha \backslash \mathcal{M}_\alpha$ for all odd $\alpha > 1$. By 8.3 and (2), $s_{G_\alpha} \in \mathcal{M}_\alpha$ for all even $\alpha > 0$.

Since s_{F_1} is a countable union of Z-sets in $\mathbb{R}^\infty$ (see the proof of 7.3), Lemma 8.10 is applicable and thus $(\mathbb{R}^\infty, s_{F_1})$ is Wadge $(I^\infty, \mathcal{A}_1)$-complete. Using 8.1 and (1), we show inductively that $(\mathbb{R}^\infty, s_{F_\alpha})$ is Wadge $(I^\infty, \mathcal{A}_\alpha)$-complete. By 6.6(b), $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_{F_\alpha}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Lambda_\alpha)$.

Since $(\mathbb{R}^\infty, s_{F_\alpha})$ is Wadge $(I^\infty, \mathcal{A}_\alpha)$-complete for all odd α, it follows from 8.1(a) and (2) that $(\mathbb{R}^\infty, s_{G_\alpha})$ is Wadge $(I^\infty, \mathcal{M}_\alpha)$-complete for even $\alpha > 0$. By 6.6(b), $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_{G_\alpha}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \Omega_\alpha)$. ■

8.11. Remark. Let F_0 be the Fréchet filter on $\mathbb{N}$ and let $A = \mathbb{FP}(2^{\mathbb{N}}, F_0)$ be the filter defining the Arens space $\mathbb{N}_A$ (see [17, Example 1.6.20]). By 8.7, we find that $c_A \cong \Omega_4$ and $s_A \cong \Lambda_3$. In particular, $c_A \in \mathcal{M}_4 \backslash \mathcal{A}_4$ and $s_A \in \mathcal{A}_3 \backslash \mathcal{M}_3$; these facts were shown with the use of different approaches by R. Pol during Winter School at Srní (Czechoslovakia), 1990. ■

8.12. Remark. If we know a filter F on $\mathbb{N}$ that belongs to $\mathcal{A}_2$ so that (X_F, c_F) is Wadge $(I^\infty, \mathcal{A}_2)$-complete, then we could repeat the inductive construction preceding Lemma 8.6 to obtain filters $\widetilde{F}_\alpha \in \mathcal{A}_\alpha \backslash \mathcal{M}_\alpha$ (resp., $\widetilde{G}_\alpha \in \mathcal{M}_\alpha \backslash \mathcal{A}_\alpha$) for all even (resp., odd) ordinals α, $2 \le \alpha < \omega$. For such α, Theorem 8.6 holds. See the next section for a construction of F. ■

The techniques of this section allow us to construct inductively linear copies of Λ_α for odd ordinals α and Ω_α for even ordinals α in $\mathbb{R}^\infty$ as follows. Let α be a countable ordinal ≥ 2. If α is a limit ordinal, let (α_n) be a sequence of ordinals such that $\alpha_n < \alpha$ and $\sup \alpha_n = \alpha$; otherwise set

$\alpha_n = \alpha - 1$ for all n. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of linear subspaces of $\mathbb{R}^\infty$ such that $(\mathbb{R}^\infty, E_n) \cong (\mathbb{R}^\infty, \Lambda_{\alpha_n})$ for all n. Let $E = \mathbb{FP}(\mathbb{R}^\infty, E_n)$ and $H = \prod_{n=1}^{\infty} E_n$.

8.13. Proposition. *We have*:

(a) $((\mathbb{R}^\infty)^\infty, E) \cong (\mathbb{R}^\infty, \Lambda_{\alpha+1})$,
(b) $((\mathbb{R}^\infty)^\infty, H) \cong (\mathbb{R}^\infty, \Omega_\alpha)$.

Proof. Evidently, $E \in \mathcal{A}_{\alpha+1}$ and $H \in \mathcal{M}_\alpha$. By 8.1, $((\mathbb{R}^\infty)^\infty, E)$ is Wadge $(I^\infty, A_{\alpha+1})$-complete and $((\mathbb{R}^\infty)^\infty, H)$ is Wadge $(I^\infty, \mathcal{M}_\alpha)$-complete. Since $((\mathbb{R}^\infty)^\infty, E)$ is homeomorphic to $\mathbb{R}^\infty \times ((\mathbb{R}^\infty)^\infty, \mathbb{FP}(\mathbb{R}^\infty, E_{n+1}))$, it easily follows that $\mathcal{F}_0((\mathbb{R}^\infty)^\infty, E) = (\mathcal{M}_1, \mathcal{A}_{\alpha+1})$ (cf. 4.6). Using the fact that $(\mathbb{R}^\infty, E_n) \cong (\mathbb{R}^\infty, E_n) \times \mathbb{R}$, we infer that $((\mathbb{R}^\infty)^\infty, H) \cong ((\mathbb{R}^\infty)^\infty, H) \times \mathbb{R}^\infty$; consequently, $\mathcal{F}_0((\mathbb{R}^\infty)^\infty, H) = (\mathcal{M}_1, \mathcal{M}_\alpha)$. Since E and H are Z_σ-spaces, 3.1 is applicable, for according to 3.8 the strong universality of both $((\mathbb{R}^\infty)^\infty, E)$ and $((\mathbb{R}^\infty)^\infty, H)$ follows. ■

9. Filters generated by subsets of I^∞.

We shall adapt the construction of filters described in [22] to I^∞. This is done so that for every subset A of I^∞, there exists a filter F_A such that c_{F_A} contains a closed copy of A.

Let d be a metric on I^∞ that is bounded by 1. For $k \ge 1$, let Q_k be a finite subset of I^∞ such that for every $q \in I^\infty$ there exists $q' \in Q_k$ with $d(q, q') < 1/k$. Assume $\{Q_k\}_{k=1}^{\infty}$ is pairwise disjoint. Let $Q = \bigcup_{k=1}^{\infty} Q_k$, and let $\{q_n\}_{n=1}^{\infty}$ be an enumeration of Q. For $k \ge 1$, put $N_k = \{i \in \mathbb{N} \mid q_i \in Q_k\}$; $\mathbb{N}$ has thus been decomposed into finite sets N_k. For every $q \in I^\infty$, set

$$B_q = \bigcup_{k=1}^{\infty} \{n \in N_k \mid d(q, q_n) \le 2/k\}\,.$$

Note that, for $p, q \in I^\infty$, $p \ne q$, the set $B_p \cap B_q$ is finite. For $A \subseteq I^\infty$, let F_A be the filter on $\mathbb{N}$ generated by the sets of the form

$$\mathbb{N} \backslash (B_{q_1} \cup \ldots \cup B_{q_n} \cup S)\,,$$

where $n = 1, 2, \ldots$, $q_i \in A$ for $i = 1, \ldots, n$ and S is a finite subset of $\mathbb{N}$. If $q \ne q_i$ for $i = 1, \ldots, n$, then $B_q \cap B_{q_i}$ is finite; hence $\mathbb{N} \backslash (B_{q_1} \cup \ldots \cup B_{q_n} \cup S) \ne \emptyset$. This shows that F_A is well defined. Write $F = F_{I^\infty}$.

9.1. Lemma. *There exists an embedding $\varphi : I^\infty \to \mathbb{R}^\infty$ such that, for every subset A of I^∞, we have*

(i) $\varphi(A) \subseteq s_{F_A}$,
(ii) $\varphi(I^\infty \backslash A) \subset u_{F_A}$.

Proof. Define φ as follows:

$$\varphi(q)(n) = k \max(0, 2 - k d(q, q_n)) \qquad \text{if } n \in N_k\,,$$

$k = 1, 2, \ldots$ It is clear that φ is a map of I^∞ into $\mathbb{R}^\infty$. Let $q \in I^\infty$ and $n \in N_k$ with $\varphi(q)(n) \neq 0$. Then $2 - kd(q, q_n) > 0$; hence $d(q, q_n) < 2/k$ and n belongs to B_q. Consequently, if $q \in A$ then $\varphi(q) \in s_{F_A}$; this shows (i). For each k there exists $n_k \in N_k$ with $d(q, q_{n_k}) < 1/k$. It easily follows that $\varphi(q)(n_k) \geq k$; thus $\varphi(q)$ is unbounded on B_q. This shows that φ is injective. To see (ii), suppose $q \notin A$ and let

$$X = \mathbb{N} \backslash (B_{q_1} \cup \ldots \cup B_{q_n} \cup S)$$

be a basic element of F_A, where $q_1, \ldots, q_n$ are points of A and S is a finite subset of $\mathbb{N}$. Since $B_q \backslash X$ is finite, $\varphi(q)$ is unbounded on X. ∎

9.2. Proposition. *Let α be a countable ordinal ≥ 2 and n be an integer ≥ 1. For a subset A of I^∞, the following assertions are equivalent*:

(i) $A \in \mathcal{A}_\alpha$ (*resp.*, $\mathcal{M}_\alpha$ *or* $\mathcal{P}_n$),

(ii) $F_A \in \mathcal{A}_\alpha$ (*resp.*, $\mathcal{M}_\alpha$ *or* $\mathcal{P}_n$),

(iii) $s_{F_A} \in \mathcal{A}_\alpha$ (*resp.*, $\mathcal{M}_\alpha$ *or* $\mathcal{P}_n$).

Moreover, *this holds for the class $\mathcal{A}_1$ provided* (iii) *is replaced by* (iii)$'$: $s_{F_A} \in \mathcal{A}_1(\mathbb{R}^\infty)$.

Proof. (i)⇒(iii). For $A \subseteq I^\infty$ and $l, m \geq 1$, we put

$$P(A, l, m) = \{f \in \mathbb{R}^\infty \mid \exists_{p \in A^m}\ \mathbb{N} \backslash f^{-1}(\{0\}) \subseteq B_{p_1} \cup \ldots \cup B_{p_m} \cup \{1, \ldots, l\}\},$$

where $p = (p_1, \ldots, p_m)$. Note that

$$s_{F_A} = \bigcup_{l,m=1}^{\infty} P(A, l, m). \tag{1}$$

If $\pi : \mathbb{R}^\infty \times (I^\infty)^m \to \mathbb{R}^\infty$ is the natural projection, then $P(A, l, m) = \pi(C_{m,l} \cap (\mathbb{R}^\infty \times A^m))$, where $C_{m,l} = \{(f, p) \in \mathbb{R}^\infty \times (I^\infty)^m \mid \forall_{n > l} [n \in N_k \to \exists_{1 \leq i \leq m}\ d(q_n, p_i) \leq 2/k \text{ or } (f(n) = 0)]\}$. The set $C_{m,l}$ is closed. If A is an F_σ-set, then the same is true of $C_{m,l} \cap (\mathbb{R}^\infty \times A^m)$, therefore also of its projection, $P(A, l, m)$ on $\mathbb{R}^\infty$. Apply (1), to get $s_{F_A} \in \mathcal{A}_1(\mathbb{R}^\infty)$.

Recall that $\sigma = \{(x_i) \in \mathbb{R}^\infty \mid x_i = 0 \text{ a.e.}\}$. Put $R(A, l, 1) = (A, l, 1) \backslash \sigma$ and $R(A, l, m) = P(A, l, m) \backslash \bigcup_{j=1}^\infty P(A, j, m-1)$ for $m \geq 2$; hence

$$s_{F_A} = \sigma \cup \bigcup_{l,m=1}^{\infty} R(A, l, m). \tag{2}$$

Since σ is σ-compact, to examine the case of $A \in \mathcal{A}_\alpha$, $\alpha \geq 2$ (resp., $A \in \mathcal{P}_n$), it suffices to show that each $R(A, l, m)$ belongs to $\mathcal{A}_\alpha$ (resp., $\mathcal{P}_n$).

Since $P(I^\infty, l, m) = \pi(C_{m,l})$ is closed, each $R(I^\infty, l, m)$ is a G_δ-subset of $\mathbb{R}^\infty$. Let Q_m be the closed subset of 2^{I^∞} consisting of elements that contain no more than m points. For $f \in R(I^\infty, l, m)$, there exists a unique

set $H(f)=\{p_1,\ldots,p_m\}\in Q_m$ such that
$$\mathbb{N}\setminus f^{-1}(\{0\})\subset B_{p_1}\cup\ldots\cup B_{p_m}\cup\{1,\ldots,l\}\,.$$
We will check that H is continuous. Fix $f\in R(I^\infty,l,m)$ and $\varepsilon>0$ and write $H(f)=\{p_1,\ldots,p_m\}$. Let $\delta=\min\{d(p_i,p_j)\mid 1\le i,j\le m,\ i\ne j\}$. Choose $k>l$ with $1/k<\min(\delta/8,\varepsilon/4)$. Since $f\notin\bigcup_{j=1}^\infty P(A,j,m-1)$, we can find $n_1,\ldots,n_m\in\bigcup_{j\ge k}N_j$ such that $f(n_i)\ne 0$ and that $n_i\in B_{p_i}$ for $i=1,\ldots,m$. By definition of B_{p_i}, we have
$$d(q_{n_i},p_i)\le\frac{2}{k}\,.$$
Then
$$\begin{aligned}d(q_{n_i},q_{n_j})&\ge d(p_i,p_j)-d(p_i,q_{n_i})-d(p_j,q_{n_j})\\&\ge\delta-\frac{2}{k}-\frac{2}{k}>\delta-\frac{\delta}{2}>\frac{4}{k}\,.\end{aligned}$$
If $q\in R(I^\infty,l,m)$ is so close to p that $q(n_i)\ne 0$ for $i=1,\ldots,m$, then, writing $H(q)=\{r_1,\ldots,r_m\}$, for every $i\le m$ there exists an index $j(i)$ such that $n_i\in B_{r_{j(i)}}$. Then $d(q_{n_i},r_{j(i)})\le 2/k$ and, by (3), $i\to j(i)$ is a bijection. Moreover, we have
$$d(p_i,r_{j(i)})\le d(p_i,q_{n_i})+d(q_{n_i},r_{j(i)})\le\frac{2}{k}+\frac{2}{k}<\varepsilon\,;$$
the continuity of H follows. Let $A_m=\{\{p_1,\ldots,p_m\}\in Q_m\mid p_i\ne p_j$ for $i\ne j$ and $p_i\in A$ for $i=1,\ldots,m\}$. Every point of A_m has an open neighborhood in A_m that is homeomorphic to an open set in A^m. Consequently, A_m belongs to $\mathcal{A}_\alpha$ (resp., $\mathcal{P}_n$). One can easily verify that $R(A,l,m)=H^{-1}(A_m)$. Since $R(I^\infty,l,m)$ is an absolute G_δ-set, it follows that $R(A,l,m)\in\mathcal{A}_\alpha$ (resp., $\mathcal{P}_n$).

Suppose now that $A\in\mathcal{M}_\alpha$, $\alpha\ge 2$. Then $A=\bigcap_{n=1}^\infty B_n$, where $B_n\in\bigcup_{\beta<\alpha}\mathcal{A}_\beta$. We have
$$\begin{aligned}s_{F_A}&=\{f\in s_F\mid\forall_{p\in I^\infty\setminus A}\ (\mathbb{N}\setminus f^{-1}(\{0\}))\cap B_p\text{ is finite}\}\\&=\{f\in s_F\mid\forall_{p\in I^\infty\setminus\bigcap_{n=1}^\infty B_n}\ (\mathbb{N}\setminus f^{-1}(\{0\}))\cap B_p\text{ is finite}\}\\&=\bigcap_{n=1}^\infty\{f\in s_F\mid\forall_{p\in I^\infty\setminus B_n}\ (\mathbb{N}\setminus f^{-1}(\{0\}))\cap B_p\text{ is finite}\}\\&=\bigcap_{n=1}^\infty s_{F_{B_n}}\,.\end{aligned}$$
Since $s_{F_{B_n}}\in\bigcup_{\beta<\alpha}\mathcal{A}_\beta(\mathbb{R}^\infty)$, applying the additive case, we conclude that $s_{F_A}\in\mathcal{M}_\alpha$.

The implication (iii)⇒(ii) follows from the fact that F_A is homeomorphic to a closed subset of s_{F_A}.

(ii)⇒(i). If $F_A \in \mathcal{M}_\alpha$ then, by [14, Lemma 4.2], $c_{F_A} \in \mathcal{M}_\alpha$. The argument of [14, Lemma 4.2] (see also our Corollary 5.3) shows that if $F_A \in \mathcal{P}_n$ then $c_{F_A} \in \mathcal{P}_n$. By 9.1, condition (i) follows for $\mathcal{M}_\alpha$ and $\mathcal{P}_n$.

Assume $F_A \in \mathcal{A}_\alpha$. Then, by [14, Lemma 4.2], c_{F_A} is Borelian. In view of 9.1, so is A. If A does not belong to $\mathcal{A}_\alpha$, by [19], there exists a Cantor set $C \subset I^\infty$ such that $C \cap A$ does not belong to $\mathcal{A}_\alpha$. The fact below contradicts the assumption that $F_A \in \mathcal{A}_\alpha$.

9.3. Lemma. *Let $C \subset I^\infty$ be a Cantor set. There exists $\psi : C \to 2^{\mathbb{N}}$ such that $\psi^{-1}(F_A) = A \cap C$ for all $A \subseteq I^\infty$.*

Proof. We identify each subset of $\mathbb{N}$ with its characteristic function. For $k \geq 1$, let $\mathcal{C}_k = \{C_k^1, C_k^2, \ldots, C_k^{m_k}\}$ be a partition of C into closed subsets of diameters $< 1/k$. Let $C_k(q)$ be the unique element of $\mathcal{C}_k$ which contains q. Define

$$\psi(q)(n) = \begin{cases} 0 & \text{if } d(q_n, C_k(q)) \leq 1/k, \\ 1 & \text{if } d(q_n, C_k(q)) > 1/k, \end{cases}$$

for $n \in N_k$, $k = 1, 2, \ldots$ Since $\psi(q)(n)$ is constant on $C_k(q)$, $\psi(q)$ is continuous. It is easy to see that ψ is injective. Let $q \in C$ and $n \in N_k$ with $\psi(q)(n) = 0$. Using the fact that the diameters of $C_k(q)$ are less than $1/k$, we infer that $d(q_n, q) < 2/k$; hence n belongs to B_q. This shows that $\psi(q)$ contains $\mathbb{N} \backslash B_q$; consequently, $\psi(q) \in F_A$ provided $q \in A$. Let $q \notin A$ and let

$$X = \mathbb{N} \backslash (B_{q_1} \cup \ldots \cup B_{q_n} \cup S)$$

be a basic element of F_A, where $q_1, \ldots, q_n \in A$ and S is a finite subset of $\mathbb{N}$. The set $B_q \backslash X$ is finite. For every k, there exists $n_k \in N_k$ with $d(q, q_{n_k}) < 1/k$. Then $n_k \in B_q$ and $\psi(q)(n_k) = 0$; hence n_k does not belong to $\psi(q)$. Consequently, $B_q \backslash \psi(q)$ is infinite. This yields $X \backslash \psi(q) \neq \emptyset$. As a consequence $\psi(q)$ does not contain any basic element of F_A; hence it does not belong to F_A. ■

Here is our main result of this section. In particular, when applied to the pair $(\overline{\mathbb{R}}^\infty, A)$, $A = \varLambda_\alpha$ (resp., $A = \varOmega_\alpha$), it shows that for every $\alpha \geq 2$ there exist filters $F_\alpha \in \mathcal{A}_\alpha$ (resp., $G_\alpha \in \mathcal{M}_\alpha$) such that $c_{F_\alpha} \cong \varOmega_{\alpha+1}$ (resp., $c_{G_\alpha} \cong \varOmega_\alpha$). Moreover, when applied to $(\overline{\mathbb{R}}^\infty, \varPi_n)$, it provides filters $F_n \in \mathcal{P}_n$ such that $c_{F_n} \cong \varPi_n$.

9.4. Theorem. *Let A be a subset of I^∞.*

(a) *If $A \in \mathcal{M}_\alpha$, $\alpha \geq 2$, and (I^∞, A) is Wadge $(I^\infty, \mathcal{M}_\alpha)$-complete, then $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_{F_A}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_{F_A}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \varOmega_\alpha)$.*

(b) *If $A \in \mathcal{A}_\alpha$, $\alpha \geq 2$, and (I^∞, A) is Wadge $(I^\infty, \mathcal{A}_\alpha)$-complete, then $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_{F_A}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \varLambda_\alpha)$ and $(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_{F_A}) \cong (\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, \varOmega_{\alpha+1})$.*

(c) *If* $A \in \mathcal{P}_n$, $n \geq 1$, *and* (I^∞, A) *is Wadge* $(I^\infty, \mathcal{P}_n)$*-complete*, *then* $\mathcal{F}_0(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, c_{F_A}) = \mathcal{F}_0(\overline{\mathbb{R}}^\infty, \mathbb{R}^\infty, s_{F_A}) = (\mathcal{M}_0, \mathcal{M}_1, \mathcal{P}_n)$. *In particular*, $(\mathbb{R}^\infty, c_{F_A}) \cong (\mathbb{R}^\infty, s_{F_A}) \cong (\mathbb{R}^\infty, \Pi_n)$ *and* $(\overline{\mathbb{R}}^\infty, c_{F_A}) \cong (\overline{\mathbb{R}}^\infty, s_{F_A}) \cong (Q, \Pi'_n)$.

Proof. (a) It follows from 9.1 that if (I^∞, A) is Wadge $(I^\infty, \mathcal{M}_\alpha)$-complete, then so are $(\mathbb{R}^\infty, c_{F_A})$ and $(\mathbb{R}^\infty, s_{F_A})$. By 9.2 and [14, Lemma 4.2], c_{F_A} and s_{F_A} belong to $\mathcal{M}_\alpha$. Now, to get the result apply 6.6(a).

(b) This follows in the same way (use 7.1(b)).

(c) As above, using our assumption, we infer that $(\mathbb{R}^\infty, c_{F_A})$ and $(\mathbb{R}^\infty, s_{F_A})$ are Wadge $(I^\infty, \mathcal{P}_n)$-complete. Since c_F is a Z_σ-space, c_{F_A} (as a linear dense subspace of c_F) is also a Z_σ-space. Now, 6.6(c) is applicable. ∎

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