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Weakly normal ideals on $\mathcal{P}_\kappa\lambda$ and the singular cardinal hypothesis

by

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Abstract. In §1, we observe that a weakly normal ideal has a saturation property; we also show that the existence of certain precipitous ideals is sufficient for the existence of weakly normal ideals. In §2, generalizing Solovay's theorem concerning strongly compact cardinals, we show that $\lambda^{<\kappa}$ is decided if $\mathcal{P}_\kappa\lambda$ carries a weakly normal ideal and λ is regular or $\mathrm{cf}\,\lambda \le \kappa$. This is applied to solving the singular cardinal hypothesis.

0. Preliminaries. A strongly compact cardinal introduces certain regularities in the universe of set theory. For example, Solovay showed that the singular cardinal hypothesis holds above a compact cardinal.

If κ is λ-compact, $\mathcal{P}_\kappa\lambda$ carries a weakly normal fine ultrafilter. So, the existence of weakly normal ideals is a weaker hypothesis than the existence of strongly compact cardinals. In this paper, we use a weakly normal ideal to reprove those results of [14] for which Solovay used a strongly compact cardinal.

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Our set theory is ZFC and much of notation is standard (see [4], [8], [15]). Throughout the paper κ is a regular uncountable cardinal and λ is a cardinal $\ge \kappa$. Unless specified otherwise, every ideal on $\mathcal{P}_\kappa\lambda$ is assumed to be κ-complete and fine. So, every ideal I contains the smallest ideal $I_{\kappa\lambda} = \{X \subset \mathcal{P}_\kappa\lambda : X \text{ is not unbounded}\}$. Set $I^+ = \mathcal{P}(\mathcal{P}_\kappa\lambda) - I$ and let I^* be the filter dual to I. The sets in I^+ and I^* are called *I-positive* and *I-measure one* respectively. $NS_{\kappa\lambda}$ is the ideal of nonstationary sets, and $SNS_{\kappa\lambda}$ is the ideal of strongly nonstationary sets. For each $x \in \mathcal{P}_\kappa\lambda$, $\widehat{x}$ is the set $\{y \in \mathcal{P}_\kappa\lambda : x \subset y\}$. If f is a function, $f''A$ is the image of A under f.

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1. Weakly normal ideals and saturated ideals. Weakly normal fine ultrafilters as well as weakly normal ideals defined below can be seen as a weak version of normal ultrafilters. On the other hand, the form of weak normality proposed by Mignone [10], which we call here "semi-weak normality", is a weakening of normality of filters.

DEFINITION. An ideal I on $\mathcal{P}_\kappa\lambda$ is *weakly normal* if for every regressive function $f : \mathcal{P}_\kappa\lambda \to \lambda$, $\{x \in \mathcal{P}_k\lambda : f(x) \le \gamma\} \in I^*$ for some $\gamma < \lambda$. I is called *semi-weakly normal* if for all $X \in I^+$ and all regressive functions $f : X \to \lambda$, there is a $\gamma < \lambda$ such that $\{x \in X : f(x) \le \gamma\} \in I^+$.

Our weak normality is a $\mathcal{P}_\kappa\lambda$ generalization of weak normality for filters on κ due to Kanamori [7]. It appears in the proof of Theorem 2.1.

We begin by showing that weak normality is a combination of semi-weak normality and a saturation property.

LEMMA 1.1. *I is weakly normal iff I is semi-weakly normal and there is no disjoint family of* $\mathrm{cf}\,\lambda$*-many I-positive sets.*

Proof. Suppose that I is weakly normal. Let $X \in I^+$ and $f : X \to \lambda$ be regressive. We extend f to $g : \mathcal{P}_\kappa\lambda \to \lambda$ that is also regressive. Using weak normality of I, we can find $Y \in I^*$ and $\gamma < \lambda$ so that $f(x) \le \gamma$ for all $x \in Y$. Set $Z = X \cap Y$. Then $g|Z = f|Z$ and $Z \in I^+$. Thus I is semi-weakly normal.

Next, assume that there exists a disjoint family $\{A_\alpha : \alpha < \mathrm{cf}\,\lambda\}$ of I-positive sets. Let $\{\lambda_\alpha \mid \alpha < \mathrm{cf}\,\lambda\}$ be a cofinal increasing sequence in λ. We may assume that $A_\alpha \subset \widehat{\{\lambda_\alpha\}}$ for any $\alpha < \mathrm{cf}\,\lambda$. Define a regressive function $f : \mathcal{P}_\kappa\lambda \to \lambda$ by $f''A_\alpha = \{\lambda_\alpha\}$. Since I is weakly normal, $B = \{x : f(x) \le \gamma\} \in I^*$ for some $\gamma < \lambda$. Now pick a $\lambda_\alpha > \gamma$. By the definition of f, $A_\alpha \subset f^{-1}(\{\lambda_\alpha\})$ and $f^{-1}(\{\lambda_\alpha\}) \cap B = \emptyset$. This contradicts $A_\alpha \in I^+$.

Conversely, suppose that I is a semi-weakly normal ideal with no disjoint family of $\mathrm{cf}\,\lambda$-many positive sets. If I is not weakly normal, there is a regressive function $f : \mathcal{P}_\kappa\lambda \to \lambda$ such that $\{x : f(x) \ge \gamma\} \in I^+$ for any $\gamma < \lambda$. Since I is semi-weakly normal, we can find a $\gamma_0 < \lambda$ such that $A_0 = \{x : f(x) < \gamma_0\} \in I^+ - I^*$. Since $\mathcal{P}_\kappa\lambda - A_0 \in I^+$, we have a $\gamma_1 < \lambda$ so that $A_1 = \{x : \gamma_0 \le f(x) < \gamma_1\} \in I^+ - I^*$. In the same way, for any $\alpha < \mathrm{cf}\,\lambda$, we can define $\gamma_{\alpha+1} < \lambda$ such that $A_{\alpha+1} = \{x : \gamma_\alpha \le f(x) < \gamma_{\alpha+1}\} \in I^+$. For α a limit ordinal less than $\mathrm{cf}\,\lambda$, let $\eta_\alpha = \sup\{\gamma_\beta : \beta < \alpha\} < \lambda$. Since $\{x : \eta_\alpha \le f(x)\}$ is I-positive, there is a γ_α so that $A_\alpha = \{x : \eta_\alpha \le f(x) < \gamma_\alpha\} \in I^+ - I^*$.

Contrary to our hypothesis, we now have a pairwise disjoint family $\{A_\alpha : \alpha < \mathrm{cf}\,\lambda\}$ of I-positive sets. ∎

COROLLARY 1.2. *If* $\mathrm{cf}\,\lambda = \kappa$, *then I is weakly normal iff it is semi-weakly normal and κ-saturated.*

COROLLARY 1.3. *Let* $\mathrm{cf}\,\lambda < \kappa$. *Then I is weakly normal iff I is* $\mathrm{cf}\,\lambda$-*saturated.*

Proof. It is easy to show that every ideal is semi-weakly normal if $\mathrm{cf}\,\lambda < \kappa$. For more on semi-weak normality, see [10] and [11]. ∎

1.2 and 1.3 show κ is large in some inner model if $\mathcal{P}_\kappa\lambda$ carries a weakly normal ideal provided that $\mathrm{cf}\,\lambda \le \kappa$. It will be shown in [3] that the existence of weakly normal ideals on $\mathcal{P}_\kappa\lambda$ is possible for κ with various degree of largeness.

Here we only state that some familiar ideals are not weakly normal.

COROLLARY 1.4. *None of* $I_{\kappa\lambda}$, $SNS_{\kappa\lambda}$, $NS_{\kappa\lambda}$ *is weakly normal.*

Proof. It is known that $\mathcal{P}_\kappa\lambda$ is a disjoint union of λ stationary subsets (see [8] for example) and every extension of a weakly normal ideal is also weakly normal. ∎

For normal ideals, easy observations suggest that:

COROLLARY 1.5. *Every* $\mathrm{cf}\,\lambda$-*saturated normal ideal is weakly normal.*

Proof. Let $f : \mathcal{P}_\kappa\lambda \to \lambda$ be regressive and $A = \{\gamma < \lambda : f^{-1}(\{\gamma\}) \in I^+\}$. Since I is $\mathrm{cf}\,\lambda$-saturated, $|A| < \mathrm{cf}\,\lambda$. Set $\delta = \sup A$. Then $\delta < \lambda$ and it is clear that $\{x : f(x) \le \delta\} \in I^*$. ∎

Conversely, saturated ideals produce weakly normal ideals under certain conditions. We already know some cases (1.3, 1.5). In fact, Corollary 1.8 below was proved in [2] using an analogue of Solovay's construction of incompressible functions (see [13]).

We use here a generic ultrapower which makes the proof much simpler.

DEFINITION. Let I and J be ideals.

(1) $J \le_{RK} I$ if $J = f_*(I) = \{X : f^{-1}(X) \in I\}$ for some $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\lambda$.
(2) For $X \in I^+$, $I|X = \{Y \subset \mathcal{P}_\kappa\lambda : Y \cap X \in I\}$, which is also an ideal.
(3) $I_\delta = f_*(I)$ for $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\delta$ such that $f(x) = x \cap \delta$.

PROPOSITION 1.6. *Suppose that I is a precipitous ideal on $\mathcal{P}_\kappa\lambda$. Then there is a semi-weakly normal ideal $J \le_{RK} I|X$ for some $X \in I^+$.*

Proof. Let G be a generic filter for P_I, the poset of I-positive subsets of $\mathcal{P}_\kappa\lambda$, and let $j : V \to M$ be the corresponding generic elementary embedding. Pick a name $\underset{\sim}{f}$ such that $1 \Vdash_{P_I} \underset{\sim}{f}$ represents $\sup j''\lambda$ in M. There are $X \in I^+$ and $f : X \to V$ with $X \Vdash_{P_I} \underset{\sim}{f} = \check{f}$. Note that for every $\alpha < \lambda$, $\{x \in X : f(x) \le \alpha\} \in I$.

Suppose $Y = \{x \in X : g(x) < f(x)\} \in I^+$. Since $Y \le_{P_I} X$, $Y \Vdash_{P_I} \check{f}$ represents $\sup j''\lambda$ and $[g]_G < [f]_G$. Thus $Y \Vdash_{P_I} \exists\alpha < \lambda\,([g]_G < j(\alpha))$. So, $\{x \in Y : g(x) < \alpha\} \in I^+$ for some $\alpha < \lambda$.

Now if h is defined by $h(x) = x \cap f(x)$ for $x \in X$, the above observation shows that $J = h_*(I|X)$ is a semi-weakly normal ideal. ∎

As a corollary, we get the next theorem.

THEOREM 1.7. *If I is a precipitous ideal on $\mathcal{P}_\kappa\lambda$ with no pairwise disjoint family of* $\operatorname{cf}\lambda$*-many I-positive sets, then for any $X \in I^+$ we can find a $Y \in P(X) \cap I^+$ and a weakly normal ideal $J \leq_{RK} I|Y$.*

Proof. Use Lemma 1.1 and Proposition 1.6. ∎

Yo Matsubara taught the author a simpler construction. Let $J = \{Y \subset \mathcal{P}_\kappa\lambda : 1 \Vdash_{P_I} [\mathrm{id}] \cap j''\lambda \in j(\mathcal{P}_\kappa\lambda - Y)\}$. Then J is weakly normal.

Recall that any countably complete ideal with the disjointing property is precipitous, and every κ-complete κ^+-saturated ideal has the disjointing property. (See Foreman [5].)

COROLLARY 1.8. (i) *If* $\operatorname{cf}\lambda \geq \kappa$ *and $\mathcal{P}_\kappa\lambda$ carries a κ-saturated ideal, then there exists a κ-saturated weakly normal ideal.*

(ii) *If* $\operatorname{cf}\lambda \geq \kappa^+$ *and there is a κ^+-saturated ideal on $\mathcal{P}_\kappa\lambda$, then there exists a weakly normal ideal on $\mathcal{P}_\kappa\lambda$.*

If κ is λ-compact, then it is δ-compact for all $\kappa \leq \delta < \lambda$. So, one can ask whether the existence of a weakly normal filter on $\mathcal{P}_\kappa\lambda$ assures the existence of one on $\mathcal{P}_\kappa\delta$ for any $\delta < \lambda$.

If I is a normal ideal on $P_\kappa\lambda$, then I_δ is also normal. But the situation is not clear for weak normality. We can only prove:

THEOREM 1.9. (1) *If I is a weakly normal ideal on $\mathcal{P}_\kappa\lambda$ and* $\operatorname{cf}\lambda \leq \kappa$, *then there is a weakly normal ideal for any $\kappa \leq \delta < \lambda$ such that* $\operatorname{cf}\delta \geq \kappa$.

(2) *If there is a $\kappa^+(\kappa)$-saturated ideal on $\mathcal{P}_\kappa\lambda$, then we have a weakly normal ideal on $\mathcal{P}_\kappa\delta$ for all $\delta < \lambda$ with* $\operatorname{cf}\delta \geq \kappa^+(\kappa)$.

(3) *If $\mathcal{P}_\kappa\lambda$ carries a weakly normal ideal and* $\operatorname{cf}\lambda > \kappa$, *then* $\mathcal{P}_\kappa \operatorname{cf}\lambda$ *also bears a weakly normal ideal.*

(4) *If $\kappa < \delta < \lambda$, $\kappa <$* $\operatorname{cf}\delta = \operatorname{cf}\lambda$ *and $\mathcal{P}_\kappa\lambda$ carries a weakly normal ideal, then there exists a weakly normal ideal on $\mathcal{P}_\kappa\delta$.*

Proof. (1) and (2) are clear from 1.2, 1.3, 1.8, and the fact that I_δ is also $\kappa^+(\kappa)$-saturated for any $\kappa^+(\kappa)$-saturated ideal I on $\mathcal{P}_\kappa\lambda$.

(3) Let $\kappa < \delta = \operatorname{cf}\lambda < \lambda$, let $\{\lambda_\alpha \mid \alpha < \delta\}$ be a cofinal normal sequence in λ, and $K_\alpha = [\lambda_\alpha, \lambda_{\alpha+1})$. If $f(\beta) =$ the unique ordinal α such that $\beta \in K_\alpha$, then f is a mapping from λ onto δ, and $g : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\delta$ defined by $g(x) = f''x$ is also onto. For a weakly normal ideal I on $\mathcal{P}_\kappa\lambda$, define J by

$$X \in J \quad \text{iff} \quad X \subset \mathcal{P}_\kappa\delta \text{ and } g^{-1}(X) \in I\,.$$

Then J is a κ-complete proper ideal. For $\alpha < \delta$,

$$g^{-1}(\{x : \alpha \notin x\}) = \{x \in \mathcal{P}_\kappa\lambda : f(\beta) \neq \alpha \text{ for all } \beta \in x\}$$
$$= \{x : x \cap K_\alpha = \emptyset\} \in I\,.$$

So, J is fine.

To see that J is weakly normal, let $h : \mathcal{P}_\kappa\delta \to \delta$ be regressive. We have $h \circ g(x) \in g(x)$ for all $x \in \mathcal{P}_\kappa\lambda$ and $g(x) = f''x$. Thus $h \circ g(x) = f(\gamma_x)$ for some $\gamma_x \in x$. Using weak normality of I, we can find a $\gamma < \lambda$ such that $X = \{x : \gamma_x \le \gamma\} \in I^*$. By our definition, f is increasing. Hence $f(\gamma_x) \le f(\gamma)$ for all $x \in X$, which means that $\{x \in \mathcal{P}_\kappa\delta : h(x) \le f(\gamma)\} \in J^*$.

(4) Set $\eta = \operatorname{cf}\lambda$, let $\{\lambda_\alpha \mid \alpha < \eta\}$ and $\{\delta_\alpha \mid \alpha < \eta\}$ be cofinal normal sequences of cardinals in λ and δ respectively such that $\lambda_0 \ge \delta$, and let $K_\alpha = [\lambda_\alpha, \lambda_{\alpha+1})$ and $L_\alpha = [\delta_\alpha, \delta_{\alpha+1})$ for each $\alpha < \eta$.

Define $f : \lambda \to \delta$ and $g : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\kappa\delta$ so that $f''K_\alpha = L_\alpha$ and $g(x) = f''x$. Then g is surjective and $J = g_*(I)$ is weakly normal if I is weakly normal. ∎

For the existence of weakly normal ideals, we give another construction in 2.7 and 2.10.

2. $\lambda^{<\kappa}$ **and the singular cardinal hypothesis.** Solovay [14], using fine ultrafilters, proved that the size of $\mathcal{P}_\kappa\lambda$ is decided if κ is λ-compact. Here we show that the existence of weakly normal filters is enough to get his results in several cases; we also consider the singular cardinal hypothesis.

THEOREM 2.1. *If λ is regular and there is a weakly normal filter U on $\mathcal{P}_\kappa\lambda$, then $\lambda^{<\kappa} = \lambda \cdot 2^{<\kappa}$.*

We just follow Solovay's argument. For the reader's convenience, we present the complete proof.

A minor observation on weakly normal filters is needed.

LEMMA 2.2. *$\{x : \operatorname{cf}(\sup x) < \kappa\} \in U$ for every weakly normal filter U.*

Proof. We only have to show that $\{x : \sup x \in x\}$ has U-measure 0. Then $\{x : x \text{ is cofinal in } \sup x\} \in U$ and the lemma is proved.

Suppose that $\{x : \sup x \in x\} \in U^+$. Since U is semi-weakly normal, there is a $\gamma < \lambda$ such that $\{x : \sup x \le \gamma\} \in U^+$. Now $\{x : x \subset \gamma + 1\} \in I^+$, contrary to U being fine. ∎

We define a filter D on λ by

$$X \in D \quad \text{iff} \quad X \subset \lambda \text{ and } \{x : \sup x \in X\} \in U\,.$$

LEMMA 2.3. *D is a κ-complete weakly normal filter on λ and $\{\alpha : \operatorname{cf}\alpha < \kappa\} \in D$.*

Proof. It is clear that D is a κ-complete filter. For any $\alpha < \lambda$, $\{x : \sup x \ge \alpha\}$ is a member of U, hence $\{\beta : \beta \ge \alpha\}$ is in D. So D is uniform.

Suppose that $f : \lambda \to \lambda$ is regressive. Define $g : \mathcal{P}_\kappa\lambda \to \lambda$ by $g(x) = f(\sup x)$. Then $g(x) < \sup x$ for every $x \in \mathcal{P}_\kappa\lambda$. Pick a $\gamma < \lambda$ such that $A = \{x : g(x) \le \gamma\} \in U$. Then $B = \{\sup x : x \in A\} \in D$ and $f(\alpha) \le \gamma$ for any $\alpha \in B$. This says that D is weakly normal.

By the previous lemma, $\{x : \mathrm{cf}(\sup x) < \kappa\} \in U$. This obviously yields $\{\alpha < \lambda : \mathrm{cf}\,\alpha < \kappa\} \in D$. ■

Let A_α be a cofinal subset of α whose cardinality is less than κ if $\mathrm{cf}\,\alpha < \kappa$, and $A_\alpha = 0$ otherwise.

Since D is uniform, $X_\eta = \{\alpha : A_\alpha - (\eta+1) \ne \emptyset\} \in D$ for every $\eta < \lambda$. By the weak normality of D, there is an $\eta' < \lambda$ such that $\{\alpha : A_\alpha \cap [\eta, \eta') \ne \emptyset\} \in D$. With this in mind, we can define inductively a sequence $\{\eta_\xi \mid \xi < \lambda\} \subset \lambda$ as follows:

$$\begin{aligned}&\eta_0 = 0,\\ &\eta_\xi = \sup\{\eta_\beta : \beta < \xi\} \quad \text{for } \xi \text{ a limit ordinal},\\ &\eta_{\xi+1} \text{ is chosen so that } \{\alpha : A_\alpha \cap [\eta_\xi, \eta_{\xi+1}) \ne \emptyset\} \in D.\end{aligned}$$

Let $I_\xi = [\eta_\xi, \eta_{\xi+1})$ and $\mathcal{M}_\alpha = \{\xi < \lambda : I_\xi \cap A_\alpha \ne \emptyset\}$. Since I_ξ's are disjoint and $|A_\alpha| < \kappa$, we have $|\mathcal{M}_\alpha| < \kappa$ for every $\alpha < \lambda$. Moreover, for each $\xi < \lambda$, $\{\alpha : A_\alpha \cap I_\xi \ne \emptyset\} = \{\alpha : \xi \in \mathcal{M}_\alpha\} \in D$.

Let $\{x_\zeta : \zeta < \delta\}$ enumerate $x \in \mathcal{P}_\kappa\lambda$. Since D is κ-complete and $|\delta| < \kappa$ and $\{\alpha : x_\zeta \in \mathcal{M}_\alpha\} \in D$ for all $\zeta < \delta$, we have $\{\alpha : x \subset \mathcal{M}_\alpha\} \in D$. Hence $\mathcal{P}_\kappa\lambda = \bigcup\{\mathcal{P}(\mathcal{M}_\alpha) : \alpha < \lambda\}$. Now we have got $\lambda^{<\kappa} = |\mathcal{P}_\kappa\lambda| = \lambda \cdot 2^{<\kappa}$. The proof of Theorem 2.1 is complete. ■

Thus, as seen in [9], the following seems to be the most natural generalization of Solovay's theorem: if λ is regular and there is a precipitous λ-saturated ideal on $\mathcal{P}_\kappa\lambda$ then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.

COROLLARY 2.4. *If $\mathcal{P}_\kappa\lambda$ carries a λ-saturated normal ideal with* $\mathrm{cf}\,\lambda \ge \kappa$, *then* $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.

Proof. In case λ is regular, we can use the above theorem and Corollary 1.5. Suppose that $\mathrm{cf}\,\lambda = \delta$, $\kappa \le \delta < \lambda$, and I is a normal λ-saturated ideal on $\mathcal{P}_\kappa\lambda$. Then I is in fact η-saturated for some regular cardinal $\eta < \lambda$. For each regular cardinal ϱ between η and λ, I_ϱ is also normal η-saturated, hence weakly normal by 1.5. So, $\varrho^{<\kappa} = 2^{<\kappa} \cdot \varrho$.

Since $\lambda^{<\kappa} = \sup\{\varrho^{<\kappa} : \varrho \text{ is a regular cardinal} < \lambda\}$, we get $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$. ■

By a similar argument, we get

PROPOSITION 2.5. *If $\lambda >$* $\mathrm{cf}\,\lambda = \kappa$ *and there is a weakly normal ideal on $\mathcal{P}_\kappa\lambda$, then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.*

P r o o f. Let I be a weakly normal ideal on $\mathcal{P}_\kappa\lambda$. Then I is κ-saturated and I_ϱ is also κ-saturated for all regular ϱ between κ and λ. Hence we can find a weakly normal ideal on $\mathcal{P}_\kappa\varrho$ by Corollary 1.8, and $\varrho^{<\kappa} = 2^{<\kappa} \cdot \varrho$. ■

COROLLARY 2.6. *If there is a* κ^+*-saturated ideal on* $\mathcal{P}_\kappa\lambda$ *and* λ *is a limit cardinal with* $\operatorname{cf}\lambda \geq \kappa$, *then* $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda$.

The assumption of normality in 2.4 may be necessary to produce weakly normal ideals on $\mathcal{P}_\kappa\varrho$ for $\varrho < \lambda$. The author was not able to get a weakly normal ideal on $\mathcal{P}_\kappa\varrho$ from one on $\mathcal{P}_\kappa\lambda$ although we have a weakly normal ideal on $\mathcal{P}_\kappa \operatorname{cf}\lambda$ as seen in 1.9.

If κ is λ-compact and $\operatorname{cf}\lambda < \kappa$, Solovay's theorem says that $\lambda^{<\kappa} = \lambda^+$. We propose a generalization with a somewhat complicated proof. Note that it is easier when κ is inaccessible.

THEOREM 2.7. *Assume that* $\mathcal{P}_\kappa\lambda$ *bears a weakly normal filter and* $\operatorname{cf}\lambda < \kappa$. *Then* $\lambda^{<\kappa} = (\lambda^+)^{<\kappa} = 2^{<\kappa} \cdot \lambda^+$.

P r o o f. Without loss of generality we may assume that $2^{<\kappa} < \lambda$.

Note that $\lambda^{<\kappa} \geq \lambda^+$. Let $\{x_\alpha : \alpha < \lambda^{<\kappa}\}$ be an enumeration of $\mathcal{P}_\kappa\lambda$ and U a weakly normal filter on $\mathcal{P}_\kappa\lambda$. For each $x \in \mathcal{P}_\kappa\lambda$ we define $f(x) = \{\alpha < \lambda^+ : x_\alpha \subset x\}$. Thus, $f(x) \subset \lambda^+$ and $|f(x)| \leq |\mathcal{P}(x)| = 2^{|x|}$. Let δ be the least cardinal such that $2^\alpha < \delta$ for every $\alpha < \kappa$. Since $\operatorname{cf}\lambda < \kappa \leq \operatorname{cf}\delta$ and we have assumed that $2^{<\kappa} < \lambda$, we obtain $\delta < \lambda$.

Now f is a function from $\mathcal{P}_\kappa\lambda$ into $\mathcal{P}_\delta\lambda^+$. In the following, we also use $\operatorname{cf}\delta \geq \kappa$.

Let F be defined by

$$X \in F \quad \text{iff} \quad X \subset \mathcal{P}_\delta\lambda^+ \text{ and } f^{-1}(X) \in U\,.$$

LEMMA 2.8. F *is a* κ*-complete filter with the following properties*:

(i) $\{x : \alpha \in x\} \in F$ *for all* $\alpha < \lambda^+$.

(ii) F *is* $\operatorname{cf}\lambda$*-saturated.*

P r o o f. (i) For every $\alpha < \lambda^+$, $\{x \in \mathcal{P}_\kappa\lambda : x_\alpha \subset x\} \in U$, and $\alpha \in f(x)$ if $x_\alpha \subset x$.

(ii) is clear since U is $\operatorname{cf}\lambda$-saturated. ■

Now we apply Theorem 1.7. Since F is a κ-complete κ-saturated fine filter on $\mathcal{P}_\delta\lambda^+$, F is precipitous. We have a κ-complete weakly normal ideal I on $\mathcal{P}_\delta\lambda^+$ and a κ-complete uniform weakly normal filter D on λ^+ such that $\{\alpha < \lambda^+ : \operatorname{cf}\alpha < \delta\} \in D$ as in the proof of Theorem 2.1. Then we get $\{M_\alpha : \alpha < \lambda^+\}$ such that $|M_\alpha| < \delta$ for all $\alpha < \lambda^+$, and $\mathcal{P}_\kappa\lambda^+ = \bigcup\{\mathcal{P}_\kappa(M_\alpha) : \alpha < \lambda^+\}$.

Hence $(\lambda^+)^{<\kappa} \leq \lambda^+ \cdot \delta^{<\kappa}$.

LEMMA 2.9. $\delta^{<\kappa} = \delta$.

P r o o f. $\delta = 2^{<\kappa}$ or $(2^{<\kappa})^+$. If $\delta = 2^{<\kappa}$, then $\delta^{<\kappa} = (2^{<\kappa})^{<\kappa} = 2^{<\kappa} = \delta$. Otherwise

$$\delta^{<\kappa} = ((2^{<\kappa})^+)^{<\kappa} = (2^{<\kappa})^{<\kappa} \cdot (2^{<\kappa})^+ = 2^{<\kappa} \cdot (2^{<\kappa})^+ = (2^{<\kappa})^+ = \delta\,. \ \blacksquare$$

Now $\lambda^{<\kappa} \le (\lambda^+)^{<\kappa} = \delta^{<\kappa} \cdot \lambda^+ = \delta \cdot \lambda^+ = \lambda^+$. The proof of Theorem 2.7 is complete. ■

Open question. Can one compute $\lambda^{<\kappa}$ if $\kappa < \operatorname{cf} \lambda < \lambda$ and there exists a weakly normal filter on $\mathcal{P}_\kappa\lambda$?

Finally, we consider, normal λ^+-saturated ideals. Before stating the theorem, we need a definition and a lemma.

Definition. Let $\kappa \le \mu \le \nu$. An ideal I on $\mathcal{P}_\kappa\nu$ is *μ-normal* if I is closed under diagonal unions of $<\mu$-sequences, that is, if $\{X_\alpha : \alpha < \eta < \mu\} \subset I$, then $\nabla\{X_\alpha : \alpha < \eta\} = \{x \in \mathcal{P}_\kappa\nu : \exists \alpha \in x\, (x \in X_\alpha)\} \in I$.

Let $\eta(\mu)$ be the least cardinal $\ge \mu$. (μ is not necessarily a cardinal.)

Lemma 2.10. *Assume that $\kappa \le \operatorname{cf} \delta \le \delta \le \operatorname{cf} \nu$ and $\eta(\mu) \le \operatorname{cf} \nu$. Every κ-complete, fine, $\eta(\mu)$-saturated, μ-normal ideal on $\mathcal{P}_\delta\nu$ is precipitous. Hence, if such an ideal exists, then there is a weakly normal ideal on $\mathcal{P}_\delta\nu$.*

P r o o f. It suffices to show such an ideal I has the disjointing property. Let $\{X_\alpha : \alpha < \gamma\}$ be an almost disjoint family. We may assume $\gamma \le \mu$ and $X_\alpha \subset \widehat{\{\alpha\}}$ for all $\alpha < \gamma$. Set $Y_\alpha = X_\alpha - \nabla\{X_\xi \cap X_\alpha : \xi < \alpha\}$. Since $\alpha < \mu$ and I is μ-normal, Y_α is also in I^+. It is routine to show that $\{Y_\alpha : \alpha < \gamma\}$ is a pairwise disjoint family and $(Y_\alpha - X_\alpha) \cup (X_\alpha - Y_\alpha) \in I$. ■

Lemma 2.11. *Suppose that $\lambda < \lambda^{<\kappa}$ and δ is the least cardinal such that $\delta > 2^\alpha$ for all $\alpha < \kappa$. If there is a normal λ^+-saturated ideal on $\mathcal{P}_\kappa\lambda$, then there is a κ-complete, $(\lambda + 1)$-normal, λ^+-saturated, fine ideal on $\mathcal{P}_\delta\lambda^+$.*

P r o o f. Let $\{x_\alpha : \lambda \le \alpha < \lambda^{<\kappa}\}$ be an enumeration of $\mathcal{P}_\kappa\lambda$ and $f(x) = x \cup \{\alpha < \lambda^+ : x_\alpha \subset x\}$ for $x \in \mathcal{P}_\kappa\lambda$. By our assumption $|f(x)| \le 2^{|x|} < \delta$. Hence $f : \mathcal{P}_\kappa\lambda \to \mathcal{P}_\delta\lambda^+$.

Suppose that I is a normal λ^+-saturated ideal on $\mathcal{P}_\kappa\lambda$ and define J by $X \in J$ iff $X \subset \mathcal{P}_\delta\lambda^+$ and $f^{-1}(X) \in I$. Since $f^{-1}(\mathcal{P}_\delta\lambda^+) = \mathcal{P}_\kappa\lambda$, J is a proper κ-complete λ^+-saturated ideal on $\mathcal{P}_\delta\lambda^+$.

If $\alpha < \lambda$, then $\{x \in \mathcal{P}_\kappa\lambda : \alpha \in x\} \in I^*$. For $\lambda \le \alpha < \lambda^+$, $\{x : x_\alpha \subset x\}$ is also in I^*. Hence $\{x : \alpha \in f(x)\} \in I^*$ for all $\alpha < \lambda^+$, which shows J is fine.

Suppose that $\{X_\alpha : \alpha < \lambda\} \subset J$ and $X = \nabla\{X_\alpha : \alpha < \lambda\}$. Then

$$\begin{aligned} f^{-1}(X) &= \{x : f(x) \in X_\alpha \text{ for some } \alpha \in f(x)\} \\ &= \{x : f(x) \in X_\alpha \text{ for some } \alpha \in f(x) \cap \lambda\} \\ &= \{x : f(x) \in X_\alpha \text{ for some } \alpha \in x\} = \nabla\{f^{-1}(X_\alpha) : \alpha < \lambda\} \in I\,. \end{aligned}$$

Thus J is $(\lambda + 1)$-normal. ■

THEOREM 2.12. *If $\mathcal{P}_\kappa\lambda$ carries a normal λ^+-saturated ideal, then $\lambda^{<\kappa} \le (\lambda^+)^{<\kappa} = 2^{<\kappa} \cdot \lambda^+$.*

Proof. Without loss of generality, we may assume that $2^{<\kappa} < \lambda < \lambda^{<\kappa}$. By Lemma 2.11, there is a λ^+-saturated κ-complete $(\lambda+1)$-normal ideal on $\mathcal{P}_\delta\lambda^+$ with $\kappa \le \operatorname{cf}\delta \le \delta \le \lambda^+$. Using Lemma 2.10, we conclude that there is a κ-complete weakly normal ideal on $\mathcal{P}_\delta\lambda^+$.

Note that $\delta^{<\kappa} = \delta$. As an easy application of the $\mathcal{P}_\kappa\lambda^+$ case, we have $(\lambda^+)^{<\kappa} = \lambda^+ \cdot \delta^{<\kappa} = \lambda^+$. Hence $\lambda^{<\kappa} \le 2^{<\kappa} \cdot \lambda^+$. ■

COROLLARY 2.13. *If $\operatorname{cf}\lambda < \kappa$ and there is a normal λ-saturated ideal on $\mathcal{P}_\kappa\lambda$, then $\lambda^{<\kappa} = 2^{<\kappa} \cdot \lambda^+$.*

Note that Matsubara [8] already proved a somewhat stronger form of our theorem: If $\mathcal{P}_\kappa\lambda$ carries a normal λ^+-saturated ideal and GCH holds below κ, then $2^\lambda = \lambda^+$. Furthermore, if this ideal is λ-saturated, then $2^{<\lambda} = \lambda$.

With these results, we consider the singular cardinal hypothesis (SCH): if $2^{\operatorname{cf}\tau} < \tau$, then $\tau^{\operatorname{cf}\tau} = \tau^+$.

Solovay's result [14] is: SCH holds above a strongly compact cardinal. We prove SCH holds in some interval under the existence of weakly normal ideals on $\mathcal{P}_\kappa\lambda$.

THEOREM 2.14. (i) *If $\mathcal{P}_\kappa\lambda$ carries a normal η-saturated ideal and $\eta < \lambda$, then SCH holds between $2^{<\kappa} \cdot \eta$ and λ.*

(ii) *If there is a κ^+-saturated ideal on $\mathcal{P}_k\lambda$, then SCH holds between $2^{<\kappa}$ and λ.*

(iii) *If $\operatorname{cf}\lambda \le \kappa$ and there exists a weakly normal ideal on $\mathcal{P}_\kappa\lambda$, then SCH holds between $2^{<\kappa}$ and λ.*

Proof. In any case, by Silver's results [12], we only have to know that $\delta^{<\kappa} = \delta$ for every regular δ in each interval.

(i) As we have already seen in Corollary 2.4, there is a normal δ-saturated ideal on $\mathcal{P}_\kappa\delta$. So, Theorem 2.1 and Corollary 1.5 work.

(ii) $\mathcal{P}_\kappa\delta$ also carries a κ^+-saturated ideal, and hence, by Corollary 1.8, a weakly normal ideal as well.

(iii) Here, every weakly normal ideal is $\operatorname{cf}\lambda$-saturated and $\operatorname{cf}\lambda \le \kappa$. Thus, this is contained in (ii). ■

Remark. It can also be shown that the combinatorial principle E^η_λ fails for every regular $\eta < \kappa$ if there is a weakly normal filter on $\mathcal{P}_\kappa\lambda$ and λ is regular.

Another weakening of strong compactness which implies the failure of E^η_λ has been found in Johnson [6].

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