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Transference and restriction of maximal multiplier operators on Hardy spaces

by

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Abstract. The aim of this paper is to establish transference and restriction theorems for maximal operators defined by multipliers on the Hardy spaces $H^p(\mathbb{R}^n)$ and $H^p(\mathbb{T}^n)$, $0 < p \le 1$, which generalize the results of Kenig–Tomas for the case $p > 1$. We prove that under a mild regulation condition, an $L^\infty(\mathbb{R}^n)$ function m is a maximal multiplier on $H^p(\mathbb{R}^n)$ if and only if it is a maximal multiplier on $H^p(\mathbb{T}^n)$. As an application, the restriction of maximal multipliers to lower dimensional Hardy spaces is considered.

1. Introduction. Let $H^p(\mathbb{R}^n)$, $0 < p < \infty$, be the Hardy spaces defined as follows (see Fefferman–Stein [3]):

$$H^p(\mathbb{R}^n) = \{f \in S'(\mathbb{R}^n) : \|\sup_{t>0} |\varphi_t * f(x)|\,\|_{L^p(\mathbb{R}^n)} < \infty\},$$

where $\varphi \in S(\mathbb{R}^n)$, $\int \varphi = 1$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$. (For convenience, we fix such a φ with $\operatorname{supp}\varphi \subset \{x : |x| \le 1\}$ once for all in the following.) The corresponding periodic Hardy spaces are

$$H^p(\mathbb{T}^n) = \{f \in S'(\mathbb{T}^n) : \|\sup_{t>0} |\varphi_t * f(x)|\,\|_{L^p(\mathbb{T}^n)} < \infty\},$$

where $\varphi_t * f(x) = \sum_{k\in\mathbb{Z}^n} \widehat{\varphi}(tk)a_k(f)e^{2\pi i k\cdot x}$ and $\widehat{\varphi}(u) = \int_{\mathbb{R}^n} \varphi(x)e^{-2\pi i u\cdot x}\,dx$ is the Fourier transform of φ.

Let $m \in L^\infty(\mathbb{R}^n)$. For each $s > 0$, define

$$(T_s f)^\wedge(u) = m(su)\widehat{f}(u), \qquad f \in L^2(\mathbb{R}^n) \cap H^p(\mathbb{R}^n),$$

and

$$\widetilde{T}_s f(x) = \sum_{k\in\mathbb{Z}^n} m(sk)a_k(f)e^{2\pi i k\cdot x}, \qquad f \in L^2(\mathbb{T}^n) \cap H^p(\mathbb{T}^n).$$

We call m a *maximal multiplier on* $H^p(\mathbb{R}^n)$ if $T^* f(x) = \sup_{s>0} |T_s f(x)|$ can be extended to a bounded operator from $H^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$. Similarly, m is

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called a *maximal multiplier on* $H^p(\mathbb{T}^n)$ if $\widetilde{T}^*f(x)=\sup_{s>0}|\widetilde{T}_sf(x)|$ can be extended to a bounded operator from $H^p(\mathbb{T}^n)$ to $L^p(\mathbb{T}^n)$.

The aim of this paper is to study the relationship between maximal multipliers on $H^p(\mathbb{R}^n)$ and $H^p(\mathbb{T}^n)$, that is, their transference relation as well as the boundedness of their restrictions to lower dimensional Hardy spaces. Results of this type were first established by de Leeuw [6] for L^p-multipliers. Coifman–Weiss [2] studied systematically the transference problem for multiplier and maximal multiplier operators on L^p $(p\ge 1)$ spaces and presented many important applications in analysis. See [1, 7, 8, 11] for recent developments in this respect. The main results of this paper read as follows:

THEOREM 1. *Let* $0<p\le 1$, $m\in L^\infty(\mathbb{R}^n)$.

(i) *Suppose that* m *is a maximal multiplier on* $H^p(\mathbb{R}^n)$ *such that*

(1) $$m \text{ is continuous on } \mathbb{R}^n \text{ and } \lim_{|x|\to\infty} m(x)=\alpha \text{ exists.}$$

Then m *is a maximal multiplier on* $H^p(\mathbb{T}^n)$.

(ii) *If* m *is a maximal multiplier on* $H^p(\mathbb{T}^n)$ *and is continuous on* $\mathbb{R}^n\setminus\{0\}$, *then* m *is a maximal multiplier on* $H^p(\mathbb{R}^n)$.

THEOREM 2. *Let* $0<p\le 1$, *let* $1\le d<n$ *be an integer and let* m *satisfy condition* (1). *Suppose* m *is a maximal multiplier on* $H^p(\mathbb{R}^n)$ $(H^p(\mathbb{T}^n))$. *Then the restriction of* m *to* $\mathbb{R}^d$ *is a maximal multiplier on* $H^p(\mathbb{R}^d)$ $(H^p(\mathbb{T}^d))$.

For the case $p>1$, Kenig–Tomas [5] proved the same results corresponding to Theorems 1 and 2 with an even weaker regulation condition on m. However, their method depends heavily on the duality between L^p and L^q spaces, by means of which the maximal operators were linearized. Since the H^p spaces are not normed linear spaces for $0<p<1$, that method does not apply to the present situation. Here we take a different approach.

For abbreviation, denote always by C a positive constant which may vary at each of its occurrences.

2. Proof of the theorems.

We need some lemmas first.

LEMMA 1. *Let* $0<p<\infty$, $G\in S(\mathbb{R}^n)$ *with* $\operatorname{supp}\widehat{G}\subset\{u\in\mathbb{R}^n:|u|\le 1\}$ *and* $\|G\|_{L^p(\mathbb{R}^n)}=1$. *Writing* $G_{s,p}(x)=s^{-n/p}G(x/s)$, *we have for every* $f\in L^p(\mathbb{T}^n)$,

$$\lim_{s\to\infty}\|fG_{s,p}\|_{L^p(\mathbb{R}^n)}=\|f\|_{L^p(\mathbb{T}^n)}.$$

Proof. Since $f(x+k)=f(x)$ for every $k\in\mathbb{Z}^n$, we have

$$\lim_{s\to\infty}\|fG_{s,p}\|^p_{L^p(\mathbb{R}^n)}=\lim_{s\to\infty}\sum_{k\in\mathbb{Z}^n}\int_{\mathbb{T}^n}|f(x)|^p|G_{s,p}(x+k)|^p\,dx$$
$$=\lim_{s\to\infty}\int_{\mathbb{T}^n}|f(x)|^p\sum_{k\in\mathbb{Z}^n}|G_{s,p}(x+k)|^p\,dx.$$

Note that $G\in S(\mathbb{R}^n)$ and $\lim_{s\to\infty}\sum_{k\in\mathbb{Z}^n}|G_{s,p}(x+k)|^p=\|G\|^p_{L^p(\mathbb{R}^n)}=1$. By the dominated convergence theorem we get

$$\lim_{s\to\infty}\|fG_{s,p}\|^p_{L^p(\mathbb{R}^n)}=\int_{\mathbb{T}^n}|f(x)|^p\,dx,$$

which concludes the proof.

LEMMA 2. *Let* $0<p\le 1$, *and let* G *and* $G_{s,p}$ *be as in Lemma* 1. *Then for any trigonometric polynomial* $f(x)=\sum_k a_ke^{2\pi ik\cdot x}$ *with vanishing constant term, we have*

$$\lim_{s\to\infty}\|fG_{s,p}\|_{H^p(\mathbb{R}^n)}=\|f\|_{H^p(\mathbb{T}^n)}.$$

LEMMA 3. *Let* $0<p\le 1$, $f\in H^p(\mathbb{R}^n)$. *If we write* $f_{t,p}(x)=t^{-n/p}f(x/t)$, $\widetilde{f}(x)=\sum_{k\in\mathbb{Z}^n}\widehat{f}(k)e^{2\pi ik\cdot x}$ *(in* $S'(\mathbb{T}^n)$*), then* $\widetilde{f}_{t,p}\in H^p(\mathbb{T}^n)$ *for each* $t>0$ *and*

$$\lim_{t\to 0_+}\|\widetilde{f}_{t,p}\|_{H^p(\mathbb{T}^n)}=\|f\|_{H^p(\mathbb{R}^n)}.$$

Proof of Lemma 2. By definition,

$$\|fG_{s,p}\|^p_{H^p(\mathbb{R}^n)}=\int_{\mathbb{R}^n}\sup_{t>0}|\varphi_t*(fG_{s,p})(x)|^p\,dx.$$

We see that Lemma 2 follows immediately from the following two equalities:

(2) $$\lim_{s\to\infty}\int_{\mathbb{R}^n}\sup_{t\le\sqrt{s}}|\varphi_t*(fG_{s,p})(x)|^p\,dx=\|f\|^p_{H^p(\mathbb{T}^n)},$$

(3) $$\lim_{s\to\infty}\int_{\mathbb{R}^n}\sup_{t\ge\sqrt{s}}|\varphi_t*(fG_{s,p})(x)|^p\,dx=0.$$

Now, (2) can be further reduced to

(4) $$\lim_{s\to\infty}\int_{\mathbb{R}^n}\sup_{t\le\sqrt{s}}|\varphi_t*f(x)G_{s,p}(x)|^p\,dx=\|f\|^p_{H^p(\mathbb{T}^n)}$$

and

(5) $$\lim_{s\to\infty}\int_{\mathbb{R}^n}\sup_{t\le\sqrt{s}}|\varphi_t*(fG_{s,p})(x)-G_{s,p}(x)(\varphi_t*f)(x)|^p\,dx=0.$$

Observe that $\varphi_t * f(x) = \sum_k a_k \widehat{\varphi}(tk) e^{2\pi i k\cdot x}$ is uniformly bounded in x and t, and

$$\lim_{s\to\infty} \sum_{q\in\mathbb{Z}^n} |G_{s,p}(x+q)|^p \equiv \int_{\mathbb{R}^n} |G(u)|^p\,du = 1\,.$$

Writing $Q_0 = \{x \in \mathbb{R}^n : -1/2 < x_j \le 1/2,\ j = 1,\ldots,n\}$ for the fundamental cube in $\mathbb{R}^n$, from the dominated convergence theorem we get

$$\begin{aligned}\lim_{s\to\infty} \int_{\mathbb{R}^n} \sup_{t\le\sqrt{s}} |\varphi_t * f(x) G_{s,p}(x)|^p\,dx &= \lim_{s\to\infty} \int_{Q_0} \sum_{q\in\mathbb{Z}^n} |G_{s,p}(x+q)|^p \sup_{t\le\sqrt{s}} |\varphi_t * f(x)|^p\,dx \\ &= \int_{Q_0} \sup_{t>0} |\varphi_t * f(x)|^p\,dx = \|f\|^p_{H^p(\mathbb{T}^n)}\,.\end{aligned}$$

Thus (4) is proved. As for (5), noting that

$$\varphi_t * (fG_{s,p})(x) - G_{s,p}(x)(\varphi_t * f)(x) = \sum_k a_k e^{2\pi i k\cdot x} \int_{\mathbb{R}^n} \varphi_t(y) e^{-2\pi i k\cdot y}[G_{s,p}(x-y) - G_{s,p}(x)]\,dy\,,$$

and that f is a polynomial, we see that (5) is equivalent to

(6) $$\lim_{s\to\infty} \int_{\mathbb{R}^n} \sup_{t\le\sqrt{s}} \Big| \int_{\mathbb{R}^n} \varphi_t(y) e^{-2\pi i k\cdot y}[G_{s,p}(x-y) - G_{s,p}(x)]\,dy\Big|^p\,dx = 0$$

for every nonzero $k \in \mathbb{Z}^n$. Write

$$\begin{aligned}I_s &= \int_{\mathbb{R}^n} \sup_{t\le\sqrt{s}} \Big| \int_{\mathbb{R}^n} \varphi_t(y) e^{-2\pi i k\cdot y}[G_{s,p}(x-y) - G_{s,p}(x)]\,dy\Big|^p\,dx \\ &= \int_{|x|\le 2s} + \int_{|x|>2s} = I_s' + I_s''.\end{aligned}$$

Noting that $\operatorname{supp}\varphi \subset \{x : |x| \le 1\}$, we have

$$\begin{aligned}I_s' &\le \sup_{|x|\le 2s,\,|y|\le\sqrt{s}} |G_{s,p}(x-y) - G_{s,p}(x)|^p \int_{|x|\le 2s} \Big[\sup_{t\le\sqrt{s}} \int_{\mathbb{R}^n} |\varphi_t(y)|\,dy\Big]^p\,dx \\ &= \|\varphi\|^p_{L^1(\mathbb{R}^n)} \sup_{|x|\le 2,\,|h|\le 1/\sqrt{s}} |G(x-h) - G(x)|^p \le Cs^{-p/2}\,, \\ I_s'' &\le \int_{|x|>2s} \sup_{t\le\sqrt{s}} \Big[\int_{|y|\le t} |\varphi_t(y)| \cdot |\nabla G_{s,p}(x-\theta y)| \cdot |y|\,dy\Big]^p\,dx \\ &\le \int_{|x|>2s} \sup_{t\le\sqrt{s}} \Big[\frac{t}{s} \sup_{|y|\le\sqrt{s}} |(\nabla G)_{s,p}(x-\theta y)| \int_{|y|\le t} |\varphi_t(y)| \frac{|y|}{t}\,dy\Big]^p\,dx\,.\end{aligned}$$

Since $G \in S(\mathbb{R}^n)$, it is possible to choose a constant C such that $|\nabla G(x)| \le C|x|^{-(n+1)/p}$. Thus,

$$I_s'' \le Cs^{-p/2} s^{-n} \int_{|x|>2s} \Big|\frac{x}{s}\Big|^{-n-1}\,dx \cdot \|\varphi(y)|y|\,\|^p_{L^1(\mathbb{R}^n)} \le Cs^{-p/2}\,.$$

It follows that $\lim_{s\to\infty} I_s = 0$ and this is exactly (6).

Up to now, we have proved (2). So our next task is to prove (3). It is easy to see that we need only prove the following equality for every nonzero $k \in \mathbb{Z}^n$:

(7) $$\lim_{s\to\infty} \int_{\mathbb{R}^n} \sup_{t>\sqrt{s}} |\varphi_t * (G_{s,p} e^{2\pi i k\cdot})(x)|^p\,dx = 0\,.$$

Observe that

(8) $$\int_{|x|\le 2\sqrt{s}} \sup_{t>\sqrt{s}} |\varphi_t * (G_{s,p} e^{2\pi i k\cdot})(x)|^p\,dx \le \|\varphi\|^p_{L^1(\mathbb{R}^n)} \|G_{s,p}\|^p_{L^\infty(\mathbb{R}^n)} \int_{|x|\le 2\sqrt{s}} dx \le Cs^{-n/2}\,.$$

We still need to consider the case $|x| > 2\sqrt{s}$. First, suppose that $\sqrt{s} < t \le |x|/2$. By assumption, φ has compact support and $k \ne 0$. Assume that $k_1 \ne 0$ without loss of generality. Using a technique from oscillatory integral theory, we have

$$\begin{aligned}A_s(x) &= \int_{\mathbb{R}^n} \varphi_t(y) G_{s,p}(x-y) e^{2\pi i k\cdot(x-y)}\,dy \\ &= e^{2\pi i k\cdot x} \int \varphi(y) G_{s,p}(x-ty) e^{-2\pi i k\cdot ty}\,dy \\ &= e^{2\pi i k\cdot x}(-2\pi i k_1 t)^{-1} \int \varphi(y) G_{s,p}(x-ty) \frac{\partial}{\partial y_1} e^{-2\pi i k\cdot ty}\,dy \\ &= e^{2\pi i k\cdot x}(2\pi i k_1 t)^{-1} \int \Big\{\frac{\partial}{\partial y_1}\varphi(y) G_{s,p}(x-ty) \\ &\qquad - \frac{t}{s}\varphi(y)\Big(\frac{\partial}{\partial y_1}G\Big)_{s,p}(x-ty)\Big\} e^{-2\pi i t k\cdot y}\,dy\,.\end{aligned}$$

Choose a constant C such that

$$|G(x)| + \Big|\frac{\partial}{\partial y_1}G(x)\Big| \le C(1+|x|)^{-(n+1)/p}\,.$$

Then

$$|A_s(x)| \le \frac{C}{t} s^{-n/p} \int_{|y|\le 1} \Big\{\Big|\frac{\partial}{\partial y_1}\varphi(y)\Big| + \frac{t}{s}|\varphi(y)|\Big\}\Big(1 + \frac{|x-ty|}{s}\Big)^{-(n+1)/p}\,dy\,.$$

But $\sqrt{s} < t \le |x|/2$, $|y| \le 1$, and we see that $|x - ty| \ge |x|/2$. Thus,

$$|A_s(x)| \le C\left(\left\|\frac{\partial}{\partial y_1}\varphi\right\|_{L^1} + \|\varphi\|_{L^1}\right)s^{-n/p-1/2}\left(1 + \frac{1}{2}\left|\frac{x}{s}\right|\right)^{-(n+1)/p}.$$

It follows that

$$(9) \qquad \int\limits_{|x|>2\sqrt{s}} \sup_{\sqrt{s}<t\le|x|/2} |A_s(x)|^p\,dx \le Cs^{-p/2}.$$

Next, assume that $t > |x|/2 > \sqrt{s}$. Arguing similarly (assuming that $k_1 \neq 0$), we have for any positive integer N,

$$\begin{aligned} A_s(x) &= \int \varphi_t(x-y)G_{s,p}(y)e^{2\pi ik\cdot y}\,dy \\ &= t^{-n}s^{n-n/p}\int \varphi\left(\frac{x-sy}{t}\right)G(y)e^{2\pi isk\cdot y}\,dy \\ &= t^{-n}s^{n-n/p}(-2\pi isk_1)^{-N} \\ &\quad\times \int \left[\left(\frac{\partial}{\partial y_1}\right)^N\left\{\varphi\left(\frac{x-sy}{t}\right)G(y)\right\}\right]e^{2\pi isk\cdot y}\,dy \\ &= t^{-n}s^{n-n/p}(-2\pi isk_1)^{-N}\sum_{q=0}^{N}\binom{N}{q}\left(-\frac{s}{t}\right)^q I_q, \end{aligned}$$

where

$$I_q = \int \left(\frac{\partial}{\partial y_1}\right)^{N-q} G(y)\cdot\left(\left(\frac{\partial}{\partial y_1}\right)^q\varphi\right)\left(\frac{x-sy}{t}\right)e^{2\pi isk\cdot y}\,dy.$$

Since $G \in S(\mathbb{R}^n)$, $\operatorname{supp}\widehat{G} \subset \{u : |u| \le 1\}$, and $|k| \ge 1$, it follows that for any $\alpha, \beta \in \mathbb{Z}_+^n$ and $s \ge 1$,

$$(-2\pi i)^{|\alpha|}\int\limits_{\mathbb{R}^n} y^\alpha D^\beta G(y)e^{2\pi isk\cdot y}\,dy = (2\pi i)^{|\beta|}D^\alpha(u^\beta\widehat{G}(u))\big|_{u=-sk} = 0.$$

Therefore, if we set $P_j(x/t,h)$ to be the Taylor polynomial of $(\partial/\partial y_1)^q\varphi$ of order j at the point x/t, then

$$\begin{aligned} I_q &= \int\limits_{\mathbb{R}^n}\left(\frac{\partial}{\partial y_1}\right)^{N-q}G(y) \\ &\quad\times\left\{\left(\frac{\partial}{\partial y_1}\right)^q\varphi\left(\frac{x-sy}{t}\right) - P_{N-q}\left(\frac{x}{t},\frac{sy}{t}\right)\right\}e^{2\pi isk\cdot y}\,dy \\ &= \int\limits_{\mathbb{R}^n}\left(\frac{\partial}{\partial y_1}\right)^{N-q}G(y) \end{aligned}$$

$$\times \sum_{|\alpha|=N-q} C_\alpha y^\alpha\left(D^\alpha\left(\frac{\partial}{\partial y_1}\right)^q\varphi\right)\left(\frac{x-sy\theta}{t}\right)\left(-\frac{s}{t}\right)^{N-q}e^{2\pi isk\cdot y}\,dy.$$

It follows that

$$\begin{aligned} |A_s(x)| &\le (2\pi sk_1)^{-N}t^{-n}s^{n-n/p}\left(\frac{s}{t}\right)^N\sum_{q=0}^{N}\binom{N}{q} \\ &\quad\times\int |y|^{N-q}\left|\left(\frac{\partial}{\partial y_1}\right)^{N-q}G(y)\right|dy\cdot\sum_{|\alpha|=N}C_\alpha\|D^\alpha\varphi\|_{L_\infty(\mathbb{R}^n)} \\ &\le Ct^{-N-n} \le C|x|^{-N-n}. \end{aligned}$$

Choosing N such that $N + n \ge (n+1)/p$, we have

$$(10) \qquad \int\limits_{|x|>2\sqrt{s}} \sup_{t>|x|/2>\sqrt{s}} |A_s(x)|^p\,dx \le C\int\limits_{|x|>2\sqrt{s}} |x|^{-n-1}\,dx = Cs^{-1/2}.$$

From (8)–(10), we finally get

$$\begin{aligned} \int\limits_{\mathbb{R}^n}\sup_{t>\sqrt{s}}|A_s(x)|^p\,dx &\le \int\limits_{|x|\le 2\sqrt{s}}\sup_{t>\sqrt{s}}|A_s(x)|^p\,dx \\ &\quad+ \int\limits_{|x|>2\sqrt{s}}\sup_{\sqrt{s}<t\le|x|/2}|A_s(x)|^p\,dx \\ &\quad+ \int\limits_{|x|>2\sqrt{s}}\sup_{t>|x|/2}|A_s(x)|^p\,dx \\ &\le Cs^{-p/2} \quad (s \ge 1). \end{aligned}$$

That is,

$$\lim_{s\to\infty}\int\limits_{\mathbb{R}^n}\sup_{t>\sqrt{s}}|A_s(x)|^p\,dx = 0,$$

and (7) follows immediately. The lemma is thus proved.

Proof of Lemma 3. For $f \in H^p(\mathbb{R}^n)$, we see that $|\widehat{f}(y)| \le c|y|^{n(1/p-1)}$ (see [9]) and $\widetilde{f}(x) = \sum_{k\in\mathbb{Z}^n}\widehat{f}(k)e^{2\pi ik\cdot x}$ defines a periodic distribution on $\mathbb{R}^n$. On the other hand, $f_{t,p} \in H^p(\mathbb{R}^n)$ and $\|f_{t,p}\|_{H^p(\mathbb{R}^n)} = \|f\|_{H^p(\mathbb{R}^n)}$ for each $t > 0$. Observing that $\varphi \in S(\mathbb{R}^n)$, we have $\varphi_y * f_{t,p} \in L^1(\mathbb{R}^n)$. It follows from the Poisson summation formula that

$$(11) \qquad \varphi_y * \widetilde{f}_{t,p}(x) = \sum_{k\in\mathbb{Z}^n}\varphi_y * f_{t,p}(x+k) = \sum_{k\in\mathbb{Z}^n}(\varphi_{y/t} * f)_{t,p}(x+k).$$

Since $f \in H^p(\mathbb{R}^n)$ and

$$\lim_{t\to 0_+} \int_{\mathbb{T}^n} \sum_{k\neq 0} \sup_{y>0} |(\varphi_{y/t} * f)_{t,p}(x+k)|^p\,dx = \lim_{t\to 0_+} \int_{\mathbb{R}^n\setminus t^{-1}Q_0} \sup_{y>0} |\varphi_y * f(x)|^p\,dx = 0\,.$$

It follows from (11) that

$$\begin{aligned}\lim_{t\to 0_+} \|\widetilde{f}_{t,p}\|^p_{H^p(\mathbb{T}^n)} &= \lim_{t\to 0_+} \int_{\mathbb{T}^n} \sup_{y>0} |\varphi_y * \widetilde{f}_{t,p}(x)|^p\,dx \\ &= \lim_{t\to 0_+} \int_{\mathbb{T}^n} \sup_{y>0} |(\varphi_{y/t} * f)_{t,p}(x)|^p\,dx \\ &= \lim_{t\to 0_+} \int_{t^{-1}Q_0} \sup_{y>0} |\varphi_y * f(x)|^p\,dx = \|f\|^p_{H^p(\mathbb{R}^n)}\,.\end{aligned}$$

Thus the proof of the lemma is complete.

LEMMA 4. *Let $0<p<\infty, d\in\mathbb{Z}$ and $g\in S'(\mathbb{R}^n)$ with $\operatorname{supp}\widehat{g}\subset\{u\in\mathbb{R}^n : |u|\le 2^{1-d}\}$. If we write $Q_{d,q}=\{x\in\mathbb{R}^n : q_j2^d<x_j\le(q_j+1)2^d,\ j=1,\dots,n\}$, $q=(q_1,\dots,q_n)\in\mathbb{Z}^n$, then there exists a constant C depending only on p and n such that*

$$2^{nd}\sup_{x\in Q_{d,q}} |g(x)|^p \le C\sum_{v\in\mathbb{Z}^n}(1+|v|)^{-n-1}\int_{Q_{d,v+q}} |g(x)|^p\,dx\,. \tag{12}$$

Lemma 4, which belongs to Plancherel–Pólya [10], reflects the deep property of entire functions of exponential type. Its proof can be found, e.g., in [4, Lemma 2.4].

Proof of Theorem 1. (i) Suppose now that m is a maximal multiplier on $H^p(\mathbb{R}^n)$. Since the class of trigonometric polynomials forms a dense subset of $H^p(\mathbb{T}^n)$ and $L^p(\mathbb{T}^n)$, we need only prove that for every trigonometric polynomial $f(x)=\sum_k a_k e^{2\pi i k\cdot x}$,

$$\|\widetilde{T}^* f\|_{L^p(\mathbb{T}^n)} \le C\|f\|_{H^p(\mathbb{T}^n)}\,. \tag{13}$$

Note that the constant term

$$|a_0(f)| = \Big|\int_{\mathbb{T}^n} f(x)\,dx\Big| \le \|f\|_{H^p(\mathbb{T}^n)}$$

and

$$\widetilde{T}^* a_0(f) = |m(0)a_0(f)| \le |m(0)|\,\|f\|_{H^p(\mathbb{T}^n)}\,.$$

One sees that it is sufficient to prove (13) for every trigonometric polynomial with vanishing constant term. By Lemma 1,

$$\begin{aligned}\int_{\mathbb{T}^n} [\widetilde{T}^* f(x)]^p\,dx &= \lim_{t\to\infty}\int_{\mathbb{R}^n} \sup_{s>0} |\widetilde{T}_s f(x)|^p |G_{t,p}(x)|^p\,dx \\ &\le \lim_{t\to\infty}\int_{\mathbb{R}^n} \sup_{s>0} |T_s(fG_{t,p})(x) - \widetilde{T}_s f(x) G_{t,p}(x)|^p\,dx \\ &\quad + \lim_{t\to\infty}\int_{\mathbb{R}^n} \sup_{s>0} |T_s(fG_{t,p})(x)|^p\,dx\,.\end{aligned}$$

Since m is a maximal multiplier on $H^p(\mathbb{R}^n)$, it follows from Lemma 2 that

$$\begin{aligned}\lim_{t\to\infty}\int_{\mathbb{R}^n} \sup_{s>0} |T_s(fG_{t,p})(x)|^p\,dx &= \lim_{t\to\infty}\|T^*(fG_{t,p})\|^p_{L^p(\mathbb{R}^n)} \\ &\le \lim_{t\to\infty}\|T^*\|^p\cdot\|fG_{t,p}\|_{H^p(\mathbb{R}^n)} \\ &= \|T^*\|^p\cdot\|f\|^p_{H^p(\mathbb{T}^n)}\,,\end{aligned}$$

where $\|T^*\|$ denotes the $(H^p(\mathbb{R}^n), L^p(\mathbb{R}^n))$ norm of T^*. Thus, in order to prove (13), one needs only prove that

$$\lim_{t\to\infty}\int_{\mathbb{R}^n} \sup_{s>0} |T_s(fG_{t,p})(x) - (\widetilde{T}_s f(x))G_{t,p}(x)|^p\,dx = 0\,. \tag{14}$$

By assumption, f is a trigonometric polynomial with vanishing constant term. Therefore, one may take $f(x)=e^{2\pi i k\cdot x}$ in (14) for arbitrary nonzero $k\in\mathbb{Z}^n$. It is obvious that

$$\begin{aligned}F_t(x) &= \sup_{s>0} |T_s(G_{t,p}e^{2\pi i k\cdot})(x) - m(sk)e^{2\pi i k\cdot x}G_{t,p}(x)| \\ &= \sup_{s>0}\Big|\int_{\mathbb{R}^n} [m(su)-m(sk)]\widehat{G}_{t,p}(u-k)e^{2\pi i u\cdot x}\,du\Big| \\ &\le Ct^{n(1-1/p)}\sup_{s>0}\int_{|h|\le 1/t} |m(sk+sh)-m(sk)|\,dh\,.\end{aligned}$$

By assumption (1), it is not difficult to verify that for every $k\neq 0$,

$$\varepsilon(t) = \sup_{s>0} t^n \int_{|h|\le 1/t} |m(sk+sh)-m(sk)|\,dh \to 0 \tag{15}$$

as t tends to infinity. Putting $r(t)=\varepsilon(t)^{-(n+1)/p}$, one has $\lim_{t\to\infty} r(t)=\infty$. It follows that

$$\int_{x\in tr(t)Q_0} F_t(x)^p\,dx \le Ct^{-n}r(t)^{-n-1}|tr(t)Q_0| = Cr(t)^{-1}\to 0 \quad (t\to\infty)$$

where $Q_0=\{x\in\mathbb{R}^n : |x_j|\le 1/2,\ j=1,\dots,n\}$ is the fundamental cube in $\mathbb{R}^n$.

Next, we must estimate $\int_{\mathbb{R}^n\setminus tr(t)Q_0} F_t(x)^p\,dx$. We have

$$F_t(x) \le \sup_{s>0} |T_s(G_{t,p}e^{2\pi ik\cdot})(x)| + \|m\|_{L^\infty}|G_{t,p}(x)|$$

and

$$\int\limits_{\mathbb{R}^n\setminus tr(t)Q_0} |G_{t,p}(x)|^p\,dx = t^{-n}\int\limits_{x\notin tr(t)Q_0} |G(x/t)|^p\,dx$$
$$= \int\limits_{x\notin r(t)Q_0} |G(x)|^p\,dx \to 0 \quad (t\to\infty)\,.$$

It follows that one needs only prove that

$$\lim_{t\to\infty}\int\limits_{x\notin tr(t)Q_0}\sup_{s>0}|T_s(G_{t,p}e^{2\pi ik\cdot})(x)|^p\,dx = 0\,. \tag{16}$$

Choose $\psi\in S(\mathbb{R}^n)$ such that $\operatorname{supp}\widehat{\psi}\subset\{u\in\mathbb{R}^n : |u-k|\le 1/2\}$ and $\widehat{\psi}(u)=1$ for $|u-k|\le 1/4$. Obviously $\psi\in H^p(\mathbb{R}^n)$ and when $t>4$,

$$T^*(G_{t,p}e^{2\pi ik\cdot})(x) = \sup_{s>0}|T_s(\psi)*(G_{t,p}e^{2\pi ik\cdot})(x)|\,.$$

Thus following the notation in Lemma 4, one has

$$\int\limits_{x\notin tr(t)Q_0} T^*(G_{t,p}e^{2\pi ik\cdot})^p(x)\,dx$$
$$\le \sum_{v\notin tr(t)Q_0}\sup_{x\in Q_{0,v}} T^*(G_{t,p}e^{2\pi ik\cdot})^p(x)$$
$$\le \sum_{v\notin tr(t)Q_0}\sup_{x\in Q_{0,v}}\sup_{s>0}\sum_{q\in\mathbb{Z}^n}\sup_{y\in Q_{0,q}}|T_s(\psi)(x-y)|^p\sup_{y\in Q_{0,q}}|G_{t,p}(y)|^p$$
$$\le \sum_{q\in\mathbb{Z}^n}\sum_{v\notin tr(t)Q_0}\sup_{s>0}\sup_{y\in Q_{1,v-q}}|T_s(\psi)(y)|^p\sup_{y\in Q_{0,q}}|G_{t,p}(y)|^p$$
$$= \sum_{q\in\frac12 tr(t)Q_0} + \sum_{q\notin\frac12 tr(t)Q_0} = \sum\nolimits_1 + \sum\nolimits_2$$

where

$$\sum\nolimits_2 \le \sum_{q\notin\frac12 tr(t)Q_0}\sup_{y\in Q_{0,q}}|G_{t,p}(y)|^p\cdot\sum_{v\in\mathbb{Z}^n}\sup_{y\in Q_{1,v}}|T_s(\psi)(y)|^p\,.$$

Since $\operatorname{supp}\widehat{G}_{t,p}\subset\{u\in\mathbb{R}^n : |u|\le 1/t\}$, $\operatorname{supp}[T_s(\psi)e^{2\pi ik\cdot}]^\wedge\subset\{u\in\mathbb{R}^n : |u|\le 1/2\}$ and $|T_s(\psi)(y)e^{2\pi ik\cdot y}| = |T_s(\psi)(y)|$, Lemma 4 is applicable, from which it follows that

$$\sum_{v\in\mathbb{Z}^n}\sup_{s>0}\sup_{y\in Q_{1,v}}|T_s(\psi)(y)|^p$$
$$\le C\sum_{v\in\mathbb{Z}^n}\sup_{s>0}\sum_{j\in\mathbb{Z}^n}(1+|j|)^{-n-1}\int\limits_{Q_{1,v+j}}|T_s(\psi)(y)|^p\,dy$$
$$\le C\sum_{j\in\mathbb{Z}^n}(1+|j|)^{-n-1}\sum_{v\in\mathbb{Z}^n}\int\limits_{Q_{1,v}}T^*(\psi)(y)^p\,dy$$
$$\le C\int\limits_{\mathbb{R}^n}T^*(\psi)(y)^p\,dy \le C\|T^*\|^p\cdot\|\psi\|^p_{H^p(\mathbb{R}^n)}\,.$$

Using (12) once more, one has

$$\sum\nolimits_2 \le C\sum_{q\notin\frac12 tr(t)Q_0}\sum_{j\in\mathbb{Z}^n}(1+|j|)^{-n-1}\int\limits_{Q_{0,q+j}}|G_{t,p}(y)|^p\,dy$$
$$\le C\sum_{j\notin\frac14 tr(t)Q_0}(1+|j|)^{-n-1}\int\limits_{\mathbb{R}^n}|G_{t,p}(y)|^p\,dy$$
$$+ C\sum_{j\in\frac14 tr(t)Q_0}(1+|j|)^{-n-1}\sum_{q\notin\frac12 tr(t)Q_0}\int\limits_{Q_{0,q+j}}|G_{t,p}(y)|^p\,dy$$
$$\le C\|G_{t,p}\|^p_{L^p(\mathbb{R}^n)}\sum_{j\notin\frac14 tr(t)Q_0}(1+|j|)^{-n-1} + C\int\limits_{|x|\ge\frac18 tr(t)}|G_{t,p}(y)|^p\,dy$$
$$\le C(tr(t))^{-1} + C\int\limits_{|x|\ge r(t)/8}|G(y)|^p\,dy \to 0 \quad (t\to\infty)\,.$$

Similarly, for $\sum_1$ it follows from Lemma 4 that

$$\sum\nolimits_1 \le C\|G_{t,p}\|^p_{L^p(\mathbb{R}^n)}\sum_{v\notin\frac12 tr(t)Q_0}\sup_{s>0}\sup_{y\in Q_{1,v}}|T_s(\psi)(y)|^p$$
$$\le C\sum_{v\notin\frac12 tr(t)Q_0}\sup_{s>0}\sum_{q\in\mathbb{Z}^n}(1+|q|)^{-n-1}\int\limits_{Q_{1,v+q}}|T_s(\psi)(y)|^p\,dy$$
$$\le C\sum_{q\notin\frac14 tr(t)Q_0}(1+|q|)^{-n-1}\int\limits_{\mathbb{R}^n}T^*(\psi)(y)^p\,dy$$
$$+ C\sum_{q\in\frac14 tr(t)Q_0}(1+|q|)^{-n-1}\sum_{v\notin\frac14 tr(t)Q_0}\int\limits_{Q_{1,v}}T^*(\psi)(y)^p\,dy$$
$$\le C\|T^*\|^p\|\psi\|^p_{H^p(\mathbb{R}^n)}(tr(t))^{-1} + C\int\limits_{|x|>tr(t)/8}T^*(\psi)(y)^p\,dy\,.$$

The last expression tends to zero as t tends to infinity since $T^*(\psi)$ is a proper function in $L^p(\mathbb{R}^n)$, which concludes the proof of equality (16).

(ii) Next, we suppose that m is a maximal multiplier on $H^p(\mathbb{T}^n)$, and is continuous on $\mathbb{R}^n\setminus\{0\}$. We must show that m is a maximal multiplier on $H^p(\mathbb{R}^n)$. Assume $f \in H^p(\mathbb{R}^n)$ such that supp $\widehat{f}$ is compact and does not contain the origin. Then $m(su)\widehat{f}(u)$ is continuous on $\mathbb{R}^n$ and has compact support for each $s > 0$. It follows that

$$T_s f(x) = \lim_{t\to 0_+} t^n \sum_{k\in\mathbb{Z}^n} m(stk)\widehat{f}(tk)e^{2\pi i tk\cdot x}$$

because the right hand side of the above equality is exactly the Riemannian sum of $m(su)\widehat{f}(u)e^{2\pi i u\cdot x}$. Thus, for any $x \in \mathbb{R}^n$,

$$\begin{aligned}\sup_{s>0}|T_s f(x)|^p &= \sup_{s>0}\lim_{t\to 0_+} t^{np}\Big|\sum_{k\in\mathbb{Z}^n} m(stk)\widehat{f}(tk)e^{2\pi i tk\cdot x}\Big|^p\\ &= \sup_{s>0}\lim_{t\to 0_+} t^n|\widetilde{T}_{st}(\widetilde{f}_{t,p})(tx)|^p\\ &= \sup_{s>0}\lim_{t\to 0_+} t^n|\widetilde{T}_{st}(\widetilde{f}_{t,p})(tx)|^p\chi_{Q_0}(tx),\end{aligned}$$

where $\widetilde{f}_{t,p}$ is in Lemma 3 and χ_{Q_0} denotes the characteristic function of Q_0. By the Fatou Lemma we have

$$\begin{aligned}\int_{\mathbb{R}^n} T^*f(x)^p\,dx &= \int_{\mathbb{R}^n}\sup_{s>0}\lim_{t\to 0_+} t^n|\widetilde{T}_{st}(\widetilde{f}_{t,p})(tx)|^p\chi_{Q_0}(tx)\,dx\\ &\le \int_{\mathbb{R}^n}\liminf_{t\to 0_+}\sup_{s>0} t^n|\widetilde{T}_{st}(\widetilde{f}_{t,p})(tx)|^p\chi_{Q_0}(tx)\,dx\\ &\le \liminf_{t\to 0_+}\int_{\mathbb{R}^n} t^n[\widetilde{T}^*(\widetilde{f}_{t,p})(tx)]^p\chi_{Q_0}(tx)\,dx\\ &= \liminf_{t\to 0_+}\int_{\mathbb{T}^n}\widetilde{T}^*(\widetilde{f}_{t,p})(x)^p\,dx\\ &\le \liminf_{t\to 0_+}\|\widetilde{T}^*\|^p\|\widetilde{f}_{t,p}\|^p_{H^p(\mathbb{T}^n)} = \|\widetilde{T}^*\|^p\|f\|^p_{H^p(\mathbb{R}^n)},\end{aligned}$$

where the last equality is derived from Lemma 3.

Finally, note that the functions in $H^p(\mathbb{R}^n)$ whose Fourier transforms have compact supports away from the origin form a dense subset of $H^p(\mathbb{R}^n)$. It follows that m is a maximal multiplier on $H^p(\mathbb{R}^n)$.

The proof is complete.

Remark 1. By the Riesz transform characterization of $H^p(\mathbb{R}^n)$ norms [9], it is easy to show that a maximal multiplier on $H^p(\mathbb{R}^n)$ is also a multiplier on $H^p(\mathbb{R}^n)$ and is therefore continuous on $\mathbb{R}^n\setminus\{0\}$ (see [12, p. 137]). It follows that the requirement $m \in C(\mathbb{R}^n\setminus\{0\})$ in Theorem 1(ii) is necessary.

Remark 2. From the proof of Theorem 1(i) we see that condition (1) can be replaced by the following weaker one. For every nonzero $k \in \mathbb{Z}^n$,

$$\lim_{t\to\infty}\sup_{s>0} s^{-n}\int_{|h|\le s}|m(stk+h)-m(stk)|\,dh = 0. \tag{17}$$

Thus if m satisfies an inequality of Mikhlin type: $|D^\alpha m(x)| \le C|x|^{-1}$, $|\alpha| = 1$, then m satisfies (17). Condition (17) is essentially a requirement that m does not vary too fast at infinity and about the origin. The authors do not know if it can be weakened any further.

As an application of Theorem 1, we can now give a simple proof of Theorem 2.

Proof of Theorem 2. By Theorem 1, we need only prove the periodic case. Let m be a maximal multiplier on $H^p(\mathbb{T}^n)$. It follows from the compactness of $\mathbb{T}^n$ that for any $f(u) \in H^p(\mathbb{T}^d)$, if we set $F(x) = F(u,v) = f(u)$, $x = (u,v) \in \mathbb{T}^d\times\mathbb{T}^{n-d}$, then $F \in H^p(\mathbb{T}^n)$ and $\|F\|_{H^p(\mathbb{T}^n)} = \|f\|_{H^p(\mathbb{T}^d)}$. Let $m'(u) = m(u,0)$ be the restriction of m to $\mathbb{T}^d$ and T'^* the maximal operator corresponding to m'. Then for any $(u,v) \in \mathbb{T}^d\times\mathbb{T}^{n-d}$,

$$T'^*f(u) = T^*F(u,v).$$

Therefore,

$$\begin{aligned}\|T'^*f\|_{L^p(\mathbb{T}^d)} &= \|T^*F\|_{L^p(\mathbb{T}^n)} \le \|T^*\|\cdot\|F\|_{H^p(\mathbb{T}^n)}\\ &= \|T^*\|\cdot\|f\|_{H^p(\mathbb{T}^d)},\end{aligned}$$

and the desired result follows. This concludes the proof of Theorem 2.

Remark 3. From the proof above we see that for the periodic case of Theorem 2, m is not required to satisfy (1).

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Construction de p-multiplicateurs

by

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Abstract. Using characteristic functions of polyhedra, we construct radial p-multipliers which are continuous over $\mathbb{R}^n$ but not continuously differentiable through S^{n-1} and give a p-multiplier criterion for homogeneous functions over $\mathbb{R}^2$. We also exhibit fractal p-multipliers over the real line.

1. Introduction. Les critères généraux permettant de décider si une fonction bornée sur $\mathbb{R}^n$ est un p-multiplicateur ne s'appliquent le plus souvent qu'à des fonctions suffisamment dérivables ou lipschitziennes (voir [St], chap. IV). Or, comme le remarque Stein lui-même, de tels critères ne permettent même pas de traiter le cas simple des fonctions indicatrices de polyèdres.

Aussi avons-nous choisi dans cet article une démarche «constructiviste». On sait que, si une suite de p-multiplicateurs dont la norme est uniformément bornée converge presque partout, alors sa limite est aussi un p-multiplicateur. A l'aide de ce résultat, nous montrons comment, à partir de fonctions indicatrices de polyèdres, on peut construire des exemples de p-multiplicateurs de $\mathbb{R}^2$ continus mais non dérivables partout; nous établissons un critère assez général pour les fonctions homogènes sur $\mathbb{R}^2$; nous exhibons des p-multiplicateurs sur $\mathbb{R}$ continus mais nulle part dérivables, ce qui met en évidence les limites de la caractérisation due à Stechkin ([EG], p. 105).

2. Définitions, notations. Nous désignerons par p un nombre réel > 1 *quelconque* fixé.

Soient G un groupe abélien localement compact, dx une mesure de Haar sur G, $\widehat{G}$ le groupe dual de G; la transformée de Fourier d'une fonction f dans $L^1(G)$ est définie pour $\gamma \in \widehat{G}$ comme

$$\widehat{f}(\gamma) = \mathcal{F}f(\gamma) = \int_G f(x)\overline{\gamma(x)}\,dx\,.$$

On munit $\widehat{G}$ de la mesure de Haar $d\gamma$ permettant d'écrire $\|\widehat{f}\|_2 = \|f\|_2$

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