[St] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

INSTITUT DE MATHÉMATIQUES
UNIVERSITÉ DE LAUSANNE
CH-1015 LAUSANNE, SUISSE

Received April 7, 1992 (2925)

STUDIA MATHEMATICA 105 (2) (1993)

Two characterizations of automorphisms on $B(X)$

by

PETER ŠEMRL (Ljubljana)

Abstract. Let X be an infinite-dimensional Banach space, and let ϕ be a surjective linear map on $B(X)$ with $\phi(I) = I$. If ϕ preserves injective operators in both directions then ϕ is an automorphism of the algebra $B(X)$. If X is a Hilbert space, then ϕ is an automorphism of $B(X)$ if and only if it preserves surjective operators in both directions.

Let X be an infinite-dimensional Banach space and let $B(X)$ denote the algebra of all bounded linear operators on X. Recall that the *point spectrum* $\sigma_p(T)$ of an operator $T \in B(X)$ is the set of all eigenvalues of T. The *surjectivity spectrum* $\sigma_s(T)$ of an operator $T \in B(X)$ is defined as $\sigma_s(T) = \{\lambda \in \mathbb{C} : (T - \lambda)X \neq X\}$. For basic facts concerning the surjectivity spectrum we refer the reader to [7]. It should be mentioned that the surjectivity spectrum has been called other things by other authors, e.g. approximate defect spectrum in [3].

We shall consider linear mappings ϕ on $B(X)$ which satisfy $\phi(I) = I$ and preserve injective operators in both directions: i.e., for every $T \in B(X)$ the operator $\phi(T)$ is injective if and only if T is injective. It is easy to see that such mappings preserve the point spectrum, that is, $\sigma_p(T)$ equals $\sigma_p(\phi(T))$ for every $T \in B(X)$. Conversely, if $\phi : B(X) \to B(X)$ is a point spectrum preserving surjective linear mapping, then one can prove using the same approach as in [5, Lemma 3] that $\phi(I) = I$. Moreover, such a ϕ preserves injective operators in both directions. We shall prove that for every point spectrum preserving surjective linear mapping $\phi : B(X) \to B(X)$ there exists a bounded invertible linear operator $A : X \to X$ such that $\phi(T) = ATA^{-1}$ for every $T \in B(X)$. Hence, we will show that a surjective linear mapping ϕ on $B(X)$ is an automorphism if and only if ϕ preserves injective operators in both directions and satisfies $\phi(I) = I$. In the case that X is an infinite-dimensional Hilbert space we shall prove an analogous

1991 *Mathematics Subject Classification*: 47A10, 47B49.

This work was supported by the Research Council of Slovenia.

result for surjective linear mappings preserving the surjectivity spectrum (surjective operators in both directions).

It seems that the systematic study of linear mappings from the set of $n\times n$ matrices into itself which leave certain properties invariant begins with [9]. In the last few years the interest in problems of this kind, especially in operator algebras over infinite-dimensional spaces, grows again (cf. [1], [2], [4], [5], [10], [11]). In [5] it was proved that a spectrum preserving linear surjective mapping on the algebra of all bounded linear operators on a Banach space X is either an automorphism or an antiautomorphism. Instead of spectrum preserving mappings one can study mappings which preserve various parts of spectrum. It seems natural to ask which subsets of the spectrum have the property that surjective linear mappings which preserve them are necessarily automorphisms. Our results show that the point spectrum and the surjectivity spectrum have that property. Some of ideas used in our proofs are similar to those of Jafarian and Sourour [5]. So, we shall omit some parts of the proofs and refer to [5] wherever possible.

We now fix some notation. The dual of a Banach space X will be denoted by X' and the adjoint of $A\in B(X)$ by A'. For any $x\in X$ and $f\in X'$ we denote by $x\otimes f$ the bounded linear operator on X defined by $(x\otimes f)y=f(y)x$ for $y\in X$. Every operator of rank one can be written in this form. Note that $(x\otimes f)'=f\otimes(Kx)$, where K is the natural embedding of X into X''.

We start with a spectral characterization of rank-one operators which is an extension of [5, Theorem 1].

LEMMA 1. *Let A be a nonzero bounded linear operator on a Banach space X. Then the following conditions are equivalent.*

(i) *A has rank one.*
(ii) *$\sigma(T+A)\cap\sigma(T+cA)\subset\sigma(T)$ for every $T\in B(X)$ and $c\neq 1$.*
(iii) *$\sigma_p(T+A)\cap\sigma_p(T+cA)\subset\sigma_p(T)$ for every $T\in B(X)$ and $c\neq 1$.*

Proof. Jafarian and Sourour [5] proved that (i) implies (ii). They also showed that under the assumption that A has rank greater than 1 we can find an operator T and a pair of complex numbers λ, c, $c\neq 1$, such that λ is an eigenvalue of $T+A$ and $T+cA$, but $\lambda\notin\sigma(T)$. It follows that each of the conditions (ii) or (iii) implies (i). So, it remains to prove that (i) yields (iii). Let A be an operator of rank one. Then it is of the form $A=x\otimes f$. Assume that T is a bounded linear operator on X. Suppose also that c and λ are complex numbers such that $\lambda\notin\sigma_p(T)$ and $\lambda\in\sigma_p(T+A)\cap\sigma_p(T+cA)$. We have to show that $c=1$. According to our assumptions we can find nonzero vectors $z,u\in X$ such that $(T-\lambda)z=-f(z)x$ and $(T-\lambda)u=-cf(u)x$. The complex number λ is not an eigenvalue of T and consequently, $x\neq 0$, $f(z)\neq 0$, and $f(u)\neq 0$. Moreover, from $(T-\lambda)(cf(u)z-f(z)u)=0$ it follows that $cf(u)z=f(z)u$. This implies

that z and u are linearly dependent. Thus, we have necessarily $c=1$, which completes the proof.

In the sequel we shall need the following analogue of Weyl's theorem [14, Theorem 0.10].

LEMMA 2. *Let $B,C\in B(X)$ and $\lambda\in\mathbb{C}$ be given. Assume that C is a finite-rank operator. Suppose that $\lambda\notin\sigma(B)$ and $\lambda\in\sigma(B+C)$. Then $\lambda\in\sigma_s(B+C)$.*

Proof. We have $B+C-\lambda=(B-\lambda)(I+(B-\lambda)^{-1}C)$. According to our assumptions the operator $B+C-\lambda$ is not invertible. As $B-\lambda$ is invertible, $I+(B-\lambda)^{-1}C$ is not invertible. $(B-\lambda)^{-1}C$ is a finite-rank operator, and consequently, $I+(B-\lambda)^{-1}C$ is not surjective. The same must be true for $B+C-\lambda$. This concludes the proof.

MAIN THEOREM. *Let X be an infinite-dimensional complex Banach space and let $\phi:B(X)\to B(X)$ be a surjective linear mapping. Then ϕ is an automorphism of the algebra $B(X)$ if and only if ϕ preserves injective operators in both directions and satisfies $\phi(I)=I$. Assume that X is a Hilbert space. Then a surjective linear mapping ϕ on $B(X)$ is an automorphism if and only if ϕ preserves surjective operators in both directions and satisfies $\phi(I)=I$.*

This theorem follows immediately from the following two results.

THEOREM 3. *Let X be an infinite-dimensional complex Banach space. If $\phi:B(X)\to B(X)$ is a point spectrum preserving surjective linear mapping then there is a bounded invertible linear operator $A:X\to X$ such that $\phi(T)=ATA^{-1}$ for every $T\in B(X)$.*

THEOREM 4. *Let H be an infinite-dimensional complex Hilbert space. If $\phi:B(H)\to B(H)$ is a surjectivity spectrum preserving surjective linear mapping then there exists a bounded invertible linear operator $A:H\to H$ such that $\phi(T)=ATA^{-1}$ for every $T\in B(H)$.*

Proof of Theorem 3. As in [5] we prove that ϕ is injective and $\phi(I)=I$. It follows from Lemma 1 that if $A\in B(X)$ has rank one the same must be true for $\phi(A)$. Using the same approach as in [5, Theorem 2] one can see that there exist either linear bijective transformations $A:X\to X$ and $C:X'\to X'$ such that $\phi(x\otimes f)=Ax\otimes Cf$ for every $x\in X$, $f\in X'$, or linear bijective transformations $B:X'\to X$ and $D:X\to X'$ such that $\phi(x\otimes f)=Bf\otimes Dx$ for all $x\in X$, $f\in X'$.

In the first case, from $\sigma_p(x\otimes f)=\sigma_p(Ax\otimes Cf)$ we get $(Cf)(Ax)=f(x)$, which shows that C is the adjoint of A^{-1}. This forces C to be bounded, which implies that A^{-1} and finally A are bounded.

Let T be an arbitrary bounded linear operator on X such that 0 is not an eigenvalue of T. Then $\phi(T)$ is injective as well. Let x be a nonzero vector in X. Set $Tx = y$. Then $0 \in \sigma_{\rm p}(T - y \otimes f)$ for every $f \in X'$ satisfying $f(x) = 1$. From $f(x) = (Cf)(Ax)$ and the surjectivity of C we get $\{Cf : f \in X' \text{ and } f(x) = 1\} = \{g \in X' : g(Ax) = 1\}$. So, we have $0 \in \sigma_{\rm p}(\phi(T) - Ay \otimes g)$ for every $g \in X'$ satisfying $g(Ax) = 1$. Let g and h be functionals such that $g(Ax) = h(Ax) = 1$. Then we can find nonzero vectors $z_g, z_h \in X$ such that $\phi(T)z_g = g(z_g)Ay \neq 0$ and $\phi(T)z_h = h(z_h)Ay \neq 0$. Consequently, $\phi(T)(h(z_h)z_g - g(z_g)z_h) = 0$. It follows from injectivity of $\phi(T)$ that z_h and z_g are linearly dependent. This yields the existence of a nonzero vector $z \in X$ such that for every functional $g \in X'$ with $g(Ax) = 1$ we have $\phi(T)z = g(z)Ay$. We claim that z and Ax are linearly dependent. If this is not the case, then there exists $g \in X'$ such that $g(Ax) = 1$ and $g(z) = 0$, implying $\phi(T)z = 0$, which contradicts the injectivity of $\phi(T)$. So, z and Ax are linearly dependent and from $\phi(T)z = g(z)Ay$ we obtain $\phi(T)Ax = Ay$, or equivalently, $Tx = y = A^{-1}\phi(T)Ax$. Thus, $\phi(T) = ATA^{-1}$ for every T satisfying $0 \notin \sigma_{\rm p}(T)$. From $\phi(I) = I$ we get $\phi(T) = ATA^{-1}$ for every $T \in B(X)$, which completes the proof in our first case.

It remains to consider the case that there exist bijective linear mappings $B : X' \to X$ and $D : X \to X'$ such that $\phi(x \otimes f) = Bf \otimes Dx$ for all $x \in X$, $f \in X'$. As before we have $(Dx)(Bf) = f(x)$. Let K be the natural embedding of X into X''. Then D' is defined at least on the image of K and coincides there with $B^{-1}K^{-1}$. Thus, B^{-1} is closed and therefore bounded. The operators D and $(B^{-1})'$ are bijections. Obviously, $D = (B^{-1})'K$. Thus, K is a bijection, which implies the reflexivity of X. It follows that there exists a separable subspace W of X and a linear projection P from X onto W such that $\|P\| = 1$ [8, Proposition 1]. W is a separable Banach space and according to Ovsepian–Pełczyński's result on the existence of total bounded biorthogonal systems in separable Banach spaces [12, Theorem 1] there is a sequence (x_n) of vectors in W and a sequence of bounded linear functionals (f_n) on W such that

- $f_m(x_n) = \delta_{m,n}$ (the Kronecker symbol) for $m, n = 1, 2, \ldots$
- The linear span of (x_n) is dense in W in the norm topology.
- If $x \in W$ and $f_n(x) = 0$ for all $n \in \mathbb{N}$ then $x = 0$.
- $\sup_n \|x_n\|\,\|f_n\| = M < \infty$.

For every $n \in \mathbb{N}$ define a functional $g_n \in X'$ by $g_n(x) = f_n(Px)$, $x \in X$. The linear operator $S = \sum_{n=1}^{\infty} 2^{-n} x_n \otimes g_n + I - P$ is bounded and injective. Obviously, it is not invertible. As ϕ is surjective there exists $T \in B(X)$ such that $\phi(T) = S$. Moreover, T is injective. Choose a nonzero $x \in X$ and set $Tx = y$. As before we get $0 \in \sigma_{\rm p}(T - y \otimes f)$ for all $f \in X'$ satisfying $f(x) = 1$ and $0 \in \sigma_{\rm p}(S - u \otimes Dy)$ for all $u \in X$ satisfying $(Dx)(u) = 1$. Therefore, we can find for every $u \in X$ with $(Dx)(u) = 1$ a nonzero vector $w \in X$ such that $Sw = ((Dy)(w))u$. As $w \neq 0$ we have $Sw \neq 0$ and consequently, the image of S contains the linear span of $\{u \in X : (Dx)(u) = 1\}$. But the linear span of this set is X, which contradicts the noninvertibility of S. It follows that the second case cannot occur. This completes the proof.

Let H be a Hilbert space. For any $x, y \in H$ we denote the inner product of these two vectors by y^*x, while xy^* denotes the rank one operator given by $(xy^*)z = (y^*z)x$. The orthogonal complement of $K \subset H$ is denoted by $K^{\perp}$.

Proof of Theorem 4. It follows from [7, Lemma 1] that if $\sigma(T)$ is a finite set, then $\sigma(T) = \sigma_{\rm s}(T) = \sigma_{\rm s}(\phi(T)) = \sigma(\phi(T))$. The same must be true if $\sigma(\phi(T))$ is finite. In particular, $T \in B(H)$ is quasi-nilpotent if and only if $\phi(T)$ is quasi-nilpotent.

First we show that ϕ is injective. If $\phi(T) = 0$ then T is quasi-nilpotent. Assume that $T \neq 0$. Then we can find $x \in H$ such that $Tx = y \neq 0$. Clearly, x and y are linearly independent. Define a nilpotent operator $N : H \to H$ by

$$Nx = x - y, \quad Ny = x - y, \quad Nz = 0 \quad \text{for } z \in \{x, y\}^{\perp}.$$

It follows from $\phi(N) = \phi(T + N)$ that $T + N$ is quasi-nilpotent, which contradicts $(T + N)x = x$.

Our next step will be to prove that for every $T \in B(H)$ the operator $\phi(T)$ is an idempotent of rank one if and only if T is an idempotent of rank one. For this purpose we choose an idempotent operator $P \in B(H)$ of rank one, and set $\phi(P) = Q$. As $\sigma(P)$ is finite we have $\sigma(P) = \sigma(Q) = \{0, 1\}$. It follows from Lemma 1 that

$$\sigma(T + P) \cap \sigma(T + 2P) \subset \sigma(T) = \sigma_{\rm s}(T)$$

for every $T \in B(H)$ having a finite spectrum. The surjectivity spectrum of an operator is a subset of its spectrum. The mapping ϕ maps the set of all operators with finite spectrum onto itself. Consequently,

$$\sigma_{\rm s}(T + Q) \cap \sigma_{\rm s}(T + 2Q) \subset \sigma(T) \tag{1}$$

for every bounded linear operator T with finite spectrum.

We first show that for every x from H the vectors x, Qx, and Q^2x are linearly dependent. Assume on the contrary that there exists $x \in H$ such that x, Qx, Q^2x are linearly independent. Let us define a linear bounded operator S on H by

$$Sx = 3x - Qx, \quad SQx = 3Qx - 2Q^2x, \quad SQ^2x = x,$$
$$Sz = 0 \quad \text{for } z \in \{x, Qx, Q^2x\}^{\perp}.$$

It is easy to verify that $3 \in \sigma(S+Q) \cap \sigma(S+2Q)$ and $3 \notin \sigma(S)$. It follows from Lemma 2 and from $3 \notin \sigma(Q) \cup \sigma(2Q)$ that $3 \in \sigma_s(S+Q) \cap \sigma_s(S+2Q)$, which contradicts (1). Therefore, for every $x \in H$ the vectors x, Qx, and Q^2x are linearly dependent, which implies that Q satisfies a quadratic polynomial equation $p(Q) = 0$ [6]. It follows from $\sigma(Q) = \{0, 1\}$ that p is of the form $p(\lambda) = \lambda(\lambda - 1)$, which further implies that Q is an idempotent.

Next, we show that $\operatorname{rank} Q = 1$. If $\operatorname{rank} Q > 1$ we can find linearly independent vectors x, y such that $Qx = x$ and $Qy = y$. Define an operator R on H by

$$Rx = 2x, \quad Ry = y, \quad Ru = 0 \quad \text{for } u \in \{x, y\}^{\perp}.$$

As before we prove that $3 \in \sigma_s(R + Q) \cap \sigma_s(R + 2Q)$, which contradicts $3 \notin \sigma(R)$ and (1).

Thus, ϕ preserves idempotents of rank one. The same must be true for ϕ^{-1}, and consequently, ϕ preserves idempotents of rank one in both directions.

According to [10, Proposition 2.6] there exists either an invertible $A \in B(H)$ such that $\phi(T) = ATA^{-1}$ for all finite-rank operators $T \in B(H)$, or a bounded invertible conjugate-linear operator C on H such that $\phi(T) = CT^*C^{-1}$ for every $T \in B(H)$ of finite rank.

In the first case we choose $T \in B(H)$ such that $T^2 = 0$. Let x, y be arbitrary vectors in H. If $\lambda \in \mathbb{C} \setminus \{0\}$ then it follows from [5, Lemma 4] that $y^*(\lambda - T)^{-1}x = 1$ if and only if $\lambda \in \sigma(T + xy^*)$. By Lemma 2 this is equivalent to $\lambda \in \sigma_s(T + xy^*) = \sigma_s(\phi(T) + Axy^*A^{-1})$. Clearly, $\sigma(\phi(T)) = \sigma_s(\phi(T)) = \{0\}$. Thus, Lemma 2 shows that the last relation is satisfied if and only if $\lambda \in \sigma(\phi(T) + Axy^*A^{-1})$. Applying [5, Lemma 4] once again we finally conclude that this is equivalent to $y^*A^{-1}(\lambda - \phi(T))^{-1}Ax = 1$. Using linearity we get

$$y^*(\lambda - T)^{-1}x = y^*A^{-1}(\lambda - \phi(T))^{-1}Ax$$

for all $x, y \in H$ and all nonzero complex numbers λ. Applying

$$(\lambda - T)^{-1} = \sum_{k=0}^{\infty} \frac{T^k}{\lambda^{k+1}} \quad \text{and} \quad (\lambda - \phi(T))^{-1} = \sum_{k=0}^{\infty} \frac{\phi(T)^k}{\lambda^{k+1}}$$

for $|\lambda| > \max\{\|T\|, \|\phi(T)\|\}$ we get

$$y^*Tx = y^*A^{-1}\phi(T)Ax$$

for every $x, y \in H$. This further implies $\phi(T) = ATA^{-1}$. Since, by a result of Pearcy and Topping [13], every operator in $B(H)$ is a finite sum of operators with square zero, this yields $\phi(T) = ATA^{-1}$ for every $T \in B(H)$, which completes the proof in the first case.

In the second case we show similarly that $\phi(T) = CT^*C^{-1}$ for all $T \in B(H)$. It follows that an operator T is surjective if and only if T^*

is surjective. But this certainly is not true (consider, for instance, the shift operator). Thus, the second case cannot occur. The proof of the theorem is complete.

References

[1] M. D. Choi, D. Hadwin, E. Nordgren, H. Radjavi, and P. Rosenthal, *On positive linear maps preserving invertibility*, J. Funct. Anal. 59 (1984), 462–469.

[2] M. D. Choi, A. A. Jafarian, and H. Radjavi, *Linear maps preserving commutativity*, Linear Algebra Appl. 87 (1987), 227–241.

[3] C. Davis and P. Rosenthal, *Solving linear operator equations*, Canad. J. Math. 26 (1974), 1384–1389.

[4] J. C. Hou, *Rank-preserving linear maps on $B(X)$*, Science in China (Ser. A) 32 (1989), 929–940.

[5] A. A. Jafarian and A. R. Sourour, *Spectrum-preserving linear maps*, J. Funct. Anal. 66 (1986), 255–261.

[6] I. Kaplansky, *Infinite Abelian Groups*, University of Michigan Press, Ann Arbor 1954.

[7] K. B. Laursen and P. Vrbová, *Some remarks on the surjectivity spectrum of linear operators*, Czechoslovak Math. J. 39 (1989), 730–739.

[8] J. Lindenstrauss, *On nonseparable reflexive Banach spaces*, Bull. Amer. Math. Soc. 72 (1966), 967–970.

[9] M. Marcus and B. N. Moyls, *Linear transformations on algebras of matrices*, Canad. J. Math. 11 (1959), 61–66.

[10] M. Omladič, *On operators preserving commutativity*, J. Funct. Anal. 66 (1986), 105–122.

[11] —, *On operators preserving numerical range*, Linear Algebra Appl. 134 (1990), 31–51.

[12] R. I. Ovsepian and A. Pełczyński, *Existence of a fundamental total and bounded biorthogonal sequence*, Studia Math. 54 (1975), 149–159.

[13] C. Pearcy and D. Topping, *Sums of small number of idempotents*, Michigan Math. J. 14 (1967), 453–465.

[14] H. Radjavi and P. Rosenthal, *Invariant Subspaces*, Ergeb. Math. Grenzgeb. 77, Springer, Berlin 1973.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF LJUBLJANA
JADRANSKA 19
61000 LJUBLJANA, SLOVENIA

Received June 11, 1992 (2956)
Revised version January 25, 1993