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Range inclusion results for derivations on noncommutative Banach algebras

by

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Abstract. Let A be a Banach algebra, and let $D : A \to A$ be a (possibly unbounded) derivation. We are interested in two problems concerning the range of D:

1. When does D map into the (Jacobson) radical of A?
2. If $[a, Da] = 0$ for some $a \in A$, is Da necessarily quasinilpotent?

We prove that derivations satisfying certain polynomial identities map into the radical. As an application, we show that if $[a, [a, [a, Da]]]$ lies in the prime radical of A for all $a \in A$, then D maps into the radical. This generalizes a result by M. Mathieu and the author which asserts that every centralizing derivation on a Banach algebra maps into the radical. As far as the second question is concerned, we are unable to settle it, but we obtain a reduction of the problem and can prove the quasinilpotency of Da under commutativity assumptions slightly stronger than $[a, Da] = 0$.

Introduction. The interest in range inclusion results for derivations on Banach algebras goes back to I. M. Singer's and J. Wermer's paper [S–W] from 1955, in which they proved that every bounded derivation on a commutative Banach algebra maps into the (Jacobson) radical. In a footnote they conjectured that the boundedness requirement for the derivation was superfluous. It took more than thirty years until this conjecture was finally proved by M. P. Thomas ([Tho 1]).

The simple-minded attempt to extend these results to noncommutative Banach algebras obviously fails, even for bounded derivations: Let A be a noncommutative, semisimple Banach algebra, and fix some $a \in A$ which does not lie in the center $Z(A)$ of A. Then $A \ni x \mapsto [a, x] := ax - xa$ is a bounded derivation, which is nonzero, and therefore does not map into the radical. There are, however, various meaningful generalizations of the bounded Singer–Wermer theorem to the noncommutative setting (see [Yoo], [M–M] and [Vuk 1], for instance). All these results require at some point the

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following theorem by A. M. Sinclair ([Sin 1]): Every bounded derivation on a Banach algebra leaves the primitive ideals invariant. For commutative Banach algebras, the classical Singer–Wermer theorem can easily be deduced from Sinclair's result, which justifies the name noncommutative Singer–Wermer theorem for it. The big open question, which has become known as the noncommutative Singer–Wermer problem, is if *every*, possibly unbounded derivation on a Banach algebra leaves the primitive ideals invariant.

It comes as no surprise that range inclusion problems for derivations on Banach algebras are closely connected with automatic continuity problems, i.e. questions if for certain Banach algebras A every derivation $D : A \to A$ is continuous. It is well known that all derivations on semisimple Banach algebras are continuous ([J–S]), whereas it is still an open problem if the same holds true for derivations on semiprime Banach algebras (see [Cus], [Gar], [Run], and [M–R]).

Another way of extending the Singer–Wermer theorem to the noncommutative situation is to consider local phenomena: The classical Kleinecke–Shirokov theorem ([Klei], [Shi]) states that if A is a Banach algebra, and $a, b \in A$ are such that $[a,[a,b]] = 0$, then $[a,b]$ is quasinilpotent. Replacing either $A \ni x \mapsto [a,x]$ or $A \ni x \mapsto [x,b]$ by an arbitrary derivation D, two questions arise naturally:

(A) Does $D^2a = 0$ for some $a \in A$ imply that Da is quasinilpotent?

(B) Does $[a, Da] = 0$ for some $a \in A$ imply that Da is quasinilpotent?

If D is bounded, the answer is "yes" in both cases (see [M–M], for example). Further, M. Mathieu and the author proved without any boundedness assumptions for D that if we assume $[a, Da] = 0$—or even weaker: $[a, Da] \in Z(A)$—for all $a \in A$, then D maps into the radical ([M–R]). From the point of view adopted in [M–M], this result can be considered a global, unbounded Kleinecke–Shirokov theorem, whereas (A) and (B) ask for local, unbounded Kleinecke–Shirokov theorems. Recently, M. P. Thomas ([Tho 2]) gave a positive answer to (A), whereas (B) seems to be still open.

The present paper is organized as follows.

In the first section we put together some preliminary material. In particular, we discuss the connections of the noncommutative Singer–Wermer conjecture with other range inclusion and automatic continuity problems.

In Section 2, we establish a rather general theorem which asserts that derivations satisfying certain polynomial identities map into the radical. As an application, we prove a refinement of [M–R, Theorem 1].

Section 3 is devoted to the problem if a local, unbounded Kleinecke–Shirokov theorem, i.e. an affirmative answer to (B), does hold. We are unable to settle (B), but we obtain a reduction of the problem, and can prove the quasinilpotency of Da in two cases, where the assumptions are slightly stronger than $[a, Da] = 0$.

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1. Preliminaries.

We begin with a definition (compare [Tho 2]) which will make formulations easier in the sequel.

DEFINITION 1.1. Let A be a Banach algebra, and let $D : A \to A$ be a derivation. A primitive ideal $P \subset A$ is said to be *exceptional* if $DP \not\subset P$. The set of all exceptional primitive ideals of A is denoted by $\mathcal{E}_A(D)$.

In terms of Definition 1.1, [Sin 1, Theorem 2.2] asserts that $\mathcal{E}_A(D) = \emptyset$ for every bounded derivation D on a Banach algebra A, and the noncommutative Singer–Wermer conjecture claims that $\mathcal{E}_A(D) = \emptyset$ holds as well if D is unbounded.

As far as the author knows, there is no range inclusion result for bounded derivations on Banach algebras, where continuity of the derivation cannot be replaced by the (possibly redundant) condition that there are no exceptional primitive ideals. As an illustration, we give a variant of the Kleinecke–Shirokov theorem:

PROPOSITION 1.2. *Let A be a Banach algebra, let $D : A \to A$ be a derivation with $\mathcal{E}_A(D) = \emptyset$, and let $a \in A$ be such that $[a, Da] = 0$. Then Da is quasinilpotent.*

Before we prove Proposition 1.2, let us state a useful observation as a lemma (see also [Tho 2, Proposition 2.1]):

LEMMA 1.3. *Let A be a Banach algebra, and let* $\mathrm{Prim}(A)$ *denote the collection of its primitive ideals. Then, if A is unital,*

$$\sigma_A(a) = \bigcup_{P \in \mathrm{Prim}(A)} \sigma_{A/P}(a+P) \quad (a \in A),$$

and, if A has no identity,

$$\sigma_A(a) = \bigcup_{P \in \mathrm{Prim}(A)} \sigma_{A/P}(a+P) \cup \{0\} \quad (a \in A).$$

Proof. This is an immediate consequence of [Rick, Theorem 2.2.9(v)]. ∎

Proof of Proposition 1.2. Let $P \subset A$ be a primitive ideal. Then D drops to a derivation D_P on A/P, which is bounded by [J–S, Theorem 4.1]. By the Kleinecke–Shirokov theorem for bounded derivations $D_P(\pi_P(a)) =$

$\pi_P(Da)$ is quasinilpotent, where $\pi_P : A \to A/P$ is the canonical epimorphism. By Lemma 1.3, we have

$$\sigma_A(Da) \subset \bigcup_{P \in \mathrm{Prim}(A)} \sigma_{A/P}(\pi_P(Da)) \cup \{0\} = \{0\},$$

which means that Da is quasinilpotent. ∎

The following theorem appears in the published literature for the first time as [Jia, Theorem 3.6] (albeit with a fallacious proof). For a corrected proof see [Tho 2].

THEOREM 1.4. *Let A be a Banach algebra, and let $D : A \to A$ be a (possibly unbounded) derivation. Then $\mathcal{E}_A(D)$ consists of a finite number of primitive ideals of A, all of which have finite codimension.*

Remark. Unpublished work by M. P. Thomas shows—in analogy with the commutative case ([Joh, Theorem 4])—that the noncommutative Singer–Wermer problem can be reduced to the case of a radical Banach algebra R and a derivation $D : R^\# \to R^\#$, where $R^\#$ stands for the unitization of R.

COROLLARY 1.5. *Let A be an amenable Banach algebra. Then $\mathcal{E}_A(D) = \emptyset$ for every derivation $D : A \to A$.*

Proof. Assume there is a primitive ideal $P \in \mathcal{E}_A(D)$. By [Hel, Proposition VII.2.31], P has a bounded approximate identity. As a consequence of Cohen's factorization theorem ([B–D, Corollary 11.11]), we have $P^2 = P$. Since $D(P^2) \subset P$, this contradicts our assumption that $P \in \mathcal{E}_A(D)$, i.e. $DP \not\subset P$. ∎

Dealing with unbounded derivations on Banach algebras is necessarily more algebraic in flavour than the bounded case. There are in fact some striking, purely ring-theoretic theorems about derivations, which are of considerable use in the Banach algebra situation. A result we will apply in this paper is due to J. Vukman ([Vuk 1, Theorem 1]), and refines for rings with characteristic different from two, three and five a classical result by E. C. Posner ([Pos, Theorem 2]):

THEOREM 1.6. *Let A be a prime ring whose characteristic is different from two, three and five, and let $D : A \to A$ be a derivation such that $[a, [a, [a, Da]]] \in Z(A)$ for all $a \in A$. Then $D = 0$, or A is commutative.*

The following lemma may serve in some situations as a substitute for [Sin 1, Theorem 2.2], and allows to reduce range inclusion problems for derivations to the case where the domain is prime.

LEMMA 1.7. *Let A be a ring, let $D : A \to A$ be a derivation, and let $P \subset A$ be a minimal prime ideal such that the additive group of A/P is torsion-free. Then P is invariant under D.*

As it seems, Lemma 1.7 appears for the first time in the published literature as [G–W, Proposition 1.1]. It has been rediscovered later at several occasions (see [Gar], where a version for commutative Banach algebras is given, and [M–R]).

As an example of how Lemma 1.7 can be used to reduce range inclusion problems for derivations on Banach algebras to the prime case, we give a noncommutative analogue of [Run, Theorem 2]:

THEOREM 1.8. *The following are equivalent:*

(i) *For every Banach algebra A and for every derivation $D : A \to A$ we have $\mathcal{E}_A(D) = \emptyset$.*

(ii) *For every radical Banach algebra R and for every derivation $D : R^\# \to R^\#$ we have $DR \subset R$.*

(iii) *For every prime, radical Banach algebra R and for every derivation $D : R^\# \to R^\#$ we have $DR \subset R$.*

Proof. The equivalence of (i) and (ii) is the aforementioned, unpublished result by M. P. Thomas. For the proof of the equivalence of (ii) and (iii) just observe that the arguments used in the commutative case (see [Run]) carry over almost verbatim. ∎

COROLLARY 1.9. *If all derivations on prime Banach algebras are continuous, then the noncommutative Singer–Wermer conjecture is true.*

Remark. It was observed earlier by J. Cusack ([Cus]) that if every derivation on a Banach algebra had a nilpotent separating space, then the noncommutative Singer–Wermer theorem would hold. By [M–R, Theorem 2], Cusack's and our condition are equivalent.

2. Derivations satisfying polynomial identities. The main goal of this section is Theorem 2.3, which gives a rather general condition forcing a derivation on a Banach algebra to map into the radical. As an immediate consequence of Theorems 2.3 and Theorems 1.6, we then obtain Corollary 2.4, which generalizes [M–R, Theorem 1]. To prove only this corollary, we could simply proceed by more or less following the lines of the proof in [M–R] with the only difference that we would invoke Theorem 1.6 instead of [Pos, Theorem 2]. The idea, however, behind both proofs can be expressed in a more general context and is applicable to other situations.

For $n \in \mathbb{N}$, let $\mathbb{C}\{X_1, \ldots, X_n\}$ denote the polynomial algebra in n noncommuting variables, i.e. the free complex algebra on n generators. Let A be a (complex) algebra. If $a_1, \ldots, a_n \in A$ satisfy $p(a_1, \ldots, a_n) = 0$ for some nonzero $p \in \mathbb{C}\{X_1, \ldots, X_n\}$, we say that $a_1, \ldots, a_n$ satisfy a *polynomial identity*. Theorem 1.6 and Corollary 2.4 make assertions about derivations $D : A \to A$ satisfying the polynomial "identities" $p(a, Da) = 0$ and

$p(a, Da) \in \mathrm{nil}(A)$, respectively, for all $a \in A$, where $p = [X_1, [X_1, [X_1, X_2]]] \in \mathbb{C}\{X_1, X_2\}$.

Most of the work for the proof of Theorem 2.3 is done in the following two lemmas.

LEMMA 2.1. *Let $n \in \mathbb{N}$, and let $p \in \mathbb{C}\{X_1, \ldots, X_{n+1}\}$ with the following property:*

If D is a derivation on a primitive Banach algebra A such that $p(a, Da, \ldots, D^n a) = 0$ for all $a \in A$, then $D = 0$.

Then every derivation D on a Banach algebra A with $\mathcal{E}_A(D) = \emptyset$ and satisfying $p(a, Da, \ldots, D^n a) = 0$ for all $a \in A$ maps into the radical.

Proof. Let A be a Banach algebra, and let D be a derivation with $\mathcal{E}_A(D) = \emptyset$ and $p(a, Da, \ldots, D^n a) = 0$ for all $a \in A$. Let P be a primitive ideal of A. Then D drops to a derivation D_P on the primitive Banach algebra A/P. Clearly, D_P satisfies $p(a, D_P a, \ldots, D_P^n a) = 0$ for all $a \in A/P$. By assumption, $D_P = 0$, which means $DA \subset P$. Since P is an arbitrary primitive ideal of A, we have $DA \subset \mathrm{rad}(A)$. ■

For the second lemma recall some definitions.

For any ring A let $\mathrm{nil}(A)$ denote its prime radical, i.e. the intersection of all prime ideals of A. Since by Zorn's Lemma every prime ideal of A contains a minimal prime ideal, $\mathrm{nil}(A)$ equals the intersection of all minimal prime ideals of A.

Let A be a Banach algebra, and let $D : A \to A$ be a derivation. Then

$$\mathcal{S}(D) := \{y \in A : \text{ there is a sequence } \{x_n\}_{n=1}^{\infty} \text{ in } A \text{ such that } x_n \to 0 \text{ and } Dx_n \to y\}$$

is the *separating space* of D. It is a closed ideal of A, which, by the closed graph theorem, is zero if and only if D is bounded. For more information on separating spaces of linear operators see [Sin 2].

LEMMA 2.2. *Let $n \in \mathbb{N}$, and let $p \in \mathbb{C}\{X_1, \ldots, X_{n+1}\}$ have the following property:*

Every derivation D on a prime Banach algebra A such that $p(a, Da, \ldots, D^n a) = 0$ for all $a \in A$ maps into the radical.

Then every derivation D on a Banach algebra A such that

$$p(a, Da, \ldots, D^n a) \in \mathrm{nil}(A) \quad (a \in A)$$

maps into the radical.

Proof. Choose an arbitrary primitive ideal $P \subset A$, which, in particular, is a prime ideal, and therefore contains a minimal prime ideal Q. By Lemma 1.7, $DQ \subset Q$, and D drops to a derivation D_Q on A/Q.

Suppose first that $\mathcal{S}(D) \not\subset Q$. Then by [Cus, Lemma 2.3], Q is closed, and A/Q is a prime Banach algebra. Since

$$p(a, Da, \ldots, D^n a) \in \mathrm{nil}(A) \subset Q \quad (a \in A),$$

we have $p(a, D_Q a, \ldots, D_Q^n a) = 0$ for all $a \in A/Q$. By assumption, this means

$$D_Q(A/Q) \subset \mathrm{rad}(A/Q) \subset P/Q,$$

and consequently, $DA \subset P$.

Now, assume that $\mathcal{S}(D) \subset Q$. Let $\pi : A \to A/Q^-$ be the canonical epimorphism. Then

$$\mathcal{S}(\pi \circ D) = (\pi(\mathcal{S}(D)))^- = \{0\}.$$

In other words: $\pi \circ D$ is continuous. Since $(\pi \circ D)(Q) = \{0\}$, this means that $(\pi \circ D)(Q^-) = \{0\}$, i.e. Q^- is invariant under D. Therefore, D drops to a bounded derivation D_{Q^-} on the Banach algebra A/Q^-. Obviously, D_{Q^-} satisfies $p(a, D_{Q^-} a, \ldots, D_{Q^-}^n a) = 0$ for all $a \in A/Q^-$. Then we have by Lemma 2.1,

$$D_{Q^-}(A/Q^-) \subset \mathrm{rad}(A/Q^-) \subset P/Q^-,$$

and as a consequence, we obtain again $DA \subset P$. ■

THEOREM 2.3. *Let $n \in \mathbb{N}$, and let $p \in \mathbb{C}\{X_1, \ldots, X_{n+1}\}$ have the following property:*

If D is a derivation on a prime Banach algebra A such that $p(a, Da, \ldots, D^n a) = 0$ for all $a \in A$, then $D = 0$, or A is commutative.

Then every derivation D on a Banach algebra A such that

$$p(a, Da, \ldots, D^n a) \in \mathrm{nil}(A) \quad (a \in A)$$

maps into the radical.

Proof. Let A be a prime Banach algebra, and let $D : A \to A$ be a derivation satisfying $p(a, Da, \ldots, D^n a) = 0$ for all $a \in A$. If A is commutative, we have $DA \subset \mathrm{rad}(A)$ by [Tho 1]. The claim then follows from Lemma 2.2. ■

COROLLARY 2.4. *Let A be a Banach algebra, and let $D : A \to A$ be a derivation such that $[a, [a, [a, Da]]] \in \mathrm{nil}(A)$ for all $a \in A$. Then D maps into the radical of A.*

Proof. By Theorem 1.6, $p = [X_1, [X_1, [X_1, X_2]]]$ satisfies the assumptions of Theorem 2.3. ■

Remarks. 1. It is an interesting question whether the assumption in Corollary 2.4 can be weakened to $[a, [a, [a, Da]]] \in \mathrm{rad}(A)$ for all $a \in A$ as in its bounded counterpart [Vuk 1, Theorem 2]. Certainly, the boundedness assumption of [Vuk 1, Theorem 2] can be replaced by $\mathcal{E}_A(D) = \emptyset$. Hence, if the noncommutative Singer–Wermer conjecture were true, [Vuk 1, Theorem 2] would hold for arbitrary derivations. Now, let $A = R^{\#}$, where R is a radical Banach algebra. Then every element $a \in A$ is of the form $a = r + \lambda 1$ for $r \in R$ and $\lambda \in \mathbb{C}$, and consequently for every derivation $D : A \to A$,

$$[a, Da] = [r + \lambda 1, D(r + \lambda 1)] = [r, Dr] \in R = \mathrm{rad}(A).$$

In particular, the assumption of [Vuk 1, Theorem 2] that $[a, [a, [a, Da]]] \in \mathrm{rad}(A)$ for all $a \in A$ holds automatically. By Theorem 1.8, a proof for [Vuk 1, Theorem 2] without assuming $\mathcal{E}_A(D) = \emptyset$ would therefore entail the correctness of the noncommutative Singer–Wermer conjecture.

2. It is possible to prove a "multivariable version" of Theorem 2.3:

Let $n, m, k \in \mathbb{N}$, where $k \le m$, and let $p \in \mathbb{C}\{X_1, \ldots, X_{nm+1}\}$ have the following property:

If $D_1, \ldots, D_m$ are derivations on a prime Banach algebra A such that

$$p(a, D_1 a, \ldots, D_m a, \ldots, D_1^n a, \ldots, D_m^n a) = 0 \quad (a \in A),$$

then $D_1 = \ldots = D_k = 0$, or A is commutative.

Then, if $D_1, \ldots, D_m$ are derivations on a Banach algebra A such that

$$p(a, D_1 a, \ldots, D_m a, \ldots, D_1^n a, \ldots, D_m^n a) \in \mathrm{nil}(A) \quad (a \in A),$$

the derivations $D_1, \ldots, D_k$ map into the radical.

The necessary adjustments in the proof of Theorem 2.3 are easily made. Using this theorem, it is not hard to harvest more unbounded range inclusion results: For example, [Vuk 2, Theorem 1] holds for unbounded derivations as well, as does [B–V, Theorem 1] if in the assumption $\mathrm{rad}(A)$ is replaced by $\mathrm{nil}(A)$.

3. The local situation. In this section we turn to the problem of proving a local, unbounded Kleinecke–Shirokov theorem as asked for in (B).

First we will give a reduction of the problem to the case where a is quasinilpotent, and Da is invertible. For this purpose we require a lemma which is analogous with [Tho 2, Corollary 2.2].

LEMMA 3.1. *Let A be a Banach algebra, let $D : A \to A$ be a derivation, and let $a \in A$ be such that $[a, Da] = 0$. Then $\sigma_A(Da)$ is finite.*

Proof. Fix a primitive ideal $P \subset A$. If P is not exceptional, then we obtain as in Proposition 1.2 that $\sigma_{A/P}(\pi_P(Da)) = \{0\}$, where $\pi_P : A \to A/P$ is again the canonical epimorphism. If P is exceptional, A/P is finite-dimensional, which implies that $\sigma_{A/P}(\pi_P(Da))$ is finite. By Lemma 1.3, we have

$$\sigma_A(Da) \subset \bigcup_{P \in \mathrm{Prim}(A)} \sigma_{A/P}(\pi_P(Da)) \cup \{0\}$$

$$= \bigcup_{P \in \mathcal{E}_A(D)} \sigma_{A/P}(\pi_P(Da)) \cup \{0\},$$

and since there are only a finite number of exceptional primitive ideals, $\sigma_A(Da)$ must be finite. ∎

PROPOSITION 3.2. *The following are equivalent:*

(i) *There is a Banach algebra A and a derivation $D : A \to A$ such that there is an element $a \in A$ with $[a, Da] = 0$ and $Da \notin \mathcal{Q}(A)$.*

(ii) *There is a unital Banach algebra A and a derivation $D : A \to A$ such that there is an element $a \in \mathcal{Q}(A)$ with $[a, Da] = 0$ and Da invertible. In this situation,* $\mathrm{Prim}(A) = \mathcal{E}_A(D)$, *and there is $r \in \mathrm{rad}(A)$ with $[r, Dr] = 0$ and $Dr \notin \mathrm{rad}(A)$.*

Proof. Let A be a Banach algebra, let $D : A \to A$ be a derivation, and assume there is $a \in A$ such that $[a, Da] = 0$, but $Da \notin \mathcal{Q}(A)$. By Lemma 3.1, $\sigma_A(Da) = \{\lambda_1, \ldots, \lambda_n\}$ for distinct $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. We may assume without loss of generality that $\lambda_1 = 1$. For $\varepsilon > 0$ and $j = 1, \ldots, n$ let $B(\lambda_j, \varepsilon)$ be the open disc in $\mathbb{C}$ with center λ_j and radius ε. Choose $\varepsilon > 0$ so small that $B(\lambda_j, \varepsilon) \cap B(\lambda_k, \varepsilon) = \emptyset$ for $j \ne k$. Define a function

$$f : \bigcup_{j=1}^{n} B(\lambda_j, \varepsilon) \to \mathbb{C}, \qquad f(z) = \begin{cases} 1, & z \in B(1, \varepsilon), \\ 0, & z \in \bigcup_{j=2}^{n} B(\lambda_j, \varepsilon). \end{cases}$$

Then f is analytic in a neighborhood of $\sigma_A(Da)$, and the Riesz functional calculus yields an idempotent $e_1 := f(Da)$ that commutes with both a and Da. Put $A_1 := e_1 A e_1$, $a_1 := e_1 a e_1$ and define a derivation $D_1 : A_1 \to A_1$, $D_1 x = e_1 (Dx) e_1$. Then we have

$$\begin{aligned}[a_1, D_1 a_1] &= [e_1 a e_1, e_1 D(e_1 a e_1) e_1] = [e_1 a, e_1 D(e_1 a) e_1] \\ &= [e_1 a, e_1 Da] + [e_1, e_1 (De_1) e_1 a] = e_1 [a, Da] = 0,\end{aligned}$$

and from the choice of e_1 it is clear that $\sigma_{A_1}(D_1 a_1) = \{1\}$, i.e. $D_1 a_1$ is invertible.

If there were any primitive ideal $P \in \mathrm{Prim}(A_1) \setminus \mathcal{E}_{A_1}(D_1)$, then again the classical Kleinecke–Shirokov theorem would yield $\sigma_{A_1/P}(\pi_P(D_1 a_1)) = \{0\}$. The invertibility of $D_1 a_1$ in A_1, however, implies that $\pi_P(D_1 a_1)$ is invertible in A_1/P. Hence, we have $\mathrm{Prim}(A_1) = \mathcal{E}_{A_1}(D_1)$. From Lemma 1.3 we deduce that a_1 —as any element of A_1—has finite spectrum, $\sigma_{A_1}(a_1) = \{\mu_1, \ldots, \mu_m\}$ say, for distinct $\mu_1, \ldots, \mu_m \in \mathbb{C}$. Let $p(X) := \prod_{j=1}^{m} (X - \mu_j) \in \mathbb{C}[X]$. Then by the spectral mapping theorem, we have $\sigma_{A_1}(p(a_1)) = p(\sigma_{A_1}(a_1)) = \{0\}$, i.e. $p(a_1) \in \mathcal{Q}(A_1)$. Since $[a_1, D_1 a_1] = 0$, we have $D_1(p(a_1)) = p'(a_1) D_1 a_1$, where p' is the formal derivative of p. Clearly, $p'(a_1)$ and $D_1 a_1$ commute, whence we have

$$\sigma_{A_1}(D_1(p(a_1))) \subset \sigma_{A_1}(p'(a_1)) \sigma_{A_1}(D_1 a_1) = \sigma_{A_1}(p'(a_1)).$$

From the choice of p it follows that $0 \notin \sigma_{A_1}(p'(a_1))$, which entails that $D_1(p(a_1))$ cannot be quasinilpotent. Repeating the argument we used to get e_1, we obtain an idempotent $e_2 \in A_1$ commuting with both $p(a_1)$ and $D_1(p(a_1)) = p'(a_1)D_1a_1$ such that $e_2D_1(p(a_1))e_2$ is invertible in $e_2A_1e_2$. Put $A_2 := e_2A_1e_2$, $a_2 := e_2p(a_1)e_2$, and define a derivation $D_2 : A_2 \to A_2$, $D_2x = e_2(D_1x)e_2$. Then $[a_2, D_2a_2] = 0$, $a_2 \in \mathcal{Q}(A_2)$, and D_2a_2 is invertible in A_2. As before, we see $\mathrm{Prim}(A_2) = \mathcal{E}_{A_2}(D_2)$.

Let $P_1, \ldots, P_k$ be the primitive ideals of A_2. Then

$$A_2/\mathrm{rad}\,(A_2) \cong A_2/P_1 \oplus \cdots \oplus A_2/P_k\,,$$

i.e $A_2/\mathrm{rad}\,(A_2)$ is finite-dimensional. Hence, the coset of a_2 in $A_2/\mathrm{rad}\,(A_2)$ is not only quasinilpotent, but nilpotent in the algebraic sense, i.e. there is $n \in \mathbb{N}$ such that $a_2^n \in \mathrm{rad}(A_2)$. Choose n minimal with respect to this property, and put $r := a_2^n$. Then $Dr = na_2^{n-1}D_2a_2$, which implies $[r, Dr] = 0$. If $Dr \in \mathrm{rad}(A_2)$, the invertibility of D_2a_2 in A_2 would yield $a_2^{n-1} \in \mathrm{rad}(A_2)$ contradicting the minimal choice of n. ∎

Remark. Let $n \in \mathbb{N}$, and let $p \in \mathbb{C}\{X_1, \ldots, X_{n+1}\}$ with the following property: If A is a Banach algebra, $D : A \to A$ a bounded derivation, and $a \in A$ such that $p(a, Da, \ldots, D^n a) = 0$, then Da is quasinilpotent. By the two bounded Kleinecke–Shirokov theorems $p = [X_1, X_2]$ and $p = X_3$ have this property. As an inspection of their proofs shows, Lemma 3.1 and Proposition 3.2 hold for every p satisfying the condition above.

Unfortunately, the strategy employed by Thomas to prove [Tho 2, Theorem 2.9] resists straightforward adaptation to our problem. So far, we have to put up with two poor man's versions of a local, unbounded Kleinecke–Shirokov theorem, both of which require stronger commutativity assumptions than $[a, Da] = 0$.

Let A be a ring, and let $S \subset A$ be an arbitrary subset. Recall that

$$Z(S) := \{x \in A : [s, x] = 0 \text{ for all } s \in S\}$$

is called the *centralizer* of S in A. In case $S = A$, this notation is consistent with the one we used earlier for the center of A.

Our next lemma puts together some basic information about centralizers in rings, much of which is well known.

LEMMA 3.3. *Let A be a ring, and let S be a subset of A. Then:*

(i) *$Z(S)$ is a full subring of A. If A is a Banach algebra, $Z(S)$ is a Banach subalgebra of A such that $\sigma_{Z(S)}(a) = \sigma_A(a)$ for all $a \in Z(S)$.*

(ii) *If S is commutative, $Z(Z(S))$ is a commutative subring of A containing S and contained in $Z(S)$.*

(iii) *$Z(Z(Z(S))) = Z(S)$.*

(iv) *If $D : A \to A$ is a derivation, then $D(Z(S \cup DS)) \subset Z(S)$. In particular, if $DS \subset S$, we have $D(Z(S)) \subset Z(S)$.*

Proof. (i), (ii) and (iii) are routine (compare [B–D, Proposition 15.2]). It is obviously sufficient to show (iv) if $S = \{a\}$ for some $a \in A$. Let $x \in Z(\{a, Da\})$. Then

$$0 = D([a, x]) = [Da, x] + [a, Dx] = [a, Dx]\,,$$

i.e. $Dx \in Z(\{a\})$. ∎

THEOREM 3.4. *Let A be a Banach algebra, let $D : A \to A$ be a derivation, and let $a \in A$ be such that $Da \in Z(Z(\{a\}))$. Then Da is quasinilpotent.*

Proof. Since $\{a, Da\} \subset Z(Z(\{a\}))$, we have

$$Z(\{a, Da\}) \supset Z(Z(Z(\{a\}))) = Z(\{a\})\,.$$

Since obviously $Z(\{a, Da\}) \subset Z(\{a\})$, equality holds. Therefore, by Lemma 3.3(iv), $D(Z(\{a, Da\})) \subset Z(\{a, Da\})$ and consequently $D(Z(Z(\{a, Da\}))) \subset Z(Z(\{a, Da\}))$. It follows from Lemma 3.3 that $Z(Z(\{a, Da\}))$ is a commutative Banach algebra containing a, which by the above argument is invariant under D. By the unbounded, commutative Singer–Wermer theorem, D maps $Z(Z(\{a, Da\}))$ into its radical. In particular, Da is quasinilpotent. ∎

Remarks. 1. Let A be a Banach algebra, let $S \subset A$ be any set, and let $D : A \to A$ be a derivation. Let $y \in \mathcal{S}(D|_{Z(S)})$, i.e. there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that

$$x_n \to 0 \quad \text{and} \quad Dx_n \to y\,.$$

Then for any $s \in S$,

$$0 = D([x_n, s]) = [Dx_n, s] + [x_n, Ds] \to [y, s]\,,$$

i.e. $\mathcal{S}(D|_{Z(S)}) \subset Z(S)$. This observation is another unpublished result by M. P. Thomas. Consequently, $\mathcal{S}(D|_{Z(Z(S))}) \subset Z(Z(S))$, and if B is a commutative Banach subalgebra of A, then $\mathcal{S}(D|_B) \subset Z(Z(B))$. Hence, if there is an $a \in A$ such that $[a, Da] = 0$, and if B denotes the Banach subalgebra of A generated by a, it would be sufficient for the proof of a local unbounded Kleinecke–Shirokov theorem to show that $Da \in \mathcal{S}(D|_B)$.

2. Let A be a unital Banach algebra, let $D : A \to A$ be a derivation, and suppose there is $a \in A$ such that $[a, Da] = 0$, and Da is invertible. Then, if B denotes again the closed subalgebra of A generated by a, $\mathcal{S}(D|_B) \neq \{0\}$. Otherwise, define a map $d : A \to A$, $dx = (Da)^{-1}Dx$, which, by continuity, leaves B invariant, and whose restriction to B is a derivation. However, $da = 1$, which violates the classical Singer–Wermer theorem.

THEOREM 3.5. *Let A be a Banach algebra, let $D : A \to A$ be a derivation, and let $a \in A$ be such that $[a, D^n a] = 0$ for all $n \in \mathbb{N}$. Then Da is quasinilpotent.*

Proof. First, we prove that $[D^m a, D^n a] = 0$ for all $m, n \in \mathbb{N}$. For $m = 0$ the claim is clear by assumption. Now, let $m > 0$, and assume $[D^{m-1}a, D^n a] = 0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} 0 = D([D^{m-1}a, D^n a]) &= [D^m a, D^n a] + [D^{m-1}a, D^{n+1}a] \\ &= [D^m a, D^n a] \quad (n \in \mathbb{N}). \end{aligned}$$

By induction on m, it follows that $S = \{a, Da, D^2a, \ldots\}$ is a commutative subset of A. Obviously, $DS \subset S$, whence $D(Z(S)) \subset Z(S)$ and $D(Z(Z(S))) \subset Z(Z(S))$. Since S is commutative, $Z(Z(S))$ is a commutative Banach algebra containing a which is left invariant by D. As in the proof of the previous theorem, $Da \in \mathcal{Q}(A)$ follows. ■

Remark. If only $[a, Da] = 0$ is assumed, we still obtain

$$0 = D([a, Da]) = [Da, Da] + [a, D^2a] = [a, D^2a].$$

However, there is no obvious reason why $[a, D^n a] = 0$ should hold for $n \geq 3$.

The following proposition is—at the present stage—the strongest assertion we can make about $[a, D^3a]$.

PROPOSITION 3.6. *Let A be a unital Banach algebra, let $D : A \to A$ be a derivation such that* $\mathrm{Prim}(A) = \mathcal{E}_A(D)$, *and let $a \in A$ be such that $[a, Da] = 0$. Then $[a, D^3a]$ is quasinilpotent.*

Proof. For $b \in Z(\{a\})$ define $d : A \to A$, $dx = [b, Dx]$. For $x \in Z(Z(\{a\}))$, we have

$$0 = D([x, b]) = [Dx, b] + [x, Db],$$

or equivalently, $dx = [x, Db]$. Consequently, $d|_{Z(Z(\{a\}))}$ is a bounded derivation. Note that

$$[a, da] = [a, [b, Da]] = [[a, b], Da] + [b, [a, Da]] = 0.$$

By assumption, $\mathrm{Prim}(A) = \mathcal{E}_D(A)$, which by Lemma 1.3 yields that $\sigma_A(da)$ is finite. Assume $\sigma_A(da) \neq \{0\}$. A construction as in the proof of Proposition 3.2 gives a Banach subalgebra A_1 of $Z(Z(\{a\}))$ with identity, a bounded derivation $d_1 : A_1 \to A$, and an element $a_1 \in A_1$ such that $[a_1, d_1 a_1] = 0$ and $d_1 a_1$ is invertible. Let A_2 denote the unital Banach subalgebra of A_1 generated by a_1, and define $d_2 : A_2 \to A$, $d_2 x = (d_1 a_1)^{-1} d_1 x$. By continuity, $d_2 A_2 \subset A_2$, i.e. d_2 is a bounded derivation on A_2 such that $d_2 a_1 = 1$. This contradicts the bounded, commutative Singer–Wermer theorem. Letting $b = D^2a$, we find that $[D^2a, Da]$ is quasinilpotent. Since

$$0 = D([a, D^2a]) = [Da, D^2a] + [a, D^3a],$$

we have

$$[a, D^3a] = [D^2a, Da] \in \mathcal{Q}(A)$$

as claimed. ■

We conclude with a reflection on what seems to be the main obstacle on the way to a local, unbounded Kleinecke–Shirokov theorem.

Let A be a unital Banach algebra, let $D : A \to A$ be a derivation, and let $a \in A$ be such that $[a, Da] = 0$ and Da is invertible. Then $Z(Z(\{a, Da\}))$ is a commutative Banach subalgebra of A, which, in turn, contains the commutative Banach subalgebra $Z(Z(\{a\}))$, and the map

$$d : Z(Z(\{a\})) \to Z(Z(\{a, Da\})), \qquad dx = (Da)^{-1}Dx,$$

is a derivation such that $da = 1$. This leads to the following question:

(C) Let A and B be unital, commutative Banach algebras, where A is a subalgebra of B containing the identity. Is there a derivation $D : A \to B$ and an element $a \in A$ such that $Da = 1$?

For bounded D, the answer to (C) is clearly "no", since the restriction of D to the unital Banach subalgebra of A generated by a would contradict the classical Singer–Wermer theorem. If D is unbounded, however, a similar argument—using Thomas' unbounded Singer–Wermer theorem instead—does not work. Furthermore, the techniques used in [Tho 1] to settle (C) in case $A = B$ apparently do not carry over to the general situation.

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On some conjecture concerning Gaussian measures of dilatations of convex symmetric sets

by

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Abstract. The paper deals with the following conjecture: if μ is a centered Gaussian measure on a Banach space F, $\lambda > 1$, $K \subset F$ is a convex, symmetric, closed set, $P \subset F$ is a symmetric strip, i.e. $P = \{x \in F : |x'(x)| \le 1\}$ for some $x' \in F'$, such that $\mu(K) = \mu(P)$ then $\mu(\lambda K) \ge \mu(\lambda P)$.

We prove that the conjecture is true under the additional assumption that K is "sufficiently symmetric" with respect to μ, in particular it is true when K is a ball in a Hilbert space. As an application we give estimates of Gaussian measures of large and small balls in a Hilbert space.

I. Introduction. Let us recall that a measure μ defined on Borel subsets of a separable Banach space F is called *Gaussian* if for each $x' \in F'$ the measure $x'(\mu)$ coincides with the Gaussian measure $N(a,\sigma)$ on $\mathbb{R}^1$ for some a and σ which depend on x' (σ may be 0 as well). If $a = 0$ for each $x' \in F'$ then the measure is called *centered*.

A sequence of independent random variables ξ_i, $i = 1, 2, \ldots$, such that each ξ_i is distributed by the law $N(0,1)$ is called *canonical Gaussian*. In this case the distribution of the random vector $(\xi_1, \ldots, \xi_n)$ will be denoted by γ_n and it will be called the *canonical Gaussian measure* on $\mathbb{R}^n$.

If μ is a centered Gaussian measure on a separable Banach space F then there exists a sequence x_i, $i = 1, 2, \ldots$, in F such that the series $\sum_{i=1}^{\infty} x_i \xi_i$ is a.s. convergent in F and μ is the distribution of its sum; here ξ_i, $i = 1, 2, \ldots$, is a canonical Gaussian sequence. Each such sequence (x_i) will be called a *representing sequence* for μ. For all unexplained facts about Gaussian measures which will be used in this paper we refer to one of the books [5] or [8].

A sequence x_i, $i = 1, 2, \ldots$, in F is said to be a 1-*unconditional basis* for a symmetric convex set $K \subset F$ if for each $x \in K$ there exists a unique sequence α_i, $i = 1, 2, \ldots$, of numbers such that $\sum_{i=1}^{\infty} \alpha_i x_i$ is convergent to

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