STUDIA MATHEMATICA 105 (2) (1993)

An example of a generalized completely continuous representation of a locally compact group

by

DETLEV POGUNTKE (Bielefeld)

Abstract. There is constructed a compactly generated, separable, locally compact group G and a continuous irreducible unitary representation π of G such that the image $\pi(C^*(G))$ of the group C^*-algebra contains the algebra of compact operators, while the image $\pi(L^1(G))$ of the L^1-group algebra does not contain any nonzero compact operator. The group G is a semidirect product of a metabelian discrete group and a "generalized Heisenberg group".

In [6] the following theorem was proved. Let π be an irreducible continuous unitary representation of a connected Lie group G such that $\pi(C^*(G))$ contains the algebra of compact operators, i.e., π is a generalized completely continuous representation in our terminology (apparently this notion is used in different ways in the literature). Then the image of $L^1(G)$ under π contains orthogonal projections of rank one. After the efforts at proving this result it is hard to imagine that a corresponding theorem is true for general locally compact groups G. There is even no evidence why in general $\pi(L^1(G))$ should contain nonzero compact operators if $\pi(C^*(G))$ does. However, to my best knowledge there is no example in the literature where such a pathology occurs. It is the purpose of this note to provide such an example. Clearly, such groups cannot be connected, but still they will be compactly generated and separable. In [3], Guichardet constructed an example of a discrete group and a generalized completely continuous representation π of this group such that the image of the finitely supported functions under π does not contain nonzero compact operators. In some sense, my example is an extension of his.

The basis of the construction is a discrete group S acting automorphically on a locally compact abelian group H: there is given an homomorphism $\varphi : S \to \mathrm{Aut}(H)$. Moreover, it is assumed that H contains compact open subgroups. Fix one of them and call it K. Later S, H and K will be specified. The duality between the Pontryagin dual $\widehat{H}$ and H is denoted by

1991 *Mathematics Subject Classification*: 22D20, 22D10, 22D25.

$\langle\,,\rangle$, $\langle \chi, h\rangle = \chi(h)$ for $h \in H$, $\chi \in \widehat{H}$. The group S acts on $\widehat{H}$ as well, $\langle s\chi, h\rangle = \langle \chi, \varphi(s)^{-1}(h)\rangle$. Using the duality one may form the "generalized Heisenberg group" $N = H \times \widehat{H} \times \mathbb{T}$ with multiplication law

$$(h,\chi,t)\,(h',\chi',t') = (h+h', \chi\chi', tt'\langle \chi, h'\rangle^{-1})\,.$$

Observe that the abelian group H is written additively, while the groups $\widehat{H}$ and $\mathbb{T}$ are written multiplicatively. Associated with φ there is an homomorphism $\psi : S \to \mathrm{Aut}(N)$ given by

$$\psi(s)\,(h,\chi,t) = (\varphi(s)(h), s\chi, t)\,.$$

Then one may form the semidirect product G of S and N, i.e., as a topological space G is the direct product $S \times H \times \widehat{H} \times \mathbb{T}$, and the multiplication is defined as

(1) $$(s,h,\chi,t)\,(s',h',\chi',t') = (ss', \varphi(s')^{-1}(h) + h',\ (s'^{-1}\chi)\chi', tt'\langle \chi, \varphi(s')(h')\rangle^{-1})\,.$$

Later we shall consider representations of G which coincide on the central subgroup $\mathbb{T}$ with the identity map. Hence we define $\gamma : \mathbb{T} \to \mathbb{T}$ by $\gamma(t) = t$, and we denote by $L^1(G)_\gamma$ the involutive convolution algebra of all L^1-functions f on G satisfying $f(xt) = \overline{\gamma}(t) f(x)$ for all $x \in G$ and $t \in \mathbb{T}$ where, of course, $\mathbb{T}$ is identified with $\{e\} \times \{0\} \times \{1\} \times \mathbb{T}$. The algebra $L^1(N)_\gamma$ is defined similarly; it acts by convolution on $L^1(G)_\gamma$. Moreover, S acts on $L^1(N)_\gamma$ by $f^s(x) = f(\psi(s)(x))$ for $s \in S$, $x \in N$ and $f \in L^1(N)_\gamma$.

In [5] it was shown that $L^1(N)_\gamma$ is a simple Banach algebra (this will be discussed in more detail later on) and that it contains "orthogonal projections of rank one". Using the chosen compact open subgroup K we are going to construct a particular projection $\mathfrak{p}$ in $L^1(N)_\gamma$ and to determine the algebra $\mathfrak{p} * L^1(G)_\gamma * \mathfrak{p}$.

Associated with K there is a compact open subgroup of $\widehat{H}$, namely $(H/K)^\wedge$, the annihilator of K. The Haar measures of H and $\widehat{H}$ are normalized so that K and $(H/K)^\wedge$ have measure one. The function $\mathfrak{p} : N \to \mathbb{C}$ is defined by

(2) $$\mathfrak{p}(h,\chi,t) = \begin{cases} \bar{t} & \text{if } h \in K \text{ and } \chi \in (H/K)^\wedge\,, \\ 0 & \text{otherwise.} \end{cases}$$

To describe $\mathfrak{p} * L^1(G)_\gamma * \mathfrak{p}$ we need a certain family $\mathfrak{q}_s$, $s \in S$, of functions in $L^1(N)_\gamma$. Let $\delta : S \to \mathbb{R}_+$ be the modular function of the action of S on H, which is given by

$$\int_H f(\varphi(s)(x))\,dx = \delta(s) \int_H f(y)\,dy$$

for all, say, compactly supported continuous functions f on H. Choosing f

to be the characteristic function of K, one sees that

$$\delta(s) = |\varphi(s)^{-1}(K)|\,|K|^{-1} = |\varphi(s)^{-1}(K)|$$

where $|X|$ denotes the Haar measure of a measurable subset X of H. The same notation is used for measurable subsets of $\widehat{H}$. From this description of δ one easily derives that

(3) $$\begin{aligned} \delta(s) &= \#(\varphi(s)^{-1}(K)/\varphi(s)^{-1}(K) \cap K) \cdot \#(K/\varphi(s)^{-1}(K) \cap K)^{-1} \\ &= \#(K/\varphi(s)(K) \cap K) \cdot \#(K/\varphi(s)^{-1}(K) \cap K)^{-1} \\ &= |\varphi(s)^{-1}(K) \cap K| \cdot |\varphi(s)(K) \cap K|^{-1} \end{aligned}$$

for $s \in S$.

Observe in passing that S acts via ψ on N in a measure-preserving way. In particular, one has $\|f^s\|_1 = \|f\|_1$ for $f \in L^1(N)_\gamma$ and $s \in S$ where $(f,s) \mapsto f^s$ is the action defined above.

For $s \in S$ and $\chi \in (H/K \cap \varphi(s)^{-1}(K))^\wedge$ define a character χ_s on $\varphi(s)^{-1}(K) + K$, actually on $(\varphi(s)^{-1}(K) + K)/\varphi(s)^{-1}(K) \cong K/K \cap \varphi(s)^{-1}(K)$, by

(4) $$\chi_s(l+k) = \langle \chi, k\rangle \quad \text{if } l \in \varphi(s)^{-1}(K) \text{ and } k \in K\,.$$

Then define $\mathfrak{q}_s \in L^1(N)_\gamma$ by

(5) $$\mathfrak{q}_s(h,\chi,t) = \begin{cases} t^{-1}\chi_s(h)^{-1}\delta(s)^{-1/2}|K \cap \varphi(s)^{-1}(K)| \\ \quad \text{if } h \in \varphi(s)^{-1}(K) + K \text{ and } \chi \in (H/K \cap \varphi(s)^{-1}(K))^\wedge\,, \\ 0 \quad \text{otherwise.} \end{cases}$$

For each $s \in S$ the equality

(6) $$\mathfrak{q}_s = (\mathfrak{q}_{s^{-1}})^{s*}$$

holds true for the following reasons: By definition of the involution and the action, one has

$$(\mathfrak{q}_{s^{-1}})^{s*}(h,\chi,t) = \overline{\mathfrak{q}}_{s^{-1}}(\varphi(s)(-h), s(\chi^{-1}), t^{-1}\langle \chi, h\rangle^{-1})\,.$$

This is zero unless $\varphi(s)(-h) \in K + \varphi(s)(K)$, which is equivalent to $h \in \varphi(s)^{-1}(K) + K$, and $s(\chi^{-1}) \in (H/K \cap \varphi(s)(K))^\wedge$, which is equivalent to $\chi \in H/K \cap \varphi(s)^{-1}(K)$. If the latter conditions are not satisfied, both functions $(\mathfrak{q}_{s^{-1}})^{s*}$ and $\mathfrak{q}_s$ vanish at (h,χ,t). Suppose that the conditions are satisfied. Write h as $h = l + k$ with $l \in \varphi(s)^{-1}(K)$ and $k \in K$. Then

$$\begin{aligned} &\overline{\mathfrak{q}}_{s^{-1}}(\varphi(s)(-h), s(\chi^{-1}), t^{-1}\langle \chi, h\rangle^{-1}) \\ &\quad = \{t\langle \chi, l+k\rangle (s(\chi^{-1}))_{s^{-1}}(\varphi(s)(h))\delta(s)^{1/2}|K \cap \varphi(s)(K)|\}^- \\ &\quad = t^{-1}\langle \chi, l+k\rangle^{-1}(s(\chi^{-1}))_{s^{-1}}(\varphi(s)(-l-k)) \\ &\qquad \times \delta(s)^{1/2}|K \cap \varphi(s)(K)|\,. \end{aligned}$$

Since $\varphi(s)(-l-k) = \varphi(s)(-l)+\varphi(s)(-k)$ with $\varphi(s)(-l) \in K$ and $\varphi(s)(-k) \in \varphi(s)(K)$ the middle term gives

$$(s(\chi^{-1}))_{s^{-1}}(\varphi(s)(-l-k)) = \langle s(\chi^{-1}), \varphi(s)(-l)\rangle = \langle \chi^{-1}, -l\rangle .$$

Hence

$$\begin{aligned}(\mathfrak{q}_{s^{-1}})^{s*}(h,\chi,t) &= t^{-1}\langle \chi, l+k\rangle^{-1}\langle \chi^{-1}, -l\rangle \delta(s)^{1/2}|K\cap\varphi(s)(K)| \\ &= t^{-1}\langle\chi,k\rangle^{-1}\delta(s)^{1/2}|K\cap\varphi(s)(K)|,\end{aligned}$$

which gives (5) in view of (3). Since $\mathfrak{q}_e = \mathfrak{p}$ one has in particular

$$\mathfrak{p}^* = \mathfrak{p}. \tag{7}$$

The L^1-norm of $\mathfrak{q}_s$ is easily computed:

$$\|\mathfrak{q}_s\|_1 = \delta(s)^{-1/2}|K\cap\varphi(s)^{-1}(K)|\cdot|\varphi(s)^{-1}(K)+K|\cdot|(H/K\cap\varphi(s)^{-1}(K))^\wedge|.$$

From the exact sequence

$$(H/K)^\wedge \to (H/K\cap\varphi(s)^{-1}(K))^\wedge \to (K/K\cap\varphi(s)^{-1}(K))^\wedge$$

one reads off that

$$|(H/K\cap\varphi(s)^{-1}(K))^\wedge| = \#(K/K\cap\varphi(s)^{-1}(K)) = |K\cap\varphi(s)^{-1}(K)|^{-1},$$

hence

$$\|\mathfrak{q}_s\|_1 = \delta(s)^{-1/2}|\varphi(s)^{-1}(K)+K|.$$

Since $(\varphi(s)^{-1}(K)+K)/K$ is isomorphic to $\varphi(s)^{-1}(K)/K\cap\varphi(s)^{-1}(K)$ the measure of $\varphi(s)^{-1}(K)+K$ equals

$$\begin{aligned}\#(\varphi(s)^{-1}(K)/K\cap\varphi(s)^{-1}(K)) &= |\varphi(s)^{-1}(K)|\cdot|K\cap\varphi(s)^{-1}(K)|^{-1} \\ &= \delta(s)|K\cap\varphi(s)^{-1}(K)|^{-1},\end{aligned}$$

hence

$$\begin{aligned}\|\mathfrak{q}_s\|_1 = \delta(s)^{1/2}|K\cap\varphi(s)^{-1}(K)|^{-1} &= \delta(s)^{1/2}\#(K/K\cap\varphi(s)^{-1}(K)) \\ = \#(K/\varphi(s)(K)\cap K)^{1/2} &\cdot \#(K/\varphi(s)^{-1}(K)\cap K)^{-1/2} \\ \times \#(K/K\cap\varphi(s)^{-1}(K))&\end{aligned}$$

because of (3). Therefore,

$$\|\mathfrak{q}_s\|_1 = \#(K/\varphi(s)(K)\cap K)^{1/2}\cdot\#(K/\varphi(s)^{-1}(K)\cap K)^{1/2}. \tag{8}$$

Moreover, the following identities hold true:

$$\begin{aligned}&\mathfrak{q}_s^* * \mathfrak{q}_s = \mathfrak{p}, \quad \mathfrak{q}_s * \mathfrak{q}_s^* = \mathfrak{p}^s, \\ &\mathfrak{p}^s * \mathfrak{q}_s = \mathfrak{q}_s = \mathfrak{q}_s * \mathfrak{p} \quad \text{for all } s\in S,\end{aligned} \tag{9}$$

in particular, $\mathfrak{p}*\mathfrak{p} = \mathfrak{p}$,

$$\mathfrak{p}^s * L^1(N)_\gamma * \mathfrak{p} = \mathbb{C}\mathfrak{q}_s \quad \text{for all } s\in S, \tag{10}$$

in particular, $\mathfrak{p} * L^1(N)_\gamma * \mathfrak{p} = \mathbb{C}\mathfrak{p}$,

$$\mathfrak{q}_r = \mathfrak{q}_{rs}^{s^{-1}} * \mathfrak{q}_{s^{-1}} \quad \text{for all } r,s\in S. \tag{11}$$

The easiest way to prove (9)–(11) is to apply the (partial) Fourier transform. For $f\in L^1(N)_\gamma$ define $\widehat{f}: H\times H\to\mathbb{C}$ by

$$\widehat{f}(h,k) = \int_{\widehat{H}} f(h,\chi,1)\langle\chi,k\rangle^{-1}d\chi. \tag{12}$$

This map yields an injective linear map from $L^1(N)_\gamma$ into $L^1(H, C_\infty(H))$, actually a bijective map from $L^1(N)_\gamma$ onto $L^1(H, A(H))$ where $A(H)$ denotes the Fourier algebra of H. The map is an isometric $*$-isomorphism of involutive Banach algebras if the norm, involution and convolution on $L^1(H, A(H))$ are defined by

$$\begin{aligned}\|\widehat{f}\| &= \int_H \|f(h,-)\|_{A(H)}\,dh, \\ \widehat{f}^*(h,k) &= \widehat{f}(h^{-1},k+h)^-, \\ (\widehat{f}*\widehat{g})(h,k) &= \int_H \widehat{f}(h+x,k-x)\widehat{g}(-x,k)\,dx.\end{aligned} \tag{13}$$

The proof of these facts is straightforward and omitted. For more information on algebras of this type see, for instance, [5]. In particular, it is shown there that $L^1(H, A(H))$ is a simple Banach algebra, hence $L^1(N)_\gamma$ is simple as well. Moreover, $f\mapsto\widehat{f}$ is S-equivariant if $\widehat{f}^s$ is defined by

$$\widehat{f}^s(h,k) = \delta(s)^{-1}\widehat{f}(\varphi(s)(h),\varphi(s)(k)) \tag{14}$$

for $s\in S$, $h,k\in H$, $\widehat{f}\in L^1(H,A(H))$.

To obtain (9)–(11) from considerations in $L^1(H,A(H))$ one clearly has to know $\widehat{\mathfrak{q}}_s$.

(15) If u denotes the characteristic function of K then

$$\widehat{\mathfrak{q}}_s(h,x) = \delta(s)^{-1/2}u(x)u(\varphi(s)(h+x)),$$

in particular, $\widehat{\mathfrak{p}}(h,x) = \widehat{\mathfrak{q}}_e(h,x) = u(x)u(h+x)$, for $h,x\in H$.

If $h\notin\varphi(s)^{-1}(K)+K$ then $\widehat{\mathfrak{q}}_s(h,x) = 0$. Suppose now that $h = l+k$ with $l\in\varphi(s)^{-1}(K)$ and $k\in K$. Then

$$\widehat{\mathfrak{q}}_s(l+k,x) = \int_{\{H/K\cap\varphi(s)^{-1}(K)\}^\wedge} d\chi\,\delta(s)^{-1/2}|K\cap\varphi(s)^{-1}(K)|\langle\chi,k\rangle^{-1}\langle\chi,x\rangle^{-1}.$$

This integral is zero unless $k+x\in K\cap\varphi(s)^{-1}(K)$. In that case one obtains

$$\widehat{\mathfrak{q}}_s(l+k,x) = \delta(s)^{-1/2}|K\cap\varphi(s)^{-1}(K)|\cdot|\{H/K\cap\varphi(s)^{-1}(K)\}^\wedge| = \delta(s)^{-1/2}.$$

But for a given pair (h,x) in $H\times H$ the conditions: $h\in\varphi(s)^{-1}(K)+K$ and for some (any) decomposition $h=l+k$ with $l\in\varphi(s)^{-1}(K)$ and $k\in K$ the sum $k+x$ lies in $K\cap\varphi(s)^{-1}(K)$, are equivalent to: $x\in K$ and $h+x\in\varphi(s)^{-1}(K)$. Therefore, $\widehat{\mathfrak{q}}_s$ is of the form as claimed in (15).

To prove $\mathfrak{q}_s^* * \mathfrak{q}_s=\mathfrak{p}$ of (9) one first observes that since, by (13), one has $(\mathfrak{q}_s^*)^\wedge(h,x)=(\mathfrak{q}_s^\wedge)^*(h,x)=\widehat{\mathfrak{q}}_s(-h,x+h)^-$, it follows that $(\mathfrak{q}_s^*)^\wedge(h,x)=\delta(s)^{-1/2}u(x+h)u(\varphi(s)(x))$ by (15). Using (13) one obtains

$$\begin{aligned}(\mathfrak{q}_s^* * \mathfrak{q}_s)^\wedge(h,k)&=\int_H(\mathfrak{q}_s^*)^\wedge(h+x,k-x)\widehat{\mathfrak{q}}_s(-x,k)\,dx\\&=\int_H\delta(s)^{-1}u(h+k)u(\varphi(s)(k-x))u(k)u(\varphi(s)(k-x))\,dx\\&=u(h+k)u(k)\int_H\delta(s)^{-1}u(\varphi(s)(x))^2\,dx\\&=\widehat{\mathfrak{p}}(h,k)\int_H u(x)^2dx=\widehat{\mathfrak{p}}(h,k)\,.\end{aligned}$$

The proof for $\mathfrak{q}_s * \mathfrak{q}_s^*=\mathfrak{p}^s$ is similar and omitted. To show $\mathfrak{p}^s * \mathfrak{q}_s=\mathfrak{q}_s$ one first uses (14) to obtain

$$\begin{aligned}\mathfrak{p}^{s\wedge}(h,k)&=\delta(s)^{-1}\widehat{\mathfrak{p}}(\varphi(s)(h),\varphi(s)(k))\\&=\delta(s)^{-1}u(\varphi(s)(k))u(\varphi(s)(h+k))\,.\end{aligned}\tag{16}$$

Hence by (13),

$$\begin{aligned}(\mathfrak{p}^s * \mathfrak{q}_s)^\wedge(h,k)&=\int_H\mathfrak{p}^{s\wedge}(h+x,k-x)\widehat{\mathfrak{q}}_s(-x,k)\,dx\\&=\int_H\delta(s)^{-1}u(\varphi(s)(k-x))u(\varphi(s)(h+k))\\&\quad\times\delta(s)^{-1/2}u(k)u(\varphi(s)(k-x))\,dx\\&=\delta(s)^{-1/2}u(k)u(\varphi(s)(h+k))\int_H\delta(s)^{-1}u(\varphi(s)(x))^2\,dx\\&=\widehat{\mathfrak{q}}_s(h,k)\,.\end{aligned}$$

Also the proof of $\mathfrak{q}_s=\mathfrak{q}_s * \mathfrak{p}$ is omitted as well as the proof of (11); they are straightforward calculations of the same type.

To show (10) one has to compute $\mathfrak{p}^{s\wedge} * \widehat{f} * \widehat{\mathfrak{p}}$ for any $\widehat{f}\in L^1(H,A(H))$. First one observes that for $\widehat{f}=\widehat{\mathfrak{q}}_s$ one has $\mathfrak{p}^{s\wedge} * \widehat{\mathfrak{q}}_s * \widehat{\mathfrak{p}}=\mathfrak{q}_s^\wedge$ by (9), hence

$\mathfrak{p}^s * L^1(N)_\gamma * \mathfrak{p}\supset\mathbb{C}\mathfrak{q}_s$. Now let $\widehat{f}$ be arbitrary. By (13),

$$(\widehat{f} * \widehat{\mathfrak{p}})(h,k)=\int_H\widehat{f}(h+x,k-x)u(k)u(k-x)\,dx\,,$$

and by (16) and (13),

$$\begin{aligned}(\mathfrak{p}^{s\wedge} * \widehat{f} * \widehat{\mathfrak{p}})(h,k)&=\int_H\mathfrak{p}^{s\wedge}(h+y,k-y)(\widehat{f} * \widehat{\mathfrak{p}})(-y,k)\,dy\\&=\delta(s)^{-1/2}\,u(\varphi(s)(h+k))u(k)\delta(s)^{-1/2}\\&\quad\times\int_H\int_H u(\varphi(s)(k-y))u(k-x)\widehat{f}(x-y,k-x)\,dx\,dy\,.\end{aligned}$$

Substituting $y'=k-y$ and $x'=k-x$ yields

$$\begin{aligned}(\mathfrak{p}^{s\wedge} * \widehat{f} * \widehat{\mathfrak{p}})(h,k)&=\delta(s)^{-1/2}u(\varphi(s)(h+k))\;u(k)\delta(s)^{-1/2}\\&\quad\times\int_H\int_H u(\varphi(s)(y))u(x)\widehat{f}(y-x,x)\,dx\,dy\,,\end{aligned}$$

which is $\mathfrak{q}_s^\wedge(h,k)$ times a scalar independent of h and k.

The map $w:S\to\mathbb{R}$ defined by

$$w(s)=\|\mathfrak{q}_s\|_1=\#(K/K\cap\varphi(s)(K))^{1/2}\cdot\#\,(K/K\cap\varphi(s)^{-1}(K))^{1/2}$$

(cf. (8)) is clearly submultiplicative and greater than or equal to one, i.e., it is a weight function. For this notion see [7]. Therefore,

$$\ell^1(S,w)=\Big\{f:S\to\mathbb{C}\,\Big|\,\sum_{s\in S}|f(s)|w(s)<\infty\Big\}$$

is a subalgebra of the convolution algebra $\ell^1(S)$. Moreover, w is symmetric, $w(s^{-1})=w(s)$, hence $\ell^1(S,w)$ is an involutive subalgebra of $\ell^1(S)$.

Using the foregoing notations and formulas one can show the following proposition.

PROPOSITION. *The map $\ell^1(S,w)\to L^1(G)_\gamma$, $\Phi\mapsto\Phi'$, given by $\Phi'(s,h,\chi,t)=\Phi(s)\mathfrak{q}_s(h,\chi,t)$ is an isometric $*$-isomorphism from $\ell^1(S,w)$ onto $\mathfrak{p} * L^1(G)_\gamma * \mathfrak{p}$. If π is a continuous unitary representation of G in $\mathfrak{H}$ with $\pi(t)=t=\gamma(t)$ for $t\in\mathbb{T}$ then π yields involutive representations of $L^1(G)_\gamma$ and of $L^1(N)_\gamma$, also denoted by π. The operator $\pi(\mathfrak{p})$ is a nonzero orthogonal projection onto $\mathfrak{H}^{\mathfrak{p}}$, say. The map $\ell^1(S,w)\ni\Phi\mapsto\pi(\Phi')|\mathfrak{H}^{\mathfrak{p}}$ is an involutive representation of $\ell^1(S,w)$. It is obtained by integrating the unitary representation $\pi^{\mathfrak{p}}$ of S given by $\pi^{\mathfrak{p}}(s)=\pi(s)\pi(\mathfrak{q}_s)|\mathfrak{H}^{\mathfrak{p}}$. The representation π is irreducible iff $\pi^{\mathfrak{p}}$ is.*

In case that π is irreducible the following equivalences hold true.

The algebra $\pi(C^*(G))$ *contains the algebra of compact operators on* $\mathfrak{H}$ *iff* $\pi^{\mathfrak{p}}(C^*(S))$ *contains the algebra of compact operators on* $\mathfrak{H}^{\mathfrak{p}}$. *The algebra* $\pi(L^1(G))$ *contains a nonzero compact operator iff* $\pi^{\mathfrak{p}}(\ell^1(S,w))$ *does.*

Proof. The equality $\|\varPhi'\|_{L^1(G)} = \|\varPhi\|_{\ell^1(S,w)}$ is an immediate consequence of the definitions. To prove the multiplicativity of $\varPhi \mapsto \varPhi'$ let $\varPhi, \varPsi \in \ell^1(S,w)$. Then

$$(\varPhi * \varPsi)'(rx) = (\varPhi * \varPsi)(r)\mathfrak{q}_r(x) \quad \text{for } r \in S,\ x \in N,$$

and

$$\begin{aligned}(\varPhi' * \varPsi')(rx) &= \sum_{s\in S} \int_N dy\, \varPhi'(rxsy)\varPsi'((sy)^{-1}) \\ &= \sum_{s\in S} \int_N dy\, \varPhi'(rss^{-1}xsy)\varPsi'(s^{-1}sy^{-1}s^{-1}) \\ &= \sum_{s\in S} \int_N dy\, \varPhi(rs)\mathfrak{q}_{rs}(s^{-1}xsy)\varPsi(s^{-1})\mathfrak{q}^s_{s^{-1}}(y^{-1}) \\ &= \sum_{s\in S} \varPhi(rs)\varPsi(s^{-1})(\mathfrak{q}_{rs} * \mathfrak{q}^s_{s^{-1}})(s^{-1}xs).\end{aligned}$$

But $(\mathfrak{q}_{rs} * \mathfrak{q}^s_{s^{-1}})(s^{-1}xs) = (\mathfrak{q}_{rs} * \mathfrak{q}^s_{s^{-1}})^{s^{-1}}(x) = (\mathfrak{q}^{s^{-1}}_{rs} * \mathfrak{q}_{s^{-1}})(x) = \mathfrak{q}_r(x)$ by (11). Therefore,

$$(\varPhi' * \varPsi')(rx) = \sum_{s\in S} \varPhi(rs)\varPsi(s^{-1})\mathfrak{q}_r(x) = (\varPhi * \varPsi)(r)\mathfrak{q}_r(x)$$

as desired.

The equality $(\varPhi')^* = (\varPhi^*)'$ for $\varPhi \in \ell^1(S,w)$ is an easy consequence of (6), we omit the details. Hence $\varPhi \mapsto \varPhi'$ is an isometric $*$-morphism from $\ell^1(S,w)$ into $L^1(G)_\gamma$.

To show that each $\varPhi'$ is contained in $\mathfrak{p} * L^1(G)_\gamma * \mathfrak{p}$ it suffices to prove that $\mathfrak{p} * \varPhi' = \varPhi'$ and $\varPhi' * \mathfrak{p} = \varPhi'$ because $\mathfrak{p} * \mathfrak{p} = \mathfrak{p}$ by (9). But

$$\begin{aligned}(\mathfrak{p} * \varPhi')(rx) &= \int_N dy\, \mathfrak{p}(y)\varPhi'(y^{-1}rx) \\ &= \int_N dy\, \mathfrak{p}(y)\varPhi'(rr^{-1}y^{-1}rx) \\ &= \varPhi(r) \int_N dy\, \mathfrak{p}(y)\mathfrak{q}_r(r^{-1}y^{-1}rx) \\ &= \varPhi(r) \int_N dy\, \mathfrak{p}(y)\mathfrak{q}_r^{r^{-1}}(y^{-1}rxr^{-1}) = \varPhi(r)(\mathfrak{p} * \mathfrak{q}_r^{r^{-1}})(rxr^{-1}) \\ &= \varPhi(r)(\mathfrak{p}^r * \mathfrak{q}_r)(x) = \varPhi(r)\mathfrak{q}_r(x) \quad \text{by (9)}.\end{aligned}$$

The proof for $\varPhi' * \mathfrak{p} = \varPhi'$ is similar, one has to use $\mathfrak{q}_s * \mathfrak{p} = \mathfrak{q}_s$ (see (9)).

To show that the image of $\varPhi \mapsto \varPhi'$ coincides with $\mathfrak{p} * L^1(G)_\gamma * \mathfrak{p}$ let f in the latter set be given. One readily computes that for $s \in S$ the function $f(s-)$ on N is in $\mathfrak{p}^s * L^1(N)_\gamma * \mathfrak{p}$. Since the latter space is one-dimensional, consisting of multiples of $\mathfrak{q}_s$ (see (10)), there is a function $\widetilde{f} : S \to \mathbb{C}$ such that

$$f(s-) = \widetilde{f}(s)\mathfrak{q}_s,$$

or

$$f(sx) = \widetilde{f}(s)\mathfrak{q}_s(x) \quad \text{for } s \in S,\ x \in N.$$

But as f is an L^1-function the function $\widetilde{f}$ has to be in $\ell^1(S,w)$, and one finds $f = (\widetilde{f})'$.

It is clear that a representation π as in the Proposition yields by integration representations of $L^1(G)_\gamma$ and $L^1(N)_\gamma$. From (7) and (9) it follows that $\pi(\mathfrak{p})$ is an orthogonal projection. If $\pi(\mathfrak{p})$ were zero, this would yield that $\pi(L^1(N)_\gamma * \mathfrak{p} * L^1(N)_\gamma)$ would be zero. But as $L^1(N)_\gamma$ is a simple Banach algebra, and hence $L^1(N)_\gamma * \mathfrak{p} * L^1(N)_\gamma$ is total in $L^1(N)_\gamma$, one would obtain $\pi(L^1(N)_\gamma) = 0$, which is impossible.

Since $\ell^1(S,w)$ is $*$-isomorphic to $\mathfrak{p} * L^1(G)_\gamma * \mathfrak{p}$ it is obvious that $\varPhi \mapsto \pi(\varPhi')|\mathfrak{H}^{\mathfrak{p}}$ is an involutive representation of $\ell^1(S,w)$. It follows easily from the definitions that this representation is obtained by integrating $\pi^{\mathfrak{p}}$ as given in the Proposition. From this fact it follows that $\pi^{\mathfrak{p}}$ is a unitary representation. This can also be verified directly, using (6) and (11).

Suppose now that π is not irreducible, i.e., that π is the direct sum $\sigma \oplus \tau$ of two nonzero unitary representations σ and τ of G with $\sigma|_{\mathbb{T}} = \gamma\,\mathrm{Id}$ and $\tau|_{\mathbb{T}} = \gamma\,\mathrm{Id}$. Then it is easily checked that $\pi^{\mathfrak{p}}$ is the direct sum of the nonzero representations $\sigma^{\mathfrak{p}}$ and $\tau^{\mathfrak{p}}$. Therefore, the irreducibility of $\pi^{\mathfrak{p}}$ implies the irreducibility of π.

On the other hand, if π is irreducible let ξ be any nonzero vector in $\mathfrak{H}^{\mathfrak{p}}$. It is sufficient to show that $\pi^{\mathfrak{p}}(\ell^1(S,w))\xi = \pi(\mathfrak{p} * L^1(G)_\gamma * \mathfrak{p})\xi$ is dense in $\mathfrak{H}^{\mathfrak{p}}$. But $\pi(\mathfrak{p} * L^1(G)_\gamma * \mathfrak{p})\xi = \pi(\mathfrak{p})\pi(L^1(G))\xi$. Since $\pi(L^1(G))\xi$ is dense in $\mathfrak{H}$ the claim follows.

For the rest of the proof let π be irreducible. Denote by $\mathcal{K}(\mathfrak{H})$ and $\mathcal{K}(\mathfrak{H}^{\mathfrak{p}})$ the algebras of compact operators on $\mathfrak{H}$ and $\mathfrak{H}^{\mathfrak{p}}$, respectively. Suppose that $\pi^{\mathfrak{p}}(\ell^1(S,w))$ contains a nonzero compact operator, say $\pi^{\mathfrak{p}}(\varPhi), \varPhi \in \ell^1(S,w)$. Then $\pi(\varPhi')$ is a nonzero compact operator. Next suppose that $\pi(L^1(G))$ contains nonzero compact operators. Then let I be the ideal of all $f \in L^1(G)_\gamma$ such that $\pi(f)$ is compact. If ξ is any nonzero vector in $\mathfrak{H}^{\mathfrak{p}}$ then $\pi(I)\xi$ is dense in $\mathfrak{H}$ as π is irreducible. In particular, $\pi(\mathfrak{p})\pi(I)\xi = \pi(\mathfrak{p})\pi(I)\pi(\mathfrak{p})\xi$ is different from zero. Hence there exist $f \in I$ such that $\pi(\mathfrak{p})\pi(f)\pi(\mathfrak{p}) = \pi(\mathfrak{p} * f * \mathfrak{p}) \neq 0$. If $\varPhi \in \ell^1(S,w)$ satisfies $\varPhi' = \mathfrak{p} * f * \mathfrak{p}$ then $\pi^{\mathfrak{p}}(\varPhi)$ is a nonzero compact operator.

Now suppose that $\pi(C^*(G)) = \pi(L^1(G))^-$ contains $\mathcal{K}(\mathfrak{H})$ where the closure is taken in the operator norm. In particular, for each $T \in \mathcal{K}(\mathfrak{H}^\mathfrak{p})$ (the latter space being considered in the most obvious way as a subset of $\mathcal{K}(\mathfrak{H})$) there exists a sequence (f_n) in $L^1(G)_\gamma$ such that $(\pi(f_n))$ converges to T. Then $(\pi(\mathfrak{p} * f_n * \mathfrak{p}))$ converges to T. If Φ_n in $\ell^1(S, w)$ is determined by $\Phi'_n = \mathfrak{p} * f_n * \mathfrak{p}$ then $(\pi^\mathfrak{p}(\Phi_n))$ converges to T.

Finally, suppose that $\pi^\mathfrak{p}(C^*(S))$ contains $\mathcal{K}(\mathfrak{H}^\mathfrak{p})$. Since $\pi^\mathfrak{p}(C^*(S)) = \pi^\mathfrak{p}(\ell^1(S))^- = \pi^\mathfrak{p}(\ell^1(S,w))^-$, for each $T \in \mathcal{K}(\mathfrak{H}^\mathfrak{p})$ there exists a sequence (Φ_n) in $\ell^1(S,w)$ such that $(\pi^\mathfrak{p}(\Phi_n))$ converges to T. Then $(\pi(\Phi'_n))$ converges to T, hence $\pi(C^*(G))$ contains $\mathcal{K}(\mathfrak{H}^\mathfrak{p})$. By the irreducibility of π it contains all of $\mathcal{K}(\mathfrak{H})$.

Remark. By a theorem of Green [2], the C^*-hull $C^*(G)_\gamma$ of $L^1(G)_\gamma$ is isomorphic to the C^*-tensor product of $C^*(S)$ and $\mathcal{K}(L^2(H))$. This explains why irreducible or irreducible generalized completely continuous representations of S correspond to those of G as long as the latter are equal to γ on $\mathbb{T}$.

To obtain the desired example the groups H, K, S and the homomorphism $\varphi : S \to \mathrm{Aut}(H)$ are now specified. Let p be any prime number, denote by $\mathbb{Q}_p$ the field of p-adic numbers and by $\mathbb{Z}_p$ the ring of p-adic integers with its usual topology. We will mainly view $\mathbb{Q}_p$ and $\mathbb{Z}_p$ as locally compact abelian groups under addition; their multiplicative structure is used to define automorphisms.

Let H be the restricted direct product of copies of $\mathbb{Q}_p$ over the integers with respect to the compact open subgroup $\mathbb{Z}_p$, i.e.,

$$H = \{h : \mathbb{Z} \to \mathbb{Q}_p \mid h(j) \in \mathbb{Z}_p \text{ for almost all } j \in \mathbb{Z}\}.$$

The subgroup K of H consisting of all maps $h : \mathbb{Z} \to \mathbb{Z}_p$, which is isomorphic to $\mathbb{Z}_p^\mathbb{Z}$, is declared to be open in H, and K is endowed with the product topology. This way H is a locally compact abelian group.

The group S is the semidirect product of $A = \mathbb{Z}$ and $B = \mathbb{Z}^{(\mathbb{Z})}$, the direct sum over $\mathbb{Z}$ of copies of $\mathbb{Z}$. The multiplication in $S = A \ltimes B$ is given by

$$(a,b)\,(a',b') = (a+a', b'')$$

where the jth component b''_j of $b'' \in \mathbb{Z}^{(\mathbb{Z})}$ is defined by $b''_j = b_{j+a'} + b'_j$.

Finally, the homomorphism $\varphi : S \to \mathrm{Aut}(H)$ is defined by

$$[\varphi(a,b)h](j) = p^{b_{j-a}} h(j-a). \tag{17}$$

Altogether, on $G = A \times B \times H \times \widehat{H} \times \mathbb{T}$ endowed with the product topology, the general formula (1) gives a group multiplication

$$(a,b,h,\chi,t)\,(a',b',h',\chi',t') = (a'',b'',h'',\chi'',t''), \tag{18}$$

where $a'' = a + a'$, $b''_j = b_{j+a'} + b'_j$, $h'' = \varphi(a',b')^{-1}(h) + h'$, $\langle \chi'', x\rangle = \langle \chi, \varphi(a',b')(x)\rangle \chi'(x)$ for $x \in H$, and $t'' = tt'\langle \chi, \varphi(a',b')(-h')\rangle$.

Lemma 1. *The locally compact group G is compactly generated and separable, i.e., it has a countable basis of the topology. The weight function w and the modular function δ of the action of $S = A \ltimes B$ on H* (compare (8) and (3)) *are given by*

$$w(a,b) = p^{\frac{1}{2}\Sigma_{j=-\infty}^{\infty}|b_j|}, \quad \delta(a,b) = p^{\Sigma_{j=-\infty}^{\infty} b_j} \quad \textit{if} \quad b = (b_j) \in B = \mathbb{Z}^{(\mathbb{Z})}.$$

Proof. Let $b^0 \in B$ be defined by $b^0_0 = 1$ and $b^0_j = 0$ for $j \neq 0$. It is evident that S is generated by $(1,0)$ and $(0,b^0)$. Hence the subgroup L of G generated by the compact set $K \times (H/K)^\wedge \times \mathbb{T} \cup \{(1,0),(0,b^0)\}$ contains S and $K \times (H/K)^\wedge \times \mathbb{T}$. Conjugating $K \times (H/K)^\wedge \times \mathbb{T}$ by elements in $B \subset S$ produces the whole of $H \times \widehat{H} \times \mathbb{T}$. Hence $L = G$.

The question of separability reduces at once to H and $\widehat{H}$. But H is a countable extension of the compact metrizable group $K = \mathbb{Z}_p^\mathbb{Z}$, hence is separable. Moreover, the group H is selfdual. This can be seen as follows. The quotient $\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to $\mathbb{Z}[1/p]/\mathbb{Z}$. The latter group can be identified with a subgroup of $\mathbb{Q}/\mathbb{Z}$ or of $\mathbb{R}/\mathbb{Z}$ which is isomorphic to $\mathbb{T}$ in the usual manner. This way we find a canonical $\kappa \in \mathbb{Q}_p^\wedge$ with $\ker\kappa = \mathbb{Z}_p$. Then define $H \times H \to \mathbb{T}$ by

$$((q_j),(r_j)) \mapsto \prod_{j=-\infty}^{\infty} \kappa(q_j r_j).$$

It is easy to see that this pairing establishes an isomorphism from H onto $\widehat{H}$.

The formulas for w and δ follow at once from the fact that for $n \in \mathbb{Z}$ the cardinality of $\mathbb{Z}_p/p^n\mathbb{Z}_p \cap \mathbb{Z}_p$ is one for $n \le 0$ and p^n for $n \ge 0$.

The Pontryagin dual $\widehat{B}$ is isomorphic to $\mathbb{T}^\mathbb{Z}$. Each $z = (z_j) \in \mathbb{T}^\mathbb{Z}$ defines a character $\eta_z \in \widehat{B}$ by

$$\eta_z(b) = \prod_{j=-\infty}^{\infty} z_j^{b_j}. \tag{19}$$

The character η_z extends to a character $\widetilde{\eta}_z$ of the subgroup $\{0\} \times B \times \{0\} \times \widehat{H} \times \mathbb{T}$ of G by $\widetilde{\eta}_z(0,b,0,\chi,t) = t\eta_z(b)$. This character is induced to obtain a representation, say π_z, of G. The representation π_z can be realized in $L^2(A \times H)$ where $A \times H$ carries the product measure of the Haar measures on $A = \mathbb{Z}$ and H. One finds that

$$\begin{aligned}\{\pi_z(a,b,h,\chi,t)\xi\}(a',h') = \delta(a,b)^{1/2} t \langle \chi, h - \varphi(a'-a,\beta)(h')\rangle \eta_z(-\beta)\\ \times \xi(a'-a, \varphi(0,\beta)(h') - \varphi(a-a',0)(h))\end{aligned}$$

where $\beta \in B$ is given by $\beta_j = -b_{j+a'-a}$.

With π_z there is associated (see the Proposition) a representation $\pi_z^{\mathfrak{p}}$ of $S = A \ltimes B$ in $\mathfrak{H}^{\mathfrak{p}}$. The space $\mathfrak{H}^{\mathfrak{p}} = \pi_z(\mathfrak{p})(L^2(A \times H))$ is easily identified. More generally, we shall compute the operator $\pi_z(\mathfrak{q}_s)$, $s \in S$; for the definition of $\mathfrak{q}_s$, see (5) and (15).

For $\xi \in L^2(A \times H)$,

$$\begin{aligned}\{\pi_z(\mathfrak{q}_s)\xi\}(a',h') &= \int_H dh \int_{\widehat{H}} d\chi \int_{\mathbb{T}} dt\, \mathfrak{q}_s(h,\chi,t) \\ &\quad \times t\langle \chi, h - \varphi(a',0)(h')\rangle \xi(a', h' - \varphi(-a',0)(h)) \\ &= \int_H dh\, \widehat{\mathfrak{q}}_s(h, \varphi(a',0)(h') - h)\xi(a', h' - \varphi(-a',0)(h)) \\ &= \int_H dh\, \delta(s)^{-1/2} u(\varphi(a',0)(h') - h) \\ &\quad \times u(\varphi(s)\varphi(a',0)(h'))\xi(a', h' - \varphi(-a',0)(h)).\end{aligned}$$

Substituting $h'' = h' - \varphi(-a',0)(h)$ yields

$$\{\pi_z(\mathfrak{q}_s)\xi\}(a',h') = \delta(s)^{-1/2} u(\varphi(s)\varphi(a',0)(h')) \int_H dh\, u(\varphi(a',0)(h))\, \xi(a',h).$$

In particular,

$$\{\pi_z(\mathfrak{p})\xi\}(a',h') = u(\varphi(a',0)(h')) \int_H dh\, u(\varphi(a',0))\, \xi(a',h).$$

One verifies easily that

(21) The map $V : \ell^2(A) \to \mathfrak{H}^{\mathfrak{p}}$ defined by

$$(V\zeta)(a',h') = u(\varphi(a',0)(h'))\zeta(a')$$

is unitary.

Transferring via V the representation $\pi_z^{\mathfrak{p}}$ of S in $\mathfrak{H}^{\mathfrak{p}}$ into the space $\ell^2(A)$ one gets a representation ϱ_z of S in $\ell^2(A)$ given by

$$\{\varrho_z(a,b)\zeta\}(a') = \eta_z((b_{j+a'-a})_j)\zeta(a'-a) = \prod_{j=-\infty}^{\infty} z_j^{b_{j+a'-a}} \zeta(a'-a). \tag{22}$$

This formula follows from the definitions of V and $\pi_z^{\mathfrak{p}}$, $\pi_z^{\mathfrak{p}}(s) = \pi_z(s)\pi_z(\mathfrak{q}_s)$, and from the above determined structure of $\pi_z(\mathfrak{q}_s)$. The easy computation is omitted. Of course, ϱ_z is nothing but $\mathrm{ind}_B^S \eta_z$ realized in $\ell^2(A)$.

LEMMA 2. *The representation π_z of G, $z \in \mathbb{T}^{\mathbb{Z}}$, is irreducible if and only if the sequence z is not periodic, i.e., there is no positive integer m such that $z_{j+m} = z_j$ for all $j \in \mathbb{Z}$. If this condition is satisfied then π_z is a generalized completely continuous representation if and only if the A-orbit $\Omega_z = \{(z_{j+a})_j \mid a \in A\}$ is locally closed in $\mathbb{T}^{\mathbb{Z}}$.*

Remark. In order to establish the relation to the results in [3] we observe that the condition "Ω_z is locally closed in $\mathbb{T}^{\mathbb{Z}}$" is equivalent to "Ω_z is a discrete subset of $\mathbb{T}^{\mathbb{Z}}$" for the following reasons. Clearly, any discrete subspace is locally closed. If the A-orbit Ω_z is locally closed then under the map $a \mapsto az = (z_{j-a})_j$ the subspace Ω_z is homeomorphic to A as the stabilizer group is trivial. Hence Ω_z is discrete.

Proof of Lemma 2. By the Proposition the questions of whether π_z is irreducible or whether π_z is a generalized completely continuous representation, can be reduced to the corresponding questions for the representation ϱ_z of S. In the latter case the answers are known (see [3]). We shall repeat here the essential arguments. This gives the opportunity to introduce some notations which will be needed later anyway.

If z is periodic, say $z_{j+m} = z_j$ for all j, then the operator $M : \ell^2(A) \to \ell^2(A)$, $(M\zeta)(a') = \zeta(a'+m)$ commutes with $\varrho_z(S)$, hence ϱ_z is not irreducible.

Now suppose that z is not periodic, and let $U : \ell^2(A) \to \ell^2(A)$ be any intertwining operator for ϱ_z. Let ε_0 be the "Dirac delta" in $\ell^2(A)$, and let $\varepsilon := U\varepsilon_0 \in \ell^2(A)$. From $U\varrho(0,b) = \varrho(0,b)U$ it follows that $\varrho(0,b)\varepsilon = \eta_z(b)\varepsilon$ for all $b \in B$. As z is not periodic the latter identity implies that ε is a scalar multiple of ε_0, say $\varepsilon = \lambda\varepsilon_0$. Since U commutes with the translations $\varrho(a,0)$, and since the translates of ε_0 span $\ell^2(A)$, one concludes that $U = \lambda\,\mathrm{Id}$.

The L^1-group algebra of the semidirect product $S = A \ltimes B$ may be considered in the usual way as the L^1-covariance algebra $\ell^1(A, \ell^1(B))$ (see [4]). Via Fourier transform the C^*-hull of $\ell^1(B)$ is nothing but $C(\widehat{B})$, and $C^*(S)$ is the C^*-covariance algebra $C^*(A, C(\widehat{B}))$. The L^1-covariance algebra $\ell^1(A, C(\widehat{B}))$ lies half way between $\ell^1(S)$ and $C^*(S)$: there are (norm-decreasing) embeddings

$$\ell^1(A, \ell^1(B)) \to \ell^1(A, C(\widehat{B})) \to C^*(A, C(\widehat{B})).$$

The representation ϱ_z yields representations of $\ell^1(A, C(\widehat{B}))$ and of $C^*(A, C(\widehat{B}))$, also denoted by ϱ_z. The image $\varrho_z(C^*(S))$ contains nonzero compact operators if and only if there exist continuous functions Φ on $\widehat{B} = \mathbb{T}^{\mathbb{Z}}$ such that Φ is not identically zero on Ω_z, but Φ is zero on $\overline{\Omega}_z \setminus \Omega_z$ where $\overline{\Omega}_z$ denotes the closure of Ω_z. Such functions exist precisely when Ω_z is locally closed. In this case for $g \in \ell^1(A, C(\widehat{B}))$ the operator $\varrho_z(g)$ is compact if and only if for all $a \in A$ the function $g(a) \in C(\widehat{B})$ vanishes on $\overline{\Omega}_z \setminus \Omega_z$.

The proof of Lemma 2 is finished. It has also shown what we have to do further. We have to specify a locally closed A-orbit Ω_z such that the above condition on g is not satisfied for functions in the image of $\ell^1(S,w)$ under the map $\ell^1(A,\ell^1(B)) \to \ell^1(A,C(\widehat{B}))$, unless $\varrho_z(g)=0$ (compare the Proposition). To this end we need a little lemma on a particular decomposition of the integers.

LEMMA 3. *Let D be a countable set. There exists a decomposition $\mathbb{Z} = \dot\bigcup_{d\in D} C_d$ of the set of integers with the following property: If n is any positive integer and if $d_{-n},\ldots,d_{-1},d_0,\ldots,d_n$ are any elements in D then the intersection*

$$\bigcap_{j=-n}^{n} (C_{d_j} - j)$$

is not empty (and hence infinite). In particular, all the sets C_d are infinite.

Proof. Let $(D_n)_{n\in\mathbb{N}}$ be an increasing sequence of finite subsets of D with $\bigcup_{n\in\mathbb{N}} D_n = D$. First we claim that for each $n\in\mathbb{N}$ there exists a collection $(C_d^{(n)})_{d\in D_n}$ of disjoint finite subsets of $\mathbb{Z}$ with the following properties:

(i) $\Big\{\bigcup_{d\in D_n} C_d^{(n)}\Big\} \cap \Big\{\bigcup_{m<n}\bigcup_{d\in D_m} C_d^{(m)}\Big\} = \emptyset.$

(ii) If $d_{-n},\ldots,d_0,\ldots,d_n$ are any elements in D_n then $\bigcap_{j=-n}^{n}(C_{d_j}^{(n)} - j)$ is not empty.

It is easy to see that such collections exist because in (ii) there are only finitely many conditions to be fulfilled; and clearly for a given n the sets $C_d^{(n)}$, $d\in D_n$, can be chosen in the complement of the previously constructed finitely many finite sets.

Then for each $d\in D$ choose an $m\in\mathbb{N}$ with $d\in D_m$ and put $C'_d = \bigcup_{n\ge m} C_d^{(n)}$. The sets C'_d, $d\in D$, are pairwise disjoint. Finally, choose any family C_d, $d\in D$, with $C'_d\subset C_d$ for each d and $\mathbb{Z} = \dot\bigcup_{d\in D} C_d$ (for instance $C'_d = C_d$ for all $d\in D$ except for a distinguished point d_0). Such a family has the claimed property.

To see that for any given n and any given sequence $d_{-n},\ldots,d_0,\ldots,d_n$ in D the intersection $\bigcap_{j=-n}^{n}(C_{d_j}-j)$ is automatically an infinite set, let t be any positive integer, let $m = n+t(2n+1)$, and define the sequence $d'_{-m},\ldots,d'_m$ in D by $d'_k = d_j$ if $k\equiv j \bmod (2n+1)$ and $|j|\le n$. As $\bigcap_{k=-m}^{m}(C_{d'_k}-k)\ne\emptyset$, we may take a number y in this intersection. It is easily verified that then the numbers $y+s(2n+1)$, $s\in\mathbb{Z}$, $|s|\le t$, are contained in $\bigcap_{j=-n}^{n}(C_{d_j}-j)$, hence the latter intersection contains at least $2t+1$ elements.

In particular, let D be a countable subset of $\mathbb{T}$ such that $1\notin D$ and that the closure $\overline{D}$ equals $D\cup\{1\}$. For each $d\in D$ choose $r_d>0$ such that

$$\{x\in\mathbb{C} \mid |x-d|\le 2r_d\}\cap\overline{D} = \{d\}. \tag{23}$$

Let $\mathbb{Z} = \dot\bigcup_{d\in D} C_d$ be a decomposition according to Lemma 3. Choose $z = (z_j)\in\mathbb{T}^{\mathbb{Z}}$ with the following properties:

(24) The map $\mathbb{Z}\ni j\mapsto z_j\in\mathbb{T}$ is injective.

(25) If $j\in C_d$ then $0<|z_j-d|<r_d$.

(26) For each $d\in D$ and each $r>0$ the set $\{j\in C_d \mid |z_j-d|\ge r\}$ is finite.

These conditions imply that d is the only cluster point of $\{z_j \mid j\in C_d\}$, that $\overline{D}$ and $\{z_j\mid j\in\mathbb{Z}\}$ are disjoint, and that $\overline{D}\cup\{z_j\mid j\in\mathbb{Z}\}$ is a closed subset of $\mathbb{T}$.

LEMMA 4. *Let $z\in\mathbb{T}^{\mathbb{Z}}$ be as above and let $\Omega=\Omega_z$ be its orbit under the "shift group", i.e., $\Omega = \{(z_{j+a})_j \mid a\in\mathbb{Z}\}$. Then Ω is locally closed in $\mathbb{T}^{\mathbb{Z}}$ and the closure $\overline{\Omega}$ equals $\Omega\cup\overline{D}^{\mathbb{Z}}$, which is a disjoint union since $\overline{D}$ and $\{z_j\mid j\in\mathbb{Z}\}$ are disjoint subsets of $\mathbb{T}$.*

Proof. To prove that $\Omega\cup\overline{D}^{\mathbb{Z}}$ is contained in $\overline{\Omega}$ it is clearly sufficient to verify that any $x=(x_j)\in D^{\mathbb{Z}}$ is contained in $\overline{\Omega}$. To this end, let any $n\in\mathbb{N}$ and $\varepsilon>0$ be given. We have to show that there exists an $a\in A=\mathbb{Z}$ such that

$$|z_{j+a}-x_j|<\varepsilon \quad \text{for } |j|\le n\,. \tag{27}$$

By Lemma 3 the set $A' := \{a\in A \mid j+a\in C_{x_j} \text{ for } |j|\le n\}$ is infinite. By (26), for almost all $a\in A'$ the inequalities (27) are true.

To prove conversely that $\overline{\Omega}$ is contained in $\Omega\cup\overline{D}^{\mathbb{Z}}$, let x be a given point in $\overline{\Omega}$. Since $\overline{D}\cup\{z_j\mid j\in\mathbb{Z}\}$ is closed in $\mathbb{T}$, each x_k is contained in this set. If each x_k is contained even in $\overline{D}$ we are done. So, assume that there is a $k_0\in\mathbb{Z}$ with $x_{k_0}=z_{j_0}$ for some j_0. We have to show that then $x\in\Omega$. By applying a suitable element in the shift group we may suppose that $x_{j_0}=z_{j_0}$ for some j_0, and our claim reduces to $x=z$. Given j_0 from the properties (24)–(26) of z it follows that there exists an $\varepsilon_0>0$ such that

$$|z_j-z_{j_0}|<\varepsilon_0 \quad \text{implies} \quad j=j_0\,. \tag{28}$$

Then take any $j\in\mathbb{Z}$ and any ε, $0<\varepsilon<\varepsilon_0$. Since x is in $\overline{\Omega}$ there is an $a=a(j,\varepsilon)\in A$ such that

$$|z_{j+a}-x_j|<\varepsilon \quad\text{and}\quad |z_{j_0+a}-x_{j_0}|<\varepsilon\,.$$

As $x_{j_0}=z_{j_0}$ from (28) we deduce that $a=0$, hence $|z_j-x_j|<\varepsilon$. Since ε and j were arbitrary, we conclude that $x=z$.

The known structure of $\overline{\Omega}$ yields $\overline{\Omega}\setminus\Omega = \overline{D}^{\mathbb{Z}}$, which is a closed subset of $\mathbb{T}^{\mathbb{Z}}$. Therefore, Ω is locally closed.

THEOREM. *Let $G = A\times B\times H\times\widehat{H}\times\mathbb{T}$ be the group as constructed above* (see in particular (17) and (18)), *and let $z\in\mathbb{T}^{\mathbb{Z}}$ be a point as above* (compare (23) through (26)). *Then the continuous unitary representation* π_z *of* G (see (20), (17) and Lemma 1) *is irreducible and* $\pi_z(C^*(G))$ *contains the algebra of compact operators, while* $\pi_z(L^1(G))$ *contains no compact operator except for zero.*

Proof. By Lemma 2, since clearly z is not periodic and since the A-orbit Ω_z of z is locally closed by Lemma 4, π_z is an irreducible generalized completely continuous representation. To prove that $\pi_z(L^1(G))$ contains no nonzero compact operator, by the Proposition it is sufficient to show the corresponding property for $\varrho_z(\ell^1(S,w))$. By what we have seen in the proof of Lemma 2, the operator $\varrho_z(f)$, $f \in \ell^1(S,w) \subset \ell^1(S)$, is compact if and only if for all $a \in A$ the function $g_a \in C(\mathbb{T}^{\mathbb{Z}})$ defined by

$$g_a(x) = \sum_{b\in B} f(ab)\eta_x(b)^{-1} = \sum_{b\in B} f(ab) \prod_{j=-\infty}^{\infty} x_j^{-b_j}$$

vanishes on $\overline{\Omega}_z \setminus \Omega_z$. Hence we have to show that if f satisfies this condition, then $\varrho_z(f) = 0$. We claim that even better: f is then necessarily identically zero.

From the structure of w, $w(ab) = p^{\frac{1}{2}\sum_{j=-\infty}^{\infty}|b_j|}$ (compare Lemma 1); it follows easily that the series $\sum_{b\in B} f(ab)\prod_{j=-\infty}^{\infty} x_j^{-b_j}$ converges not only for $x \in \mathbb{T}^{\mathbb{Z}}$, but also for $x \in Y^{\mathbb{Z}}$ where Y denotes the annulus $\{y \in \mathbb{C} \mid p^{-1/2} \le |y| \le p^{1/2}\}$. Define $\widetilde{g}_a(x)$, $x \in Y^{\mathbb{Z}}$, to be the sum of this series. For $n \in \mathbb{N}$ let $i^{(n)}$ be the canonical embedding from $Y^{2n+1} = \{(y_{-n},\ldots,y_0,\ldots,y_n) \mid y_k \in Y \text{ for } |k| \le n\}$ into $Y^{\mathbb{Z}}$, i.e.,

$$i^{(n)}(y_{-n},\ldots,y_0,\ldots,y_n)_j = \begin{cases} y_j & \text{if } |j| \le n\,, \\ 1 & \text{if } |j| > n\,. \end{cases}$$

The function $\widetilde{g}_a \circ i^{(n)}$ is continuous on Y^{2n+1} and analytic in the interior $\mathring{Y}^{2n+1}$. Since g_a vanishes on $\overline{\Omega}_z \setminus \Omega_z = \overline{D}^{\mathbb{Z}}$ (see Lemma 4) we conclude that $\widetilde{g}_a \circ i^{(n)}$ vanishes on the subset $\overline{D}^{2n+1}$ of Y^{2n+1}. As $\widetilde{g}_a \circ i^{(n)}$ is analytic this yields that $\widetilde{g}_a \circ i^{(n)}$ is identically zero. In particular, g_a vanishes on $i^{(n)}(\mathbb{T}^{2n+1})$. Since $\bigcup_{n\in\mathbb{N}} i^{(n)}(\mathbb{T}^{2n+1})$ is dense in $\mathbb{T}^{\mathbb{Z}}$, it follows that g_a is identically zero. Hence for each $a \in A$ the function $b \mapsto f(ab)$ is identically zero and, therefore, f is identically zero.

References

[1] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Gauthier-Villars, Paris 1969.

[2] Ph. Green, *The structure of imprimitivity algebras*, J. Funct. Anal. 36 (1980), 88–104.

[3] A. Guichardet, *Caractères des algèbres de Banach involutives*, Ann. Inst. Fourier (Grenoble) 13 (1963), 1–81.

[4] H. Leptin, *Verallgemeinerte L^1-Algebren und projektive Darstellungen lokal kompakter Gruppen*, Invent. Math. 3 (1967), 257–281, 4 (1967), 68–86.

[5] H. Leptin and D. Poguntke, *Symmetry and nonsymmetry for locally compact groups*, J. Funct. Anal. 33 (1979), 119–134.

[6] D. Poguntke, *Unitary representations of Lie groups and operators of finite rank*, Ann. of Math., to appear.

[7] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Clarendon, Oxford 1968.

UNIVERSITÄT BIELEFELD
FAKULTÄT FÜR MATHEMATIK
POSTFACH 100131
W-4800 BIELEFELD, GERMANY

Received December 18, 1992 (3039)