STUDIA MATHEMATICA 105 (3) (1993)

Disjointness results for some classes of stable processes

by

MICHAEL HERNÁNDEZ (College Park, Md.) and
CHRISTIAN HOUDRÉ (Stanford, Calif.)

Abstract. We discuss the disjointness of two classes of stable stochastic processes: moving averages and Fourier transforms. Results on the incompatibility of these two representations date back to Urbanik. Here we extend various disjointness results to encompass larger classes of processes.

Introduction. There are two commonly used representations of stochastic processes: a Fourier transform representation,

$$X(t) = \int_{\mathbb{R}} e^{-i\gamma t}\, dZ(\gamma)\,, \tag{1.1}$$

where a random signal is represented as a superposition of harmonics with random amplitude, and a moving average representation,

$$X(t) = \int_{\mathbb{R}} h(t-s)\, dY(s)\,, \tag{1.2}$$

where a random signal is represented as filtered noise. As is well known, a stationary Gaussian process is a moving average of Brownian motion if and only if its spectral distribution is absolutely continuous. However, for stable non-Gaussian processes, Urbanik showed that the two models can be incompatible. In Theorem 2 of [15] he characterizes the discrete time stationary moving averages as stationary completely nondeterministic sequences (cf. Theorem 4.2 of [16] for the continuous parameter case). On the other hand,

1991 *Mathematics Subject Classification*: Primary 60G99; Secondary 42A45.

This research was done in part while the second author was visiting the Department of Mathematics, University of Maryland, College Park, MD 20742, and the Center for Stochastic Processes, University of North Carolina, Chapel Hill, NC 27599 being supported by the ONR Contract No. N00014-91-J-1003 and in part by the AFOSR Contract No. F49620-92-J-0154. The paper was also revised while this author was supported by an NSF Mathematical Sciences Post-Doctoral Fellowship.

he shows that the stationary "harmonizable" completely nondeterministic sequences are null in Theorem 4.4 of [17].

Urbanik's disjointness result has recently been extended to larger classes of stationary stable processes ([2], [3], [11]). The methods of proof vary greatly. For example in [2], the disjointness relies on a characterization of $S\alpha S$ noises as (p,q)-bounded processes. Furthermore, only moving averages of α-stable Lévy motions are considered there. Here, we relax these conditions to address random noises with independent but not necessarily stationary increments, hence considering a broader class (in general not stationary) of moving averages. We also generalize the notion of (p,q)-boundedness and characterize some of these processes as Fourier integrals in the summability sense.

The disjointness result of [2] also relies on the Hausdorff–Young theorem, so it is natural to look for extensions of this theorem in order to obtain the disjointness for a larger class of stochastic processes. Here we use "weighted" extensions of Hausdorff–Young in which we employ a regular Borel measure ν in the norm of the Fourier transform side of the inequality. On the other hand, the control measure μ of the noise Y in (1.2) is the weight we use on the other side of the inequality. There are many results available for weighted norm inequalities with absolutely continuous weights, which is our focus. A theorem of Benedetto and Heinig [1] is of particular interest here. We also examine a special case of a theorem of Johnson [8], which justifies the consideration of an absolutely continuous "space measure" ν.

In Section 2 we give definitions and theorems which are used throughout. In the same section we introduce stable random variables and processes. We also modify the definition of (p,q)-boundedness by replacing Lebesgue measure with a regular Borel measure ν and present conditions on this "space measure" to characterize some Fourier integrals as ν-(p,q)-bounded processes. Lastly, in Section 3 we prove some disjointness results. We present extensions of the results found in [2] using the generalizations of Hausdorff–Young mentioned above. These are then applied to the disjointness problem for stable processes, and in particular recover all the known results. We finish the paper by showing that, nevertheless, moving averages of an α-stable Lévy motion are Fourier transforms of noises which, of course, cannot be (p,q)-bounded. This last result completes Theorem 3.1 of [2].

2. Preliminaries. In this section we give definitions, lemmas and theorems which are used throughout. We let $\mathcal{B}(\mathbb{R})$ denote the σ-ring of Borel sets on the real line, and we denote the δ-ring of bounded Borel sets by $\mathcal{B}_0(\mathbb{R})$. In the sequel, when the measures ν and μ are absolutely continuous with respect to Lebesgue measure, we use the functions v and u to denote their respective Radon–Nikodým derivatives. Norms are usually written in the form $\|\cdot\|_{p,\mu} = (\int_{\mathbb{R}} |\cdot|^p \, d\mu)^{1/p}$. If only one index is used then it is the power index, and we assume that the measure in the integration is Lebesgue. We use the symbol $L^p_\mu(\mathbb{R})$ or $L^p(\mu)$ to denote the weighted Lebesgue space of μ-integrable functions. For $1 \le p \le \infty$, the indices p and p' always denote conjugate indices; i.e., $1/p + 1/p' = 1$. We let the symbol $\mathcal{S}(\mathbb{R})$ denote the Schwartz class of rapidly decreasing $C^\infty(\mathbb{R})$ functions. Two norms $\|\cdot\|$ and $\|\cdot\|'$ are said to be *equivalent*, written $\|\cdot\| \asymp \|\cdot\|'$, if there exist positive constants C_1 and C_2 such that $C_1\|\cdot\| \le \|\cdot\|' \le C_2\|\cdot\|$.

For $f \in L^1(\mathbb{R})$, we define the Fourier transform by $\widehat{f}(\gamma) = \int_{\mathbb{R}} f(x) e^{-i\gamma x}\, dx$. $\widehat{\mathbb{R}}$ denotes the dual or character group of $\mathbb{R}$ which is isomorphic to $\mathbb{R}$. We usually use x for elements of $\mathbb{R}$ and γ for elements of $\widehat{\mathbb{R}}$ but are not compulsive about this. The range of the Fourier transform is denoted by $A(\widehat{\mathbb{R}})$; that is, $A(\widehat{\mathbb{R}}) = \{\widehat{f} : f \in L^1(\mathbb{R})\}$. Similarly, whenever the Fourier transform makes sense, $(L^\alpha(\mu))^\wedge = \{\widehat{f} : f \in L^\alpha(\mu)\}$.

The symbol $(L^q(\widehat{\mathbb{R}}))^\vee$ denotes the space of functions $f \in L^1(\mathbb{R})$ such that the Fourier transform $\widehat{f} \in L^q(\widehat{\mathbb{R}})$. Likewise, the symbol $(L^q(\nu))^\vee$ denotes the space of functions in $L^1(\mathbb{R})$ whose Fourier transforms are q-integrable with respect to the measure ν. When $q = \infty$ we use the convention $(L^\infty(\widehat{\mathbb{R}}))^\vee = L^1(\mathbb{R})$. $BV(\mathbb{R})$ is the space of functions of bounded variation on the line and the symbol $\operatorname{Var} f$ denotes the total variation of the function f. $(\Omega, \mathcal{B}, P)$ is a probability space, and $\mathcal{E}$ denotes expectation (i.e. integration with respect to P). In general, C denotes some positive constant whose value may or may not be the same from one line to the next.

2.1. *Stable processes.* We recall some basic facts about stable variables and processes (see Samorodnitsky and Taqqu [14] for more details). A real random variable X is called *symmetric α-stable*, or $S\alpha S$ for short, if its characteristic function $\mathcal{E}e^{itX}$ is of the form $e^{-c|t|^\alpha}$, where $c > 0$. We consider here the case where $1 < \alpha < 2$, since the weighted norm inequalities alluded to in the introduction are for such indices. Furthermore, this range of indices corresponds to a Banach space structure for the linear space of the process. The real random variables $\{X_j\}_{j=1}^n$ are jointly $S\alpha S$, or the real random vector $X = (X_1, \ldots, X_n)$ is $S\alpha S$, if all (real) linear combinations $\sum_{j=1}^n a_j X_j$ are $S\alpha S$. Equivalently, their joint characteristic function $\mathcal{E}e^{i\langle t, X\rangle}$ is of the form

$$\exp\Big(-\int_{S^n} |\langle t, x\rangle|^\alpha \, d\Gamma_X(x)\Big),$$

where $t = (t_1, \ldots, t_n) \in \mathbb{R}^n$, $\langle t, x\rangle$ is the usual scalar product in $\mathbb{R}^n$, and Γ_X is a symmetric measure on the n-sphere S^n (see [14]). A real stochastic process $X = \{X(t) : t \in \mathbb{R}\}$ is $S\alpha S$ if all finite linear combinations of its random variables are $S\alpha S$. Note that if $\alpha = 2$, this is nothing more than the usual definition for symmetric (zero-mean) Gaussian variables and processes.

A complex random variable $X = X_1 + iX_2$ is $S\alpha S$ if the real random variables X_1, X_2 are jointly $S\alpha S$; that is, for $t \in \mathbb{C}$, $t = t_1 + it_2$, and $c > 0$,

$$\mathcal{E}e^{i\Re(\bar{t}X)} = \mathcal{E}e^{i(t_1X_1+t_2X_2)} = \exp\Big(-\int_{S^2} |t_1x_1 + t_2x_2|^\alpha \, d\Gamma_{X_1,X_2}(x_1,x_2)\Big).$$

When the complex $S\alpha S$ random variable is invariant under rotations it is called *radially symmetric* (or *isotropic*). That is, for $\theta \in \mathbb{R}$, $e^{i\theta}X$ and X are identically distributed. In this case, Γ_{X_1,X_2} is the uniform distribution on S^2, and the characteristic function has the nicer form $\mathcal{E}e^{i\Re(\bar{t}X)} = e^{-c|t|^\alpha}$. A collection of complex random variables is $S\alpha S$ if the respective real and imaginary parts of the variables are jointly $S\alpha S$. As in the case of real processes, a complex stochastic process X is $S\alpha S$ if all finite linear combinations of its random variables are $S\alpha S$.

The linear space of $S\alpha S$ random variables will be denoted by $\mathcal{L}$. A norm for this space is given by $\|\cdot\|_\alpha$, and is equivalent to convergence in probability [14]. For a complex $S\alpha S$ random variable X we have

$$\|X\|_\alpha \asymp (\mathcal{E}|X|^p)^{1/p},$$

where $1 \le p < \alpha$ (see [14]).

We develop now some of the theory of random measures and integration which is used throughout. The vector space of random variables is denoted by the symbol $L^0(\Omega, \mathcal{B}, P)$, or $L^0(P)$ for short, equipped with the pseudonorm $\|\cdot\|_0$ ([15]). By a *random measure* Z we mean a finitely additive set function into the space of random variables: $Z : \mathcal{B}_0(\mathbb{R}) \to L^0(P)$. If for every $A \in \mathcal{B}_0(\mathbb{R})$ and $\varepsilon > 0$ there exists an open set $\mathcal{O} \in \mathcal{B}_0(\mathbb{R})$ and a compact set $K \in \mathcal{B}_0(\mathbb{R})$ such that $K \subset A \subset \mathcal{O}$ and $\|Z(B)\|_0 < \varepsilon$ for every $B \subset \mathcal{O} \setminus K$, then the random measure Z is called *regular*.

Let ν be a regular Borel measure. For random measures with values in $L^p(P)$, $1 \le p \le 2$, and $1 \le q \le \infty$, we may define the *ν-(p,q)-variation* of Z over a set $A \in \mathcal{B}_0(\mathbb{R})$ to be

$$\|Z\|(A) = \sup\Big\{\Big(\mathcal{E}\Big|\sum_{i=1}^N a_i Z(A_i)\Big|^p\Big)^{1/p} : \Big\|\sum_{i=1}^N a_i \chi_{A_i}\Big\|_{q,\nu} \le 1\Big\}, \tag{2.1}$$

where $\{A_i\} \subset \mathcal{B}_0(\mathbb{R})$ is a finite partition of A and $a_i \in \mathbb{C}$.

DEFINITION 2.1. A random measure Z has *finite ν-(p,q)-variation* if $Z(A) \in L^p(P)$ for $A \in \mathcal{B}_0(\mathbb{R})$, $1 \le p \le 2$ and if

$$\|Z\| = \sup_{A \in \mathcal{B}_0(\mathbb{R})} \{\|Z\|(A)\} < \infty. \tag{2.2}$$

When $q = \infty$, this definition reduces to the usual definition of *semi-variation*. On $\mathcal{B}(\mathbb{R})$, the vector measure Z is of bounded semi-variation if and only if it is σ-additive (see [4, IV]).

Let f be a simple, complex-valued function, $f = \sum_{j=1}^N a_j \chi_{A_j}$, $A_j \in \mathcal{B}_0(\mathbb{R})$. If Z is a random measure of finite ν-(p,q)-variation, define

$$\int_{\mathbb{R}} f \, dZ = \sum_{j=1}^N a_j Z(A_j).$$

Making the identification $A = \bigcup_{j=1}^N A_j$ then gives

$$\Big(\mathcal{E}\Big|\int_{\mathbb{R}} f \, dZ\Big|^p\Big)^{1/p} \le \|Z\|(A)\|f\|_{q,\nu}.$$

For $1 \le q < \infty$, this integral can be extended so that it is defined for functions $f \in L^q(\nu)$. Similarly, for $q = \infty$, the integral can be extended to Borel bounded functions. On $\mathcal{B}(\mathbb{R})$, when the random measure Z is σ-additive this integral is consistent with the Bartle, Dunford and Schwartz integral [4, IV].

A random measure Z is *independently scattered* if the random variables $\{Z(A_1), \ldots, Z(A_n)\}$ are mutually independent for every collection of pairwise disjoint sets $\{A_n\} \subset \mathcal{B}_0(\mathbb{R})$. Some special cases of random measures (or noises) which we consider are the $S\alpha S$ motions. The independently scattered $S\alpha S$ motions, sometimes referred to as *independently scattered stable noises* and which we denote by Y, have independent increments. The moment of order p and the $L^\alpha(\mu)$ norm are equivalent giving us

$$\Big(\mathcal{E}\Big|\int_{\mathbb{R}} f \, dY\Big|^p\Big)^{1/p} \asymp \Big(\int_{\mathbb{R}} |f|^\alpha d\mu\Big)^{1/\alpha},$$

where $1 \le p < \alpha$, and μ is called the *control measure* of Y [12]. The integral defined in this way is a special case of the integral mentioned previously. When the increments are stationary, or shift invariant, the control measure is Lebesgue and we have a $S\alpha S$ Lévy motion. This is referred to as an *α-stable Lévy motion* and is denoted by M. The stationarity and the independence of the increments allows equivalence to become equality, i.e. for $f \in L^\alpha(\mathbb{R})$,

$$\Big(\mathcal{E}\Big|\int_{\mathbb{R}} f \, dM\Big|^p\Big)^{1/p} = C_{\alpha,p}\Big(\int_{\mathbb{R}} |f|^\alpha \, dx\Big)^{1/\alpha}, \quad 1 \le p < \alpha.$$

Let Y be an independently scattered $S\alpha S$ noise, and let μ be its control measure. Then X defined via

$$X(t) = \int_{\mathbb{R}} h(t-s) \, dY(s), \quad t \in \mathbb{R},$$

is called a *moving average* of Y.

2.2. *ν-(p,q)-bounded processes*

DEFINITION 2.2. Let ν be a regular Borel measure on $\mathbb{R}$, $1 \le p < \infty$ and $1 \le q \le \infty$. An $L^p(P)$-bounded process X is called *ν-(p,q)-bounded* if X is strongly measurable and if there exists a constant $C > 0$ such that

$$(2.3)\qquad \Big(\mathcal{E}\Big|\int_{\mathbb{R}} f(t)X(t)\,dt\Big|^p\Big)^{1/p} \le C\|\widehat{f}\|_{q,\nu}, \quad \forall f \in (L^q(\nu))^{\vee}.$$

Here we use the convention that for $q = \infty$ we replace $L^\infty(\nu)$ with $C_0(\widehat{\mathbb{R}})$ (the continuous functions going to zero at infinity). This definition coincides with the definition of (p,q)-boundedness given in [7] if we take ν to be Lebesgue measure. Hence m-(p,q)-boundedness is identical to (p,q)-boundedness, and henceforth any reference to this condition will be made by the latter. The integral

$$\int_{\mathbb{R}} f(t)X(t)\,dt, \quad f \in L^1(\mathbb{R}),$$

in (2.3) is well defined as a Bochner integral when $X = \{X(t) : t \in \mathbb{R}\}$ is strongly measurable and $L^p(P)$-bounded, i.e., $\mathcal{E}|X(t)|^p \le C$.

In [7], the (p,q)-bounded processes are characterized as weak Fourier integrals. In our broader framework, where we allow a different measure to be used in the norm of the Fourier transform, we find that the characterization still holds under some assumptions on ν.

We now give half of the characterization of ν-(p,q)-bounded processes as weak Fourier integrals.

THEOREM 2.3. *Let the process $X = \{X(t) : t \in \mathbb{R}\}$ be $L^p(P)$-bounded, strongly continuous and ν-(p,q)-bounded. Then there exists a regular random measure Z of finite ν-(p,q)-variation such that*

$$(2.4)\qquad X(t) = \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \Big(1 - \frac{|\gamma|}{\lambda}\Big) e^{-it\gamma}\,dZ(\gamma).$$

The convergence is in $L^p(P)$ and is uniform on compact subsets of $\mathbb{R}$.

Proof. By definition, an $L^p(P)$-bounded process X is ν-(p,q)-bounded if it is strongly measurable and if there exists $C > 0$ such that

$$\Big(\mathcal{E}\Big|\int_{\mathbb{R}} f(t)X(t)\,dt\Big|^p\Big)^{1/p} \le C\|\widehat{f}\|_{q,\nu}, \quad \forall f \in (L^q(\nu))^{\vee}.$$

The operator $T : A(\widehat{\mathbb{R}}) \cap L^q(\nu) \to L^p(P)$, $\widehat{f} \mapsto \int_{\mathbb{R}} f(t)X(t)\,dt$, is well defined, linear and bounded on $A(\widehat{\mathbb{R}}) \cap L^q(\nu)$ since X is ν-(p,q)-bounded. $A(\widehat{\mathbb{R}}) \cap L^q(\nu)$ is dense in $L^q(\nu)$ since $C_c(\widehat{\mathbb{R}})$, the space of continuous compactly supported functions, is dense in each of these spaces. Then as in the proof of

Theorem 3.2 of [7] there exists a regular random measure $Z : \mathcal{B}_0(\nu) \to L^0(P)$ of finite ν-(p,q)-variation such that

$$(2.5)\qquad \int_{\mathbb{R}} f(t)X(t)\,dt = \int_{\mathbb{R}} \widehat{f}(t)\,dZ(t), \quad \forall f \in (L^q(\nu))^{\vee}.$$

Let $K_\lambda(t) = \int_{-\lambda}^{\lambda}(1 - |\gamma|/\lambda)e^{-it\gamma}\,d\gamma$, and let $\widehat{K}_\lambda(\gamma) = (1 - |\gamma|/\lambda)\chi_{(-\lambda,\lambda)}(\gamma)$ be its Fourier transform. Then $K_\lambda \in L^1(\mathbb{R})$ and $\widehat{K}_\lambda \in L^q(\nu)$ so from (2.5) we get

$$(2.6)\qquad \int_{\mathbb{R}} K_\lambda(t-\tau)X(\tau)\,d\tau = \int_{-\lambda}^{\lambda} \Big(1 - \frac{|\gamma|}{\lambda}\Big) e^{-it\gamma}\,dZ(\gamma).$$

Now,

$$\begin{aligned}\Big(\mathcal{E}\Big|X(t) - \int_{-\lambda}^{\lambda} \Big(1 - \frac{|\gamma|}{\lambda}\Big) e^{-it\gamma}\,dZ(\gamma)\Big|^p\Big)^{1/p} &\le \sup_{|\tau|\le\delta} (\mathcal{E}|X(t) - X(t-\tau)|^p)^{1/p} \int_{|\tau|\le\delta} |K_\lambda(\tau)|\,d\tau \\ &\quad + 2\sup_{t\in\mathbb{R}}(\mathcal{E}|X(t)|^p)^{1/p} \int_{|\tau|>\delta} |K_\lambda(\tau)|\,d\tau.\end{aligned}$$

But X is an $L^p(P)$-bounded process and $\|K_\lambda\|_1 = O(1)$ so we have

$$\begin{aligned}\Big(\mathcal{E}\Big|X(t) - \int_{-\lambda}^{\lambda} \Big(1 - \frac{|\gamma|}{\lambda}\Big) e^{-i\gamma t}\,dZ(\gamma)\Big|^p\Big)^{1/p} &\le C \sup_{|\tau|\le\delta} (\mathcal{E}|X(t) - X(t-\tau)|^p)^{1/p} + C \int_{|\tau|>\delta} |K_\lambda(\tau)|\,d\tau.\end{aligned}$$

Since X is strongly continuous, on $|\tau| \le \delta$ we get

$$(2.7)\qquad \lim_{\tau\to 0} \sup_{|\tau|\le\delta} (\mathcal{E}|X(t) - X(t-\tau)|^p)^{1/p} = 0.$$

Furthermore, the Fejér kernel (or any other approximate identity) has the property that

$$(2.8)\qquad \lim_{\lambda\to\infty} \int_{|\tau|>\delta} |K_\lambda(\tau)|\,d\tau = 0.$$

Combining (2.7) and (2.8) proves the theorem. ∎

Remark 2.4. The full characterization appears in [7] for the case $\nu = m$. Here, a converse to Theorem 2.3 is only possible under some conditions on ν: Using a generalization of the Hausdorff–Young theorem we mention in the next section, i.e., whenever $\sup_{s>0}(1/s)\int_0^s v(\gamma)\,d\gamma < \infty$, we see that a proof for the converse follows as in [7].

In contrast to the Gaussian case, not every stationary $S\alpha S$ process, $\alpha < 2$, has a Fourier representation in the above sense (*a fortiori* since ν-(p,q)-bounded processes are possibly not stationary). In the rest of the paper, we consider the class of (non-stationary) $S\alpha S$ processes which are ν-(p,q)-bounded and the class of (non-stationary) $S\alpha S$ processes which have a moving average representation (with respect to an independently scattered $S\alpha S$ random measure). Even when restricted to stationary processes, neither class exhausts the $S\alpha S$ stationary processes.

3. Moving averages vs. Fourier transforms. As it is our intention to discuss the incompatibility of the moving average and Fourier transform models, we offer the following introduction. Let us suppose that the continuous time process X is a moving average of the independently scattered $S\alpha S$ noise Y, $1<\alpha<2$, i.e., that $X(t)=\int_{\mathbb{R}} h(t-s)\,dY(s)$, $t\in\mathbb{R}$, $h\in L^\alpha(\mu)$. For $f\in L^1(\mathbb{R})$ we then have

$$\int_{\mathbb{R}} f(t)X(t)\,dt = \int_{\mathbb{R}} (f*h)(s)\,dY(s)\,, \tag{3.1}$$

where the change of order of integration is justified by a Fubini-type theorem due to Rosinski [13]. Now, the form of the characteristic function ensures that for $1\le p<\alpha$ we have

$$\Big(\mathcal{E}\Big|\int_{\mathbb{R}} (f*h)(s)\,dY(s)\Big|^p\Big)^{1/p} \asymp C_{\alpha,p}\|f*h\|_{\alpha,\mu}\,.$$

If in addition we assume that the $L^p(P)$-bounded process X is (p,q)-bounded, then using the above together with Definition 2.2 yields

$$\|f*h\|_{\alpha,\mu} \le C\|\widehat{f}\|_{q,m}, \qquad \forall f\in (L^q(\widehat{\mathbb{R}}))^\vee\,. \tag{3.2}$$

In this way we see that proving the disjointness of the two classes of processes is equivalent to showing that the inequality (3.2) cannot hold unless $h=0$.

To show that $h=0$, we now introduce a special type of operator which we term a *multiplicator*. Using harmonic and functional analysis methods we prove that for some measures μ there are only trivial multiplicators. The disjointness of the two representations then follows as a direct consequence. In subsection 2 we give an analogue of those multiplicators in which a different space measure ν is employed, and a similar disjointness theorem is proved for those *ν-multiplicators*. We finish the section by showing that although not (p,q)-bounded, a moving average of a stable Lévy motion does have a Fourier representation.

3.1. *Multiplicators.* The measurable function $\varphi:\widehat{\mathbb{R}}\to\mathbf{C}$ is termed a *multiplicator* if $\varphi f\in(L^\alpha(\mu))^\wedge$ for all $f\in L^q(\mathbb{R})$. With such a multiplicator we associate the corresponding well defined operator $T_\varphi: L^q(\mathbb{R})\to L^\alpha(\mu)$, $f\to(\varphi f)^\vee$, or equivalently $(T_\varphi f)^\wedge=\varphi f\in(L^\alpha(\mu))^\wedge$. We call a function $h\in L^\alpha(\mu)$ a *premultiplicator* if its Fourier transform $\widehat{h}$ is a multiplicator. That is, h is a premultiplicator if $(T_{\widehat{h}}f)^\wedge=\widehat{h}f\in(L^\alpha(\mu))^\wedge$, $\forall f\in L^q(\mathbb{R})$. We saw in the introduction to this section that if $X(t)$ is a moving average of an independently scattered $S\alpha S$ motion which is (p,q)-bounded, then we must have the following inequality:

$$\|f*h\|_{\alpha,\mu}\le C\|\widehat{f}\|_{q,m}\,.$$

We use the language of multiplicators to show the disjointness of the two classes, moving averages and Fourier transforms, by proving that under this (p,q)-boundedness assumption, for certain choices of μ, there are no multiplicators except the zero or trivial multiplicator. To do this we make use of the following Hausdorff–Young type inequality which is due to Benedetto and Heinig [1].

LEMMA 3.1. *Let $1<\alpha\le2$, $1/\alpha+1/\alpha'=1$, and suppose that u and v are positive, even functions such that u and $1/v$ are nondecreasing on $(0,\infty)$. Then*

$$\|\widehat{f}\|_{\alpha',v}\le C\|f\|_{\alpha,u}, \qquad \forall f\in L^1(\mathbb{R})\cap L^\alpha_u(\mathbb{R})\,,$$

if and only if

$$\sup_{s>0}\Big(\int_0^{1/s} v(\gamma)\,d\gamma\Big)^{1/\alpha'}\Big(\int_0^s u^{1-\alpha'}(x)\,dx\Big)^{1/\alpha'}<\infty\,. \tag{3.3}$$

This was proved independently by Jurkat and Sampson [9] using different methods at about the same time. Their theorem removes the evenness criterion for the necessity.

In the following lemma we show that for certain choices of μ, the multiplicator operator T_φ, as defined above, is a bounded linear operator from $L^q(\mathbb{R})$ into $L^\alpha(\mu)$.

LEMMA 3.2. *Suppose $1<\alpha\le2$, $1\le q\le\infty$, and let $\mu\ll m$ satisfy the condition* (3.3) *of Lemma* 3.1 *with $v=1$. Then the multiplicator operator $T_\varphi: L^q(\mathbb{R})\to L^\alpha(\mu)$ is bounded.*

When $q=\infty$ we replace $L^\infty(\mathbb{R})$ with $C_0(\mathbb{R})$ as previously mentioned.

Proof. To prove this we use the Closed Graph theorem. So, we take a sequence $\{f_n\}\subset L^q(\mathbb{R})$, $f_n\overset{L^q(\mathbb{R})}{\to} f$ such that $T_\varphi f_n\overset{L^\alpha(\mu)}{\to} g$, and show that $T_\varphi f=g$. By hypothesis μ satisfies the sufficiency conditions of the

Benedetto–Heinig inequality giving us

$$\|T_\varphi f_n - g\|_{\alpha,\mu} \ge \|\varphi f_n - \widehat{g}\|_{\alpha',m}.$$

As $n \to \infty$, the above left hand side approaches 0, i.e., we have $\varphi f_n \overset{L^{\alpha'}(\mathbb{R})}{\longrightarrow} \widehat{g}$. Now, for a subsequence $\{f_{n_k}\} \subseteq \{f_n\}$ we have $\varphi f_{n_k} \to \widehat{g}$ a.e. Leb. However, since we took $f_n \overset{L^q(\mathbb{R})}{\longrightarrow} f$, it must be that $f_{n_k} \to f$ a.e. Leb. Thus $\widehat{g} = \varphi f = (T_\varphi f)^\wedge$ and so $g = T_\varphi f$. Thus, by the Closed Graph theorem, the operator T_φ is continuous and hence bounded. So, $\|T_\varphi\| \le C$ where C is a positive constant and the norm $\|\cdot\|$ denotes the usual operator norm. Thus we have

$$\|T_\varphi f\|_{\alpha,\mu} \le \|T_\varphi\|\,\|f\|_{q,m}, \qquad \forall f \in L^q(\mathbb{R}). \ \blacksquare$$

LEMMA 3.3. *Let $1 < \alpha \le 2$, $1 \le q \le \infty$, and let $\mu \ll m$ satisfy the condition* (3.3) *of Lemma* 3.1 *with $v = 1$. Then h is a premultiplicator iff*

$$\|f * h\|_{\alpha,\mu} \le C\|\widehat{f}\|_{q,m}, \qquad \forall f \in (L^q(\widehat{\mathbb{R}}))^\vee.$$

As in Lemma 3.2, for $q = \infty$ we replace $L^\infty(\widehat{\mathbb{R}})$ with $C_0(\widehat{\mathbb{R}})$.

Proof. ($\Rightarrow$) First we suppose that h is a premultiplicator. Then by definition $\widehat{h}$ is a multiplicator and so from Lemma 3.2,

$$\|(\widehat{h}g)^\vee\|_{\alpha,\mu} \le C\|g\|_{q,m}, \qquad \forall g \in L^q(\mathbb{R}).$$

So, for $f \in (L^q(\widehat{\mathbb{R}}))^\vee$,

$$\|f * h\|_{\alpha,\mu} = \|(\widehat{h}\widehat{f})^\vee\|_{\alpha,\mu} \le C\|\widehat{f}\|_{q,m}.$$

($\Leftarrow$) Conversely, suppose h is such that

$$\|f * h\|_{\alpha,\mu} \le C\|\widehat{f}\|_{q,m}.$$

Take a sequence $\{g_n\} \in \mathcal{S}(\widehat{\mathbb{R}})$, the Schwartz space of rapidly decreasing C^∞ functions on $\widehat{\mathbb{R}}$, such that $g_n \overset{L^q(\widehat{\mathbb{R}})}{\longrightarrow} g$, $g \in L^q(\widehat{\mathbb{R}})$. Define the sequence $\{f_n\}$ so that $g_n = \widehat{f}_n$. In this way, since the Fourier transform $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\widehat{\mathbb{R}})$ is onto, $f_n \in \mathcal{S}(\mathbb{R})$, and so $f_n \in L^1(\mathbb{R})$. For $n, k \in \mathbb{Z}$, we have by Lemma 3.1,

$$\begin{aligned}\|\widehat{h}g_n - \widehat{h}g_k\|_{\alpha',m} &\le \|f_n * h - f_k * h\|_{\alpha,\mu} \\ &\le C\|\widehat{f}_n - \widehat{f}_k\|_{q,m} = C\|g_n - g_k\|_{q,m} \to 0.\end{aligned}$$

Thus $\{\widehat{h}g_n\}$ and $\{f_n * h\}$ are Cauchy sequences, and we know that $f_n * h \to l$ in $L^\alpha(\mu)$ and $\widehat{h}g_n \to \widehat{l}$ in $L^{\alpha'}(\mathbb{R})$. However, we took $\{g_n\}$ such that $g_n \overset{L^q(\widehat{\mathbb{R}})}{\longrightarrow} g$, so there exists a subsequence $\{g_{n_j}\}$ such that $g_{n_j} \to g$ a.e. Leb. and $\widehat{h}g_{n_j} \to \widehat{l}$ a.e. Leb. Thus $\widehat{h}g = \widehat{l}$ a.e. Leb., or $\widehat{h}g \in (L^\alpha(\mu))^\wedge$, $\forall g \in L^q(\widehat{\mathbb{R}})$. Hence h must be a premultiplicator as required. $\blacksquare$

We are now in good position to show that under some assumptions on the control measure μ, there are no nontrivial multiplicators. As we mentioned earlier, this in turn implies that the corresponding class of moving averages of independently scattered $S\alpha S$ motions is disjoint from the class of (p,q)-bounded processes.

THEOREM 3.4. *Let $1 < \alpha < 2$, $1 \le q \le \infty$, and let Y be an independently scattered $S\alpha S$ motion with control measure $\mu \ll m$ satisfying*

$$\sup_{s>0} \frac{1}{s} \int_0^s u^{1-\alpha'}(x)\,dx < \infty,$$

where u is nondecreasing on $(0,\infty)$. Let the $L^p(P)$-bounded process X be a moving average of Y,

$$X(t) = \int_{\mathbb{R}} h(t-s)\,dY(s), \qquad t \in \mathbb{R},\ h \in L^\alpha(\mu).$$

Then X cannot be (p,q)-bounded, $1 \le p < \alpha$, unless it is null; i.e. there are no nontrivial multiplicators.

Proof. Suppose φ is a multiplicator with compact support. Then by Lemma 3.2,

$$\|(\varphi g)^\vee\|_{\alpha,\mu} \le C\|g\|_{q,m}, \qquad \forall g \in L^q(\mathbb{R}).$$

Using the Benedetto–Heinig inequality, it is not difficult to see that the product $\varphi g \in L^2(\mathbb{R})$.

For $f \in L^{\alpha'}(\mu)$, by Hölder's inequality and Lemma 3.2 we have

$$\Big|\int_{\mathbb{R}} (\varphi g)^\vee \bar{f}\,d\mu\Big| \le C\|g\|_{q,m}\|f\|_{\alpha',\mu}. \tag{3.4}$$

If $f \in L^2_{u^2}(\mathbb{R})$, then by Parseval's formula we also have

$$\int_{\mathbb{R}} (\varphi g)^\vee \bar{f}\,d\mu = \int_{\mathbb{R}} \varphi(\overline{\widehat{fu}})g\,dx. \tag{3.5}$$

Thus for all $f \in L^{\alpha'}(\mu) \cap L^2_{u^2}(\mathbb{R})$ and $g \in L^q(\mathbb{R})$ by combining equation (3.5) with the inequality (3.4) we get

$$\Big|\int_{\mathbb{R}} \varphi(\overline{\widehat{fu}})g\,dx\Big| \le C\|g\|_{q,m}\|f\|_{\alpha',\mu}.$$

Since $g \in L^q(\mathbb{R})$, we may define the bounded linear functional L as follows:

$$L(g) = \int_{\mathbb{R}} \varphi(\overline{\widehat{fu}})g\,dx, \qquad g \in L^q(\mathbb{R}).$$

By the Riesz Representation Theorem, we have $\|L\| = \|\varphi(\overline{\widehat{fu}})\|_{q',m}$ leaving us with

$$\|\varphi(\overline{\widehat{fu}})\|_{q',m} \le C\|fu^{1/\alpha'}\|_{\alpha',m}. \tag{3.6}$$

Thus the inequality (3.6) reduces to

$$\|\overline{\varphi}(fu)^\wedge\|_{q',m} \le C\|fu^{1/\alpha'}\|_{\alpha',m}. \tag{3.7}$$

Let $A = \{x \in \mathbb{R} : u(x) > 0\}$, where without loss of generality we assume $m(A) \neq 0$. Making the identification $\widetilde{u} = (u\chi_A)^{1-\alpha'}$, for any $g \in L^2(\mathbb{R}) \cap L^{\alpha'}_{\widetilde{u}}(\mathbb{R})$ we have

$$f = \frac{g}{u\chi_A} \in L^2_{u^2}(\mathbb{R}) \cap L^{\alpha'}(\mu).$$

Hence using the inequality (3.7) we get

$$\|\overline{\varphi}\,\widehat{g}\|_{q',m} \le C\|g\widetilde{u}^{1/\alpha'}\|_{\alpha',m}, \qquad \forall g \in L^2(\mathbb{R}) \cap L^{\alpha'}_{\widetilde{u}}(\mathbb{R}). \tag{3.8}$$

We now make the following

CLAIM. *φ satisfying inequality* (3.8) *must necessarily be zero.*

To see this, suppose $\varphi \neq 0$ a.e. Leb. Then, since u is nondecreasing, there exist $\varepsilon > 0$ and $g \in C_c^\infty$ such that

$$0 < \|\overline{\varphi}\,\widehat{g}\|_{q',m} \le C\|g\widetilde{u}^{1/\alpha'}\|_{\alpha',m} \le C\varepsilon^{1-\alpha'}\|g\|_{\alpha',m}, \tag{3.9}$$

where we make use of the fact that $1-\alpha'$ is negative in the last inequality. We define the function g_x for $x > 0$ by its Fourier transform: $\widehat{g}_x(\gamma) = \widehat{g}(\gamma)e^{ix\gamma^2}$. Combining the fact that $|\widehat{g}_x(\gamma)| = |\widehat{g}(\gamma)e^{ix\gamma^2}| = |\widehat{g}(\gamma)|$ with (3.9) yields

$$0 < \|\overline{\varphi}\,\widehat{g}\|_{q',m} = \|\overline{\varphi}\,\widehat{g}_x\|_{q',m} \le C\|g_x\|_{\alpha',m}.$$

Furthermore, by a lemma due to Hörmander (see [2]),

$$\|g_x\|^{\alpha'}_{\alpha'} \le C|x|^{1-\alpha'/2}\|\widehat{g}\|_2^2.$$

Recall that $\alpha' > 2$, so as $x \to \infty$, $\|g_x\|_{\alpha',m} \to 0$. However, this contradicts the fact that $0 < C\|g_x\|_{\alpha',m}$. Therefore, we conclude that when φ has compact support, $\varphi = 0$.

Now suppose φ is any multiplicator. Given any compact set K, there exists a function $\psi \in C_c(\mathbb{R})$ such that $\psi|_K = 1$. So, given $g \in L^q(\mathbb{R})$, the product $\psi g \in L^q(\mathbb{R})$. Furthermore, $(\varphi\psi)g = (\psi g)\varphi \in (L^\alpha(\mu))^\wedge$. Thus, $\psi\varphi$ is a multiplicator with compact support, and so we conclude that there are no nontrivial multiplicators. ∎

Remark 3.5. There are instances in which showing that inequality (3.2) implies $h = 0$ is almost immediate. Using the Benedetto–Heinig inequality with (3.2) and following the reasoning in Theorem 3.2 of [6] yields the result for $q < \alpha'$. It is also clear (by Hölder's inequality and Plancherel's theorem) that whenever $u \in L^{2/(2-\alpha)}(\mathbb{R})$, $1 < \alpha < 2$, and $\widehat{h} \in L^{q/(q-2)}(\mathbb{R})$, $2 \le q \le \infty$, a moving average is (p,q)-bounded, $1 < p < \alpha$. It is also clear that the disjointness proof given above carries over to processes whose random measure is "bounded below" and in particular applies to corresponding classes of semistable moving averages and Fourier transforms (see Proposition 3.8 in [12]).

Another way of showing the disjointness in the case $q = \infty$ is due to Makagon and Mandrekar [11] and uses a multiplier theorem due to Edwards. A faster and simpler proof, still using Edwards' theorem, is possible employing the methods of Lemmas 3.2 and 3.3 above, as we see in the following

THEOREM 3.6. *If the process X is (p,∞)-bounded and a moving average of an α-stable Lévy motion, then $X = 0$.*

Proof. Since X is (p,∞)-bounded, we use inequality (3.2) to get

$$\|f * h\|_\alpha \le C\|\widehat{f}\|_\infty, \qquad \forall f \in \mathcal{S}(\mathbb{R}).$$

Edwards' theorem states that if a function φ on $\widehat{\mathbb{R}}$ is such that $f\varphi \in \bigcup_{1\le\alpha<2}(L^\alpha(\mathbb{R}))^\wedge$ for every $f \in C_0(\widehat{\mathbb{R}})$, then $\varphi = 0$. But if h satisfies the above inequality, it must be that $\widehat{f}\widehat{h} \in (L^\alpha(\mathbb{R}))^\wedge$. To see this, take $\{f_n\} \subset \mathcal{S}(\mathbb{R})$ such that $\widehat{f}_n \xrightarrow{C_0} \widehat{f}$. Using the above inequality together with Hausdorff–Young yields

$$\|\widehat{f}_n\widehat{h} - \widehat{f}_k\widehat{h}\|_{\alpha'} \le C\|\widehat{f} - \widehat{f}_k\|_\infty.$$

So, this sequence is Cauchy and we have $f_n * h \xrightarrow{L^\alpha} \psi$ and $\widehat{f}_n\widehat{h} \xrightarrow{L^{\alpha'}} \widehat{\psi}$. However, $\widehat{f}_n \to \widehat{f}$ a.e. Leb. and for a subsequence we also have $\widehat{f}_{n_k}\widehat{h} \to \widehat{\psi}$. Thus $\widehat{\psi} = \widehat{f}\widehat{h} \in (L^\alpha(\mathbb{R}))^\wedge$. It follows from Edwards' theorem that $h = 0$, hence $X = 0$. ∎

3.2. *ν-multiplicators.* The goal of this section is to show that even by enlarging the definition of the Fourier transform, i.e., by considering ν-(p,q)-bounded processes, disjointness can persist. To do so, we generalize the multiplicator operators defined in the previous section. We consider maps of the form $T_\varphi : L^q(\nu) \to L^\alpha(\mathbb{R})$ where φ is a ν-measurable function and where T_φ is defined in terms of its Fourier transform:

$$(T_\varphi f)^\wedge = \varphi f, \qquad \forall f \in L^q(\nu).$$

That is to say, $T_\varphi f$ is the $L^\alpha(\mathbb{R})$ function which has φf as its Fourier transform. The corresponding functions φ are then termed *ν-multiplicators*. The fact that T_φ is bounded for some choices of φ is the subject of Lemma 3.9. We label the measure ν in the weighted norm inequality the "space measure", and we concern ourselves with inequalities of the form

$$\|\widehat{f}\|_{\alpha',\nu} \le C\|f\|_{\alpha,m}\,.$$

That is, we consider Lebesgue control measure in the Hausdorff–Young type inequalities. We have already seen that Lemma 3.1 guarantees an inequality of the form we require with certain conditions on the measure ν. When considering Lebesgue control measure Theorem 3.8 shows that we need only consider absolutely continuous space measure weights. We begin with a necessary condition (see Theorem 3.1.4 in [10]) which is actually a generalization of the theorem of Jurkat and Sampson.

THEOREM 3.7. *Let ν be a positive measure on $\widehat{\mathbb{R}}$, $u \in L^1_{\mathrm{loc}}(\mathbb{R})$, $1 < \alpha, q < \infty$. If*

$$\|\widehat{f}\|_{q,\nu} \le C\|f\|_{\alpha,u}, \quad \forall f \in L^1(\mathbb{R}) \cap L^\alpha_u(\mathbb{R}),$$

then there exists C independent of x_0 and γ_0 such that

$$\sup_{s>0}\Big(\int\limits_{|\gamma_0-\gamma|\le s} d\nu(\gamma)\Big)^{1/q}\Big(\int\limits_{|x_0-x|\le 1/s} u^{1-\alpha'}(x)\,dx\Big)^{1/\alpha'} \le C.$$

We use this to prove the following special case of a theorem of Johnson [8] that Lakey proves for the case $\alpha = q = 2$. The moral of the forthcoming theorem is that in order to get disjointness results (with the same type of proof), it is necessary to have $\nu \ll m$ with $d\nu/dx \in L^\infty(\mathbb{R})$.

THEOREM 3.8. *Suppose $1/\alpha + 1/\alpha' = 1$, $1 < \alpha \le 2$. Then*

$$\|\widehat{f}\|_{\alpha',\nu} \le C\|f\|_{\alpha,m}, \quad \forall f \in L^\alpha(\mathbb{R}),$$

if and only if ν is absolutely continuous with essentially bounded Radon–Nikodým derivative, i.e., if $d\nu/dx = v \in L^\infty(\widehat{\mathbb{R}})$.

Proof. ($\Rightarrow$) If $\|\widehat{f}\|_{\alpha',\nu} \le C\|f\|_{\alpha,m}$, then Theorem 3.7 tells us that

$$\sup_{\gamma_0\in\widehat{\mathbb{R}}}\sup_{s>0}\frac{1}{2s}\int\limits_{|\gamma_0-\gamma|\le s} d\nu \le C\,. \tag{3.10}$$

To see that ν must be absolutely continuous, let $\delta > 0$ be given and let $\{I_k\}_{k=1}^N$ be a finite disjoint collection of intervals in $\widehat{\mathbb{R}}$ such that $\sum_{k=1}^N |I_k| < \delta$. Then from (3.10) we get

$$\sum_{k=1}^N \int_{I_k} d\nu(\gamma) \le C\sum_{k=1}^N |I_k| < C\delta\,.$$

Thus ν must be absolutely continuous. So, $\nu = v \in L^1_{\mathrm{loc}}$, and $v(\gamma) \le C$, $\forall \gamma \in \mathrm{Leb}(v)$, the set of Lebesgue points of v. Thus $v \in L^\infty(\widehat{\mathbb{R}})$.

($\Leftarrow$) To prove this we use Hausdorff–Young and the fact that

$$\|\widehat{f}\|_{\alpha',\nu} \le \|v\|_\infty\|\widehat{f}\|_{\alpha',m}\,. \ \blacksquare$$

As in the previous section we want to show that a multiplier inequality implies the disjointness. To do this, we modify the inequality (3.2) by considering ν-(p,q)-bounded processes. Thus our new inequality is

$$\|f * h\|_{\alpha,m} \le C\|\widehat{f}\|_{q,\nu}\,. \tag{3.11}$$

We now give a result which corresponds to Lemmas 3.2 and 3.3. This is used to show that, under some conditions on ν, this new inequality implies that $h = 0$.

LEMMA 3.9. *For $1 < \alpha \le 2$, $1 \le q \le \infty$, and $\nu \ll m$ satisfying the condition (3.3) (with also $\mu = 1$), the ν-multiplicator operator $T_\varphi : L^q(\nu) \to L^\alpha(\mathbb{R})$ is bounded. In addition, and under the same hypotheses, a function h is a ν-premultiplicator iff*

$$\|f * h\|_\alpha \le C\|\widehat{f}\|_{q,\nu}\,.$$

When $q = \infty$ we again replace $L^\infty(\nu)$ with $C_0(\mathbb{R})$.

Of course we also have the following generalization of Theorem 3.4 for ν-multiplicators, whose proof parallels the proof of Theorem 4.2(ii) of [2].

THEOREM 3.10. *Let $1 \le p < \alpha < 2$, $1 \le q \le \infty$, ν a positive Borel measure as in Lemma 3.9, and let M be an α-stable Lévy motion. If the process $X = \{X(t) : t \in \mathbb{R}\}$ is ν-(p,q)-bounded and*

$$X(t) = \int_{\mathbb{R}} h(t-s)\,dM(s), \quad h \in L^\alpha(\mathbb{R}),$$

then $h = 0$. In other words, there are no nontrivial ν-multiplicators.

Proof. Suppose φ is a ν-multiplicator with compact support. By Lemma 3.9 we have

$$\|(\varphi g)^\vee\|_\alpha \le C\|g\|_{q,\nu}, \quad \forall g \in L^1(\mathbb{R}) \cap L^q(\nu)\,.$$

As $(\varphi g)^\vee \in L^\alpha(\mathbb{R})$, by Hausdorff–Young we have $\varphi g \in L^{\alpha'}(\mathbb{R})$, and since φ has compact support, $\varphi g \in L^2$. For $f \in C_c(\mathbb{R})$ by Hölder's inequality we have

$$\Big|\int_{\mathbb{R}} (\varphi g)^\vee \overline{fv}\,dx\Big| \le \|(\varphi g)^\vee\|_{\alpha,m}\|fv\|_{\alpha',m} \le C\|g\|_{q,\nu}\|fv\|_{\alpha',m}\,.$$

On the other hand, if $f \in C_c(\mathbb{R})$, then by Parseval we have

$$\int_{\mathbb{R}} (\varphi g)^\vee \overline{fv}\,dx = \int_{\mathbb{R}} \varphi\overline{(\widehat{fv})}g\,dx\,.$$

Thus $\forall f \in C_c(\mathbb{R})$ we get

$$\int_{\mathbb{R}} \varphi\overline{(\widehat{fv})}g\,dx \le C\|g\|_{q,\nu}\|fv\|_{\alpha'}\,.$$

We now define the bounded linear functional $L(g)$:

$$L(g)=\int_{\mathbb{R}}\varphi(\overline{\widehat{fv}})g\,dx,\qquad g\in L^q(\nu).$$

By the Riesz representation theorem, $\|L\|=\|\varphi\widehat{fv}\|_{q',m}$. This gives us the inequality

$$\|\overline{\varphi}(\widehat{fv})\|_{q',m}\le C\|fv\|_{\alpha',m}. \tag{3.12}$$

From here on, the proof is exactly the same as the proof of Theorem 3.4. ∎

It is clear that disjointness results can also be obtained by combining the μ and ν results. Instead of doing so, we first give a theorem which extends Edwards' theorem, i.e., the range of T_φ can be extended to $\bigcup_{1<\alpha<2}(L^\alpha(\mu))^\wedge$ (see also Remark 4.1 and 4.2 of [2]).

THEOREM 3.11. *Let $1\le q\le\infty$ and let $(\mu,1)$ (respectively $(1,\nu)$) satisfy the condition* (3.3). *Then*

$$\Big\{\varphi \text{ measurable}: f\varphi\in\bigcup_{1<\alpha<2}(L^\alpha(\mu))^\wedge,\ \text{for all } f\in L^q(\mathbb{R})\Big\}=\{0\}$$

$$\Big(\text{resp. }\Big\{\varphi \text{ measurable}: f\varphi\in\bigcup_{1<\alpha<2}(L^\alpha(\mathbb{R}))^\wedge,\ \text{for all } f\in L^q(\nu)\Big\}=\{0\}\Big).$$

Proof. Apply a Baire category argument to Theorem 3.4 (resp. Theorem 3.10). ∎

We now end this section by stating a disjointness result removing the nondecreasing condition on the absolutely continuous control measure μ. It is clear that as soon as a Hausdorff–Young type inequality is obtained, a disjointness result follows. Lakey [10] contains many such inequalities (Theorems 1.27, 3.32, 3.34, 3.35, 4.24, 4.25 as well as Proposition 3.39). As a sample we state the following disjointness result whose proof is similar to those of Theorems 3.4 or 3.10 using the corresponding version of the Hausdorff–Young theorem (Theorem 3.34 of [10]).

THEOREM 3.12. *Let $X(t)=\int_{\mathbb{R}}h(t-s)\,dY(s)$, where Y has absolutely continuous control measure u such that $\sum_{n=0}^\infty\sup_{n\le x<n+1}u(x)<\infty$. Then $X=\{X(t):t\in\mathbb{R}\}$ cannot be (p,q)-bounded unless it is null.*

3.3. *Unbounded noise.* That a large class of moving averages are weak Fourier transforms of stable noises with dependent increments is shown in [2] (for $h\in L^\gamma\cap L^\alpha(\mathbb{R})$), $1\le\gamma<\alpha$). With the aid of an inequality due to Stechkin (see [5]) and using a proof similar to the one in [2], we extend this result to all of $L^\alpha(\mathbb{R})$. This in turn shows us that the crux of the incompatibility of the two representations lies in the (p,q)-boundedness, i.e., in the way of defining the integral.

In the following theorem, the dependently but not stationarily scattered $S\alpha S$ noise $Y=\{Y(B):B\in\mathcal{B}_1(\mathbb{R})\}$, where $\mathcal{B}_1(\mathbb{R})$ is the δ-ring of finite unions of bounded half-open intervals, is given by

$$Y(B)=\int_{\mathbb{R}}\widehat{\chi}_B*h(t)\,dM(t).$$

THEOREM 3.13. *Let M be a $S\alpha S$ Lévy motion, $1<\alpha<2$, and let $X(t)=\int_{\mathbb{R}}h(t-\tau)\,dM(\tau)$, $h\in L^\alpha(\mathbb{R})$. Then*

$$X(t)=\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\Big(1-\frac{|\gamma|}{\lambda}\Big)e^{-it\lambda}\,dY(\gamma),\qquad t\in\mathbb{R},$$

where convergence is in probability for each t.

Proof. Let $B\in\mathcal{B}_1(\mathbb{R})$; then clearly $\chi_B\in L^1(\mathbb{R})\cap BV(\mathbb{R})$. Now Stechkin's inequality states that for $\varphi\in L^1(\mathbb{R})\cap BV(\mathbb{R})$ and $h\in L^1(\mathbb{R})\cap L^\alpha(\mathbb{R})$,

$$\|\widehat{\varphi}*h\|_\alpha\le C_\alpha\max(|\varphi(0)|,\operatorname{Var}\varphi)\|h\|_\alpha.$$

Using the arguments of Lemma 3.3, we can extend this to all of $L^\alpha(\mathbb{R})$ and thus get

$$\|\widehat{\chi}_B*h\|_\alpha\le C\max(\chi_B(0),\operatorname{Var}\chi_B)\|h\|_\alpha,\qquad h\in L^\alpha(\mathbb{R}).$$

Hence $Y(B)=\int_{\mathbb{R}}(\widehat{\chi}_B*h)(t)\,dM(t)$ is well-defined and additive. In this way we see that for every step function with bounded support we have $\int_{\mathbb{R}}f\,dY=\int_{\mathbb{R}}(\widehat{f}*h)\,dM$. Let $G=\{f\in L^1(\mathbb{R})\cap BV(\mathbb{R}):\exists\{f_n\}$ step functions with bounded support, $f_n(0)=f(0)$, and $\lim_{n\to\infty}f_n=f$ in variation$\}$. Then for $f\in G$ we have

$$\int_{\mathbb{R}}f\,dY=\lim_{n\to\infty}\int_{\mathbb{R}}f_n\,dY=\lim_{n\to\infty}\int_{\mathbb{R}}(\widehat{f}_n*h)\,dM=\int_{\mathbb{R}}(\widehat{f}*h)\,dM.$$

Of course $g_{t,\lambda}(\gamma)=(1-|\gamma|/\lambda)e^{-it\gamma}\chi_{(-\lambda,\lambda)}(\gamma)\in G$ as this is just a modulation of the tent function over a compact set. This gives us

$$\int_{-\lambda}^{\lambda}\Big(1-\frac{|\gamma|}{\lambda}\Big)e^{-it\gamma}\,dY(\gamma)=\int_{\mathbb{R}}(\widehat{g}_{t,\lambda}*h)(\tau)\,dM(\tau).$$

Finally, since $\widehat{g}_{t,\lambda}$ is an approximate identity, $\widehat{g}_{t,\lambda}*h\xrightarrow{\lambda}h$ in $L^\alpha(\mathbb{R})$ and

$$X(t)=\lim_{\lambda\to\infty}\int_{-\lambda}^{\lambda}\Big(1-\frac{|\gamma|}{\lambda}\Big)e^{-it\gamma}\,dY(\gamma)\qquad\text{in probability.}$$

That Y is not stationarily scattered is obtained as in the proof of Theorem 3.1 of [2]. ∎

References

[1] J. Benedetto and H. Heinig, *Weighted Hardy spaces and the Laplace transform*, in: Lecture Notes in Math. 992, Springer, 1983, 240–277.

[2] S. Cambanis and C. Houdré, *Stable processes: moving averages versus Fourier transforms*, Probab. Theory Related Fields 95 (1993), 75–85.

[3] S. Cambanis and R. Soltani, *Prediction of stable processes: spectral and moving average representations*, Z. Warhrsch. Verw. Gebiete 66 (1984), 593–612.

[4] N. Dunford and J. Schwartz, *Linear Operators, Part I: General Theory*, Wiley Interscience, New York 1957.

[5] R. Edwards and G. Gaudry, *Littlewood–Paley and Multiplier Theory*, Springer, Berlin 1977.

[6] C. Houdré, *Harmonizability, V-boundedness, (2, p)-boundedness of stochastic processes*, Probab. Theory Related Fields 84 (1990), 39–54.

[7] —, *Linear and Fourier stochastic analysis*, ibid. 87 (1990), 167–188.

[8] R. Johnson, *Recent results on weighted inequalities for the Fourier transform*, in: Seminar Analysis of the Karl-Weierstraß-Institute 1986/87, Teubner-Texte Math. 106, B. Schulze and H. Triebel (eds.), Teubner, Leipzig 1988, 287–296.

[9] W. Jurkat and G. Sampson, *On rearrangement and weight inequalities for the Fourier transform*, Indiana Univ. Math. J. 33 (1984), 257–270.

[10] J. Lakey, *Weighted norm inequalities for the Fourier transform*, Ph.D. Thesis, University of Maryland, College Park, 1991.

[11] A. Makagon and V. Mandrekar, *The spectral representation of stable processes: harmonizability and regularity*, Probab. Theory Related Fields 85 (1990), 1–11.

[12] B. Rajput and J. Rosinski, *Spectral representations of infinitely divisible processes*, ibid. 82 (1989), 451–487.

[13] J. Rosinski, *On stochastic integral representation of stable processes with sample paths in Banach spaces*, J. Multivariate Anal. 20 (1986), 277–307.

[14] G. Samorodnitsky and M. Taqqu, *Stable Random Processes*, book to appear.

[15] K. Urbanik, *Prediction of strictly stationary sequences*, Colloq. Math. 12 (1964), 115–129.

[16] —, *Some prediction problems for strictly stationary processes*, in: Proc. 5th Berkeley Sympos. Math. Statist. Probab., Vol. 2, Part I, Univ. of California Press, 1967, 235–258.

[17] —, *Random measures and harmonizable sequences*, Studia Math. 31 (1968), 61–88.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF MARYLAND
COLLEGE PARK, MARYLAND 20742
U.S.A.

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305
U.S.A.

Received July 17, 1992
Revised version February 15, 1993 (2977)

STUDIA MATHEMATICA 105 (3) (1993)

A remark on disjointness results for stable processes

by

ALEKSANDER WERON (Wrocław)

In a recent paper M. Hernández and C. Houdré [3] have applied the Fourier analysis methods to prove disjointness results for some classes of stable stochastic processes. However, these methods forced the authors to restrict the range of the index of stability to $1 \le \alpha < 2$, instead of $0 < \alpha < 2$.

In this note we would like to show how the disjointness results, going back to the pioneering work of K. Urbanik [5], can be easily understood in the more general setup of symmetric infinitely divisible (ID) stationary processes, including stationary symmetric α-stable ($S\alpha S$) processes for all $0 < \alpha < 2$. Here we will employ some basic facts concerning the hierarchy of ergodic properties for stationary ID processes [2]. In the Gaussian case the moving averages form a subclass of harmonizable processes; however, for non-Gaussian ID processes we have

PROPOSITION. *In the class of symmetric non-Gaussian ID stationary processes a nondegenerate moving average process is never harmonizable.*

Proof. All symmetric ID moving averages are stationary and mixing [2]. Therefore they are ergodic. Harmonizable processes, i.e., the Fourier transforms of independently scattered random measures are stationary iff the random measure is rotation invariant. In sharp contrast with the Gaussian case, symmetric non-Gaussian ID harmonizable processes are not ergodic (cf. [4] and [1] for the $S\alpha S$ case). It follows that both classes are disjoint.

References

[1] S. Cambanis, C. D. Hardin Jr., and A. Weron, *Ergodic properties of stationary stable processes*, Stochastic Process. Appl. 24 (1987), 1–18.

[2] S. Cambanis, A. Ławniczak, K. Podgórski and A. Weron, *Ergodicity and mixing of symmetric infinitely divisible processes*, Tech. Rept. No. 346, Univ. North Carolina, 1991.

1991 *Mathematics Subject Classification*: Primary 60G10.