$$\sum_{i=1}^{s} u_i/(K+3B_q) \le \sum_{i=1}^{s} u_i/(K'+3B_{q'}) .$$

This means that, without loss of generality, we may assume $\dim E < \infty$. In this case, however, our lemma follows easily from the preceding one. ∎

Proof of Theorem 3. Let $(g_i)_{i\in I}$ be a strongly summable family in a nuclear group $G$. We are to show that $(g_i)_{i\in I}$ is absolutely summable. As in the proof of Theorem 1, we may assume that $G = F/K$ where $K$ is a closed subgroup of some nuclear vector group $F$. Let $\varphi : F \to F/K$ be the natural projection.

Take an arbitrary $U \in \mathcal{N}_0(F)$. Due to Lemma 2, we can find a linear subspace $E$ of $F$ and pre-Hilbert seminorms $p, q$ on $E$ such that $3B_q \subset U$, $B_p \in \mathcal{N}_0(F)$ and $\sum_{k=1}^{\infty} d_k^2(B_p, B_q) < 1$. As $(g_i)_{i\in I}$ is strongly summable, there is a finite subset $J$ of $I$ such that $\sum_{i\in L} g_i \in \varphi(B_p)$ for each finite subset $L$ of $I \setminus J$. It is enough to prove that

$$\sum_{i\in I\setminus J} g_i/\varphi(3B_q) < \infty . \tag{22}$$

For each $i \in I$, choose some $u_i \in \varphi^{-1}(g_i)$. Then $\sum_{i\in L} u_i \in K + B_p$ for each finite $L \subset I \setminus J$. Lemma 9 yields

$$\sum_{i\in I\setminus J} u_i/(K+3B_q) \le 11 ,$$

whence (22) follows because $g_i/\varphi(3B_q) = u_i/(K+3B_q)$ for every $i$. ∎

### References

[1] W. Banaszczyk, *Additive Subgroups of Topological Vector Spaces*, Lecture Notes in Math. 1466, Springer, Berlin 1991.

[2] N. Bourbaki, *Topologie générale*, Chap. III, *Groupes topologiques* (*Théorie élémentaire*), Hermann, Paris 1960.

[3] N. Kalton, *Subseries convergence in topological groups and vector spaces*, Israel J. Math. 10 (1971), 402–412.

[4] A. Pietsch, *Nuclear Locally Convex Spaces*, Springer, Berlin 1972.

INSTITUTE OF MATHEMATICS
ŁÓDŹ UNIVERSITY
90-238 ŁÓDŹ, POLAND
E-mail: WBANASZ@PLUNLO51.BITNET

*Received November 3, 1992*
*Revised version February 23, 1993*
(3019)

STUDIA MATHEMATICA 105 (3) (1993)

# An inverse Sidon type inequality

by

S. FRIDLI (Budapest)

**Abstract.** Sidon proved the inequality named after him in 1939. It is an upper estimate for the integral norm of a linear combination of trigonometric Dirichlet kernels expressed in terms of the coefficients. Since the estimate has many applications for instance in $L^1$ convergence problems and summation methods with respect to trigonometric series, newer and newer improvements of the original inequality has been proved by several authors. Most of them are invariant with respect to the rearrangement of the coefficients. Although the newest results are close to best possible, no nontrivial lower estimate has been given so far. The aim of this paper is to give the best rearrangement invariant function of coefficients that can be used in a Sidon type inequality. We also show that it is equivalent to an Orlicz type and a Hardy type norm. Examples of applications are also given.

**1. Introduction.** Let $L^1[-\pi,\pi]$ denote the set of $2\pi$-periodic Lebesgue-integrable functions with norm denoted by $\|\ \|_{L^1[-\pi,\pi]}$. Furthermore, let the real Hardy space $H[-\pi,\pi]$ be defined as the Banach space of functions $f \in L^1[-\pi,\pi]$ the trigonometric conjugate $\widetilde{f}$ of which is integrable, and

$$\|f\|_{H[-\pi,\pi]} = \|f\|_{L^1[-\pi,\pi]} + \|\widetilde{f}\|_{L^1[-\pi,\pi]} .$$

We will also need the Banach spaces $L^p[0,1]$ $(1 \le p \le \infty)$ with the usual norm denoted by $\|\ \|_p$, and the dyadic Hardy space $H[0,1]$. For any $f \in L^1[0,1]$ let the dyadic maximal function $f^*$ be defined as follows:

$$f^*(x) = \sup\left\{\frac{1}{\mu(I)}\Big|\int_I f(t)\,dt\Big| : I \in \mathcal{I}, I \ni x\right\} \quad (x \in [0,1)) ,$$

where $\mathcal{I}$ is the set of dyadic intervals, i.e.

$$\mathcal{I} = \{[k2^{-n}, (k+1)2^{-n}) : k, n \in \mathbb{N},\ 0 \le k < 2^n\} ,$$

and $\mu(I)$ denotes the length of $I$. ($\mathbb{N}$ stands for the set of natural numbers.) Then, $H[0,1]$ is the set of integrable functions $f$ for which $f^*$ is integrable,

1991 *Mathematics Subject Classification*: Primary 42A05; Secondary 42A20, 42A16.
*Key words and phrases*: Sidon type inequalities, Hardy spaces, convergence classes.
This research was supported by the Hungarian National Foundation for Scientific Research (OTKA 2080).

and the norm $\|f\|_H$ is defined to be the $L^1[0,1]$ norm of $f^*$. Equivalent definitions exist for $H[-\pi,\pi]$ and $H[0,1]$. We refer to [5], [11], [13] for details.

Let $D_k$ denote the well known Dirichlet kernel, i.e.

$$D_k = \frac{1}{2} + \sum_{j=1}^{k} \cos jx \quad (k \in \mathbb{N},\ x \in [-\pi,\pi]).$$

We will consider the expression

$$\frac{1}{n+1} \int_0^\pi \Big| \sum_{k=0}^{n} c_k D_k(x) \Big| \, dx \quad (n \in \mathbb{N}), \tag{1.1}$$

where $c_k$'s are arbitrary real numbers. The following inequality was proved by Sidon [12] in 1939:

$$\frac{1}{n+1} \int_0^\pi \Big| \sum_{k=0}^{n} c_k D_k(x) \Big| \, dx \le C \max_{0\le k\le n} |c_k| \quad (n \in \mathbb{N}). \tag{1.2}$$

($C, C_1, C_2, C_3, C_4$ will denote absolute positive constants, and $C^{(p)}$ positive constants depending only on $p$, all of them not necessarily the same at different occurrences.)

In Section 2 we give the best rearrangement invariant function of coefficients that can serve as the right side of such an inequality. As a generalization we prove a similar result for the shifted case. Then a short summary on the history of Sidon type inequalities is given.

In Section 3 we show that the rearrangement invariant function of coefficients introduced in Section 2 is equivalent to a norm generated by the dyadic Hardy norm, and also to an Orlicz type norm. We also give some applications of the results of Section 2 to $L^1$ convergence problems for trigonometric series.

## 2. Lower estimate in Sidon type inequalities. The shifted version.

For any $n \in \mathbb{N}$ denote by $P_n$ the set of permutations of $\{0,1,\ldots,n\}$.

THEOREM 1. *Let $(c_k)$ be a sequence of real numbers. Then*

(i) $$\int_0^\pi \Big| \sum_{k=0}^{N} c_k D_k(x) \Big| \, dx \le C_1 \sum_{k=0}^{N} |c_k| \left(1 + \log^+ \frac{|c_k|}{(N+1)^{-1}\sum_{j=0}^{N}|c_j|}\right),$$

(ii) $$\max_{p\in P_N} \int_0^\pi \Big| \sum_{k=0}^{N} c_{p_k} D_k(x) \Big| \, dx \ge C_2 \sum_{k=0}^{N} |c_k| \left(1 + \log^+ \frac{|c_k|}{(N+1)^{-1}\sum_{j=0}^{N}|c_j|}\right) \quad (N \in \mathbb{N}).$$



We make some historical comments and remarks on various improvements of Sidon's original result. The first step was made by Bojanic and Stanojević [1] who proved that

$$\frac{1}{n} \int_0^\pi \Big| \sum_{k=0}^{n-1} c_k D_k(x) \Big| \, dx \le C^{(p)} \left(\frac{1}{n} \sum_{k=0}^{n-1} |c_k|^p\right)^{1/p} \quad (p > 1). \tag{2.1}$$

We note that this estimate is essentially contained in Fomin [4]. It is easy to see that (2.1) is not valid for $p = 1$. Indeed, if $c_n = 1$ and $c_k = 0$ ($k \ne n$, $k \in \mathbb{N}$) then the left side is of order $(\log n)/n$ while the right side is of order $1/n$ as $n \to \infty$. Still, a certain balancing of the $p = 1$ and $p > 1$ cases is possible. Namely, Tanović-Miller [14] showed that (1.1) can be dominated by

$$C^{(p)} \left( \frac{\log \alpha}{n} \sum_{k=0}^{n-1} |c_k| + \alpha^{-1/q} \left(\frac{1}{n} \sum_{k=0}^{n-1} |c_k|^p\right)^{1/p} \right) \tag{2.2}$$
$$(\alpha \ge 1,\ 1 < p \le 2,\ 1/p + 1/q = 1).$$

Let $(c_k)$ be a real sequence, $n \in \mathbb{N}$, and let the characteristic function of a set $A$ of real numbers be denoted by $\chi_A$. If the coefficient vector $(c_k)_0^{2^n-1}$ is associated with the step function $\Gamma_n$ defined on $[0,1]$ by

$$\Gamma_n = \sum_{k=0}^{2^n-1} c_k \chi_{[k2^{-n},(k+1)2^{-n})},$$

then the right sides of (1.2), (2.1), and (2.2) can be expressed in terms of $\|\Gamma_n\|_p$ ($1 \le p \le \infty$). Indeed, for the indices $2^n$ ($n \in \mathbb{N}$) the right sides of (1.2) and (2.1) are simply $\|\Gamma_n\|_\infty$ and $\|\Gamma_n\|_p$ ($p > 1$). Similarly, (2.2) corresponds to the mixed norm

$$(\log \alpha)\|\Gamma_n\|_1 + \alpha^{-1/q}\|\Gamma_n\|_p \quad (1 < p \le 2).$$

Schipp [10] observed that the original Sidon inequality for coefficients having zero sum and the uniform boundedness of the $L^1$ norm of the Fejér kernels are closely connected with the so called atomic decomposition of $\Gamma_n$. On the basis of this observation he proved that

$$\frac{1}{2^n} \int_0^\pi \Big| \sum_{k=0}^{2^n-1} c_k D_k(x) \Big| \, dx \le \|\Gamma_n\|_H \quad (n \in \mathbb{N}). \tag{2.3}$$

Furthermore, Schipp showed that (1.2), (2.1), and (2.2) can be deduced from (2.3). He also showed that similar results hold for several systems other than the real trigonometric system. We note that he originally proved the above inequality by using the norm of the so called nonperiodic Hardy space (see e.g. [5]) which is generally smaller than the dyadic Hardy norm. However, this difference does not affect the results of this paper.

Observe that in all cases except (2.3) the right sides of the estimates are invariant with respect to the rearrangement of the coefficients. The left side of course depends on the order of the coefficients. However, the dyadic Hardy norm is not rearrangement invariant, but $\|\Gamma_n\|_H$ depends on the order of $c_k$'s $(0 \le k < 2^n)$ in a quite different way as (1.1). For instance, if one of $c_k$'s $(0 \le k < 2^n)$, say the $j$th, is 1 and all the others are 0, then (1.1) is of order $(\log j)/n$, while the Hardy norm (invariant with respect to the dyadic translation) of the corresponding step function can be estimated below by $C(\log n)/n$. Similarly, as a consequence of the uniform boundedness of the $L^1$ norm of the Fejér kernels, if $c_k$'s are positive and decreasing, then (1.1) can be dominated by the $L^1$ norm of the corresponding step function. Consequently, (1.1) and $\|\Gamma_n\|_H$ depend in an essentially different way on the order of the coefficients. Another difficulty with $\|\Gamma_n\|_H$ is that it would be very complicated to express it directly by the $c_k$'s. We note that Theorem 1(i) can be derived from the result of Schipp, i.e. from (2.3).

In some applications a shifted version of the above inequalities is useful. Móricz proved in [8] that for any $K \le N$ and $1 < p \le 2$,

$$\int_0^\pi \Big|\sum_{k=K}^N c_k D_k(x)\Big|\,dx \le C\Big(1+\log\frac{N+1}{N-K+1}\Big)\sum_{k=K}^N |c_k| + C^{(p)}(N-K+1)\Big(\sum_{k=K}^N \frac{|c_k|^p}{N-K+1}\Big)^{1/p}.$$

A modification of this inequality has been given by Buntinas and Tanović-Miller [2]. Similarly to the original case, we will give the best rearrangement invariant estimate for the shifted Sidon inequality. It is a generalization of Theorem 1 and the results of Móricz, and of Buntinas and Tanović-Miller.

THEOREM 2. *Let $(c_k)$ be a sequence of real numbers, and $K \le N$ $(K, N \in \mathbb{N})$. Then*

(i)
$$\int_0^\pi \Big|\sum_{k=K}^N c_k D_k(x)\Big|\,dx \le C_1\Big(\log\frac{N}{N-K+1}\Big|\sum_{k=K}^N c_k\Big| + \sum_{k=K}^N |c_k|\Big(1+\log^+\frac{|c_k|}{(N-K+1)^{-1}\sum_{j=K}^N |c_j|}\Big)\Big),$$

*and*

(ii)
$$\max_{p\in P_{N-K}} \int_0^\pi \Big|\sum_{k=0}^{N-K} c_{K+p_k} D_{K+k}(x)\Big|\,dx \ge C_2\Big(\log\frac{N}{N-K+1}\Big|\sum_{k=K}^N c_k\Big| + \sum_{k=K}^N |c_k|\Big(1+\log^+\frac{|c_k|}{(N-K+1)^{-1}\sum_{j=K}^N |c_j|}\Big)\Big).$$



**3. Applications.** In this section we show some consequences of the inequalities of Theorems 1 and 2. The first one concerns Hardy and Orlicz spaces. Namely, we prove that the largest rearrangement invariant subspace of $H$ is equivalent to an Orlicz space.

Let $L_0$ denote the set of measure preserving bijections from $[0,1]$ onto itself. It is known that the Hardy norm is not rearrangement invariant. Moreover (see e.g. [11]), there exist $f \in H$ and $\nu \in L_0$ such that $f\circ\nu \notin H$. Denote by $H^\sharp$ the largest subset of $H$ which is invariant with respect to rearrangements, i.e.

$$H^\sharp = \{f \in H : f\circ\nu \in H,\ \nu \in L_0\}.$$

Then a rearrangement invariant norm on $H^\sharp$ can be defined by

$$\|f\|_{H^\sharp} = \sup\{\|f\circ\nu\|_H : \nu \in L_0\} \qquad (f \in H^\sharp).$$

We note that the finiteness of $\|f\|_{H^\sharp}$ for every $f \in H^\sharp$ will be shown later.

Define the Young function $M$ as the integral function of

$$p(t) = \begin{cases} t & \text{if } 0 \le t < 1, \\ 1+\log t & \text{if } t \ge 1. \end{cases}$$

Then

$$M(x) = \int_0^{|x|} p(t)\,dt = \begin{cases} (1/2)|x|^2 & \text{if } 0 \le |x| < 1, \\ 1/2 + |x|\log^+|x| & \text{if } |x| \ge 1. \end{cases}$$

The Young function $N$ complementary to $M$ is the integral function of

$$q(t) = \begin{cases} t & \text{if } 0 \le t < 1, \\ e^{t-1} & \text{if } t \ge 1, \end{cases}$$

i.e.

$$N(x) = \begin{cases} (1/2)|x|^2 & \text{if } 0 \le |x| < 1, \\ e^{|x|-1} - 1/2 & \text{if } |x| \ge 1. \end{cases}$$

The Orlicz space $L_M$ is the collection of functions $f \in L^1[0,1]$ for which

$$\int_0^1 M(f(x))\,dx < \infty.$$

The corresponding norm is

$$\|f\|_M = \int_0^1 p(k|f(x)|)|f(x)|\,dx \quad (f \in L_M),$$

where the real number $k$ is determined by

$$\int_0^1 N(p(k|f(x)|))\,dx = 1.$$

For the theory of Orlicz spaces we refer to [7]. The next theorem shows that $H^\sharp$ and $L_M$ are isomorphic.

THEOREM 3. *$L_M = H^\sharp$, and*

$$C_1\|f\|_M \le \|f\|_{H^\sharp} \le C_2\|f\|_M \quad (f \in L_M).$$

Remark. In view of Theorem 3 the result of Theorem 1 can be formulated as follows. The rearrangement invariant norm generated by (1.1) in the $n$-dimensional space is equivalent (uniformly in $n$) to an Orlicz type norm.

Sidon type inequalities can also be used in the investigation of the integrability problems of cosine and sine series. Indeed, if $(a_k)$ is a sequence of real numbers, then the partial sums of the series $\sum_{k=0}^\infty a_k \cos kx$ can be expressed as

$$\sum_{k=0}^n a_k \cos kx = \sum_{k=0}^{n-1} \Delta a_k D_k + a_n D_n \quad (n \in \mathbb{N}),$$

where $\Delta a_k = a_k - a_{k+1}$. The Fourier series of $f \in L^1[-\pi,\pi]$ will be denoted by $Sf(x)$. Similarly, $S_nf(x)$ denotes the $n$th Fourier partial sum of $f$.

The Fourier series of a $2\pi$-periodic even function integrable on $[0,\pi]$ is a cosine series. Let the collection of sequences of Fourier coefficients of all such functions be denoted by $\mathcal{L}$. The subsets of $\mathcal{L}$ are called *integrability classes*. Suppose $\mathcal{J} \subset \mathcal{L}$, $(a_k) \in \mathcal{J}$ and denote by $f$ the function whose Fourier series is $\sum_{k=0}^\infty a_k \cos kx$. If $\lim_{n\to\infty}\|S_nf - f\|_1 = 0$ holds if and only if $\lim_{n\to\infty} a_n \log n = 0$ then $\mathcal{J}$ is called an *$L^1$ convergence class*.

No characterization is known for the elements of $\mathcal{L}$ in terms of the properties of the sequences; however, several sufficient conditions exist for a class to be an $L^1$ convergence class. In the following we define conditions that generate integrability and $L^1$ convergence classes. Conditions for convergence of cosine series in Hardy norm will also be given. By conjugation these results can be transferred to trigonometric and sine series.



Let $\mathcal{S}$ denote the collection of null sequences $(a_k)$ for which there exists a strictly increasing sequence $(N_j)$ of natural numbers such that

$$\sum_{j=0}^\infty \Big(\log \frac{N_{j+1}}{N_{j+1}-N_j}\Big|\sum_{k=N_j}^{N_{j+1}-1} \Delta a_k\Big|\Big) < \infty \tag{3.1}$$

and

$$\sum_{j=0}^\infty \sum_{k=N_j}^{N_{j+1}-1} |\Delta a_k|\Big(1 + \log^+ \frac{|\Delta a_k|}{(N_{j+1}-N_j)^{-1}\sum_{l=N_j}^{N_{j+1}-1}|\Delta a_l|}\Big) < \infty. \tag{3.2}$$

Furthermore, denote by $\mathcal{S}^*$ the subset of $\mathcal{S}$ the elements of which satisfy (3.2) and

$$\sum_{j=0}^\infty \Big(\log \frac{N_{j+1}}{N_{j+1}-N_j}\sum_{k=N_j}^{N_{j+1}-1} |\Delta a_k|\Big) < \infty. \tag{3.3}$$

Thus $\mathcal{S}$ is an integrability class, and $\mathcal{S}^*$ is an $L^1$ convergence class.

THEOREM 4. *Suppose that $(a_k) \in \mathcal{S}$. Then*

(i) *the function $f(x) = \sum_{k=0}^\infty a_k \cos kx$ $(x \ne 0)$ is in $L^1[-\pi,\pi]$,*

(ii) *$\sum_{k=0}^\infty a_k \cos kx$ is the Fourier series of $f$,*

(iii) *if $(a_k) \in \mathcal{S}^*$ then $\sum_{k=0}^n a_k \cos kx$ converges to $f$ in mean if and only if $a_n \log n = o(1)$ as $n \to \infty$.*

Now, we consider the convergence of trigonometric series in Hardy norm. By taking $N_j = 2^j$ $(j \in \mathbb{N})$ in the definition of $\mathcal{S}$, (3.1) and (3.2) reduce to the form

$$\sum_{j=0}^\infty \sum_{k=2^j}^{2^{j+1}-1} |\Delta a_k|\Big(1 + \log^+ \frac{|\Delta a_k|}{2^{-j}\sum_{l=2^j}^{2^{j+1}-1}|\Delta a_l|}\Big) < \infty. \tag{3.4}$$

If $f \in H$ is an even function with Fourier coefficients $(a_k)$, then $(a_k)$ satisfies the so called Hardy inequality, i.e.

$$\sum_{k=1}^\infty \frac{|a_k|}{k} < \infty. \tag{3.5}$$

It is easy to see that Hardy's inequality cannot be deduced from (3.4). However, if condition (3.5) is added to (3.4) then the following theorem holds about $H$ convergence.

THEOREM 5. *Suppose that $(a_k)$ satisfies (3.4). Then*

(i) *the function $f(x) = \sum_{k=0}^\infty a_k \cos kx$ $(x \ne 0)$ is in $H[-\pi,\pi]$ if and only if $(a_k)$ satisfies (3.5), and in that case,*

(ii) *$\sum_{k=0}^n a_k \cos kx$ converges to $f$ in Hardy norm if and only if $a_n \log n = o(1)$ as $n \to \infty$.*

Applying the above result to sine series we have the following corollary.

Corollary 1. *Suppose that* $(b_k)$ *satisfies both* (3.4) *and* (3.5). *Then* $\sum_{k=1}^{\infty} b_k \sin kx$ *is the Fourier series of a function* $g \in H[-\pi,\pi]$. *Moreover*, $\sum_{k=1}^{n} b_k \sin kx$ *converges to* $f$ *as* $n \to \infty$ *in Hardy norm if and only if* $b_n \log n = o(1)$.

Remark. Several sufficient conditions have been given for the integrability of odd trigonometric series, for instance in the papers of Telyakovskiĭ [16] and Móricz [9]. We note that their results can be deduced from Theorem 5.

The combination of the above results yields

Corollary 2. *Suppose that* $(a_k)$ *satisfies* (3.4), *and* $(b_k)$ *satisfies* (3.4) *and* (3.5). *Then* $\sum_{k=1}^{\infty}(a_k \cos kx + b_k \sin kx)$ *is the Fourier series of an* $f \in L^1[-\pi,\pi]$. *Moreover*, $\sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$ *converges to* $f$ *in mean as* $n \to \infty$ *if and only if* $(|a_n| + |b_n|)\log n = o(1)$. *If also* $(a_k)$ *satisfies* (3.5) *then* $f \in H[-\pi,\pi]$, *and the Fourier partial sums of* $f$ *converge to* $f$ *in Hardy norm if and only if* $(|a_n| + |b_n|)\log n = o(1)$.

In order to show how $\mathcal{S}$ relates to the previously known integrability and $L^1$ convergence classes we need some historical comments. In the rest of this section we will always assume that the sequence in question is a null sequence.

Young [17] proved that the class $\mathcal{K}$ of convex sequences (i.e. $\Delta^2 a_k \ge 0$) is an integrability and $L^1$ convergence class. The same is true for the set $\mathcal{Q}$ of quasiconvex sequences (i.e. with $\sum_{k=1}^{\infty} k|\Delta^2 a_k| < \infty$), as was proved by Kolmogorov [6]. The first conditon based on a Sidon type inequality was given by Telyakovskiĭ [15]. That leads to the class $\mathcal{T}$ of sequences $(a_k)$ for which there exists a monotone sequence $(A_k)$ such that $|\Delta a_k| \le A_k$ and $\sum_{k=0}^{\infty} A_k < \infty$. Fomin [4] extended the above class by using the estimate (2.1) in the following way. A sequence $(a_k)$ belongs to the class $\mathcal{F}_p$ if

$$\sum_{n=0}^{\infty} 2^n \Big(\sum_{k=2^n}^{2^{n+1}-1} \frac{|\Delta a_k|^p}{2^n}\Big)^{1/p} < \infty \quad (1 < p < \infty).$$

On the basis of (2.2) Tanović-Miller enlarged $\mathcal{F}_p$ to a class denoted by $\mathcal{F}_p^*$. Recently, Buntinas and Tanović-Miller introduced the classes $hv^p \supset \mathcal{F}_p^*$ $(1 < p < \infty)$. $hv^p$ is the set of sequences $(a_k)$ for which there exist sequences of natural numbers: $(\nu_k)$ nondecreasing and $(k_j)$ increasing, such that $\nu_j \le k_{j+1}$ and

$$\sum_{j=0}^{\infty} \log\frac{k_{j+1}}{\nu_j} \sum_{k=k_j}^{k_{j+1}-1} |\Delta a_k| + \nu_j^{1/q}\Big(\sum_{k=k_j}^{k_{j+1}-1} |\Delta a_k|^p\Big)^{1/p} < \infty .$$



As was proved in [4], [14], [2],

$$\mathcal{K} \subset \mathcal{Q} \subset \mathcal{F}_p \subset \mathcal{F}_p^* \subset hv^p .$$

$hv^p$ is based on a Sidon type inequality which is invariant with respect to the rearrangements of the coefficients, in the same way as $\mathcal{S}$. Therefore, $hv^p \subset \mathcal{S}$ is a consequence of Theorem 2(ii), i.e. the following result is true.

Theorem 6. $\mathcal{S}$ *and* $\mathcal{S}^*$ *contain all of the above listed classes.*

Remark. We note that several convergence classes have been defined which do not rely on Sidon type inequalities. A thorough survey is given in the paper of Buntinas and Tanović-Miller [3].

## 4. Lemmas.

In order to prove our theorems we need some lemmas.

Lemma 1. *Let* $f$ *be a nonnegative decreasing function on* $[a,b]$ $(a > 0)$. *Then*

$$\Big|\int_a^b f(x)t_\delta(x)\,dx\Big| \le 4\frac{f(a)}{\delta},$$

*where* $t_\delta(x) = \cos\delta x$ *or* $\sin\delta x$ $(\delta > 0)$.

Proof. One may assume that $\delta = 1$. Let $t(x) = \cos x$ or $\sin x$, and

$$T(x) = \int_0^x t(y)\,dy = \begin{cases} \sin x & \text{if } t(x) = \cos x, \\ -\cos x + 1 & \text{if } t(x) = \sin x. \end{cases}$$

Integration by parts yields

$$\int_a^b f(x)t(x)\,dx = \int_a^b f(x)\,dT(x) = f(b)T(b) - f(a)T(a) - \int_a^b T(x)\,df(x).$$

Thus

$$\Big|\int_a^b f(x)t(x)\,dx\Big| \le 2(|f(b)| + |f(a)|) + 2|f(a) - f(b)| = 4f(a).$$

Consequently, $|\int_a^b f(x)t_\delta(x)\,dx| \le 4f(a)/\delta$. ∎

Lemma 2. *Let* $n_k = 2^{n-k}$, $m_k < 2^{n-2^k-1}$ $(n,k,m_k \in \mathbb{N},\ n > 4,\ 2 \le k < \log_2 n)$. *Set*

$$T_{n_k,m_k,K}(x) = \sum_{j=-m_k}^{m_k} D_{n_k+j+K}(x) \quad (x \in [0,\pi],\ K \in \mathbb{N})$$

*and*

$$J_l = [2^{-(n-2^{l-1})}\pi, 2^{-(n-2^l)}\pi) \quad (1 \le l < \log_2 n).$$

*Then*

(i) $$\int_{J_k} T_{n_k,m_k,K}(x)\sin(n_k+K+\tfrac12)x\,dx \ge C_1(2m_k+1)2^k\,,$$

(ii) $$\int_{J_k} (T_{n_k,m_k,K}(x)-T_{n_k,0,K}(x))\sin(n_k+K+\tfrac12)x\,dx \ge C_2 m_k 2^k\,,$$

(iii) $$\Big|\int_{J_l} T_{n_k,m_k,K}(x)\sin(n_l+K+\tfrac12)x\,dx\Big| < C_3(2m_k+1)2^{-(2^{l-1}-2l)} \qquad (l\ne k)\,.$$

Proof. By an easy computation we have

$$T_{n_k,m_k,K}(x)=\sum_{j=-m_k}^{m_k}\frac{\sin(n_k+j+K+\frac12)x}{2\sin\frac12 x} = \frac{\sin(n_k+K+\frac12)x\sin(m_k+\frac12)x}{2\sin^2\frac12 x}\,.$$

Consequently,

$$\int_{J_k} T_{n_k,m_k,K}(x)\sin(n_k+K+\tfrac12)\,dx = \frac12\int_{J_k}\frac{1}{\sin^2\frac12 x}\sin(m_k+\tfrac12)x\sin^2(n_k+K+\tfrac12)x\,dx\,.$$

Similarly,

$$\int_{J_k}(T_{n_k,m_k,K}(x)-T_{n_k,0,K}(x))\sin(n_k+K+\tfrac12)\,dx = \frac12\int_{J_k}\frac{1}{\sin^2\frac12 x}(\sin(m_k+\tfrac12)x-\sin\tfrac12 x)\,\sin^2(n_k+K+\tfrac12)x\,dx\,.$$

Since

$$(m_k+\tfrac12)x<\tfrac12\cdot 2^{n-2^k}x\le \pi/2 \qquad (x\in J_k)\,,$$

by applying the elementary inequality $(2/\pi)\alpha<\sin\alpha<\alpha \quad (0<\alpha<\pi/2)$ we have

$$\frac{1}{\sin^2\frac12 x}>4\frac{1}{x^2} \quad\text{and}\quad \sin(m_k+\tfrac12)x>\frac{2}{\pi}(m_k+\tfrac12)x \quad (x\in J_k)\,.$$

Similarly, $\sin(m_k+\frac12)x-\sin\frac12 x\ge Cm_k x$. Thus

(4.1) $$\int_{J_k} T_{n_k,m_k,K}(x)\sin(n_k+K+\tfrac12)x\,dx \ge C(2m_k+1)\int_{J_k}\frac1x\sin^2(n_k+K+\tfrac12)x\,dx\,,$$

and

(4.2) $$\int_{J_k}(T_{n_k,m_k,K}(x)-T_{n_k,0,K}(x))\sin(n_k+K+\tfrac12)x\,dx \ge Cm_k\int_{J_k}\frac1x\sin^2(n_k+K+\tfrac12)x\,dx\,.$$

Since $1/x$ is decreasing and $\sin^2(n_k+K+\frac12)x\ge0$, we have

$$\int_{J_k}\frac1x\sin^2(n_k+K+\tfrac12)x\,dx \ge \sum_{j=0}^{2^{k-1}-1}\frac1\pi\cdot 2^{n-2^{k-1}-j-1}\int_{2^{-(n-2^{k-1}-j)}\pi}^{2^{-(n-2^{k-1}-j-1)}\pi}\sin^2(n_k+K+\tfrac12)x\,dx\,.$$

An elementary computation and $2n_k+2K+1\ge 2\cdot 2^{n-2^{k-1}-j}$ yield that the last integral is bounded below by

$$\frac12\cdot 2^{-(n-2^{k-1}-j)}\pi-\frac{1}{2n_k+2K+1}\ge C2^{-(n-2^{k-1}-j)}\,.$$

Consequently,

(4.3) $$\int_{J_k}\frac1x\sin^2(n_k+K+\tfrac12)x\,dx\ge C2^k.$$

The proof of (i) and (ii) can be completed by substituting (4.3) into (4.1) and (4.2).

To prove (iii) we first suppose that

(4.4) $$(m_k+\tfrac12)2^{-(n-2^l)}\le\tfrac12\,.$$

We will use Lemma 1. To this end write

(4.5) $$\int_{J_l} T_{n_k,m_k,K}(x)\sin(n_l+K+\tfrac12)x\,dx = \frac14\int_{J_l}\frac{\sin(m_k+\frac12)x}{\sin^2\frac12 x}(\cos(n_l-n_k)x-\cos(n_l+n_k+2K+1)x)\,dx\,.$$

By (4.4) we have

$$(m_k+\tfrac12)x\le\frac\pi2 \qquad (x\in J_l)\,.$$

Then $\cos jx$ $(1\le j\le m_k)$, and hence also $\sin(m_k+\frac12)x/\sin\frac12 x=2(\frac12+\sum_{j=1}^{m_k}\cos jx)$, are decreasing in $J_l$. On the other hand, $1/\sin\frac12 x$ is obviously decreasing in $J_l$.

Applying Lemma 1 twice to $f(x)=\sin(m_k+\frac12)x/\sin^2\frac12x$ and $t_{\delta_1}(x)=\cos|n_l-n_k|x$, $t_{\delta_2}(x)=\cos(n_l+n_k+2K+1)x$, we obtain

$$\Big|\int_{J_l}T_{n_k,m_k,K}(x)\sin(n_l+K+\tfrac12)x\,dx\Big|$$
$$\le\frac{\sin(m_k+\frac12)2^{-(n-2^{l-1})}\pi}{\sin^2\frac12\cdot2^{-(n-2^{l-1})}\pi}\Big(\frac{1}{|n_l-n_k|}+\frac{1}{n_l+n_k+2K+1}\Big).$$

The first factor can be estimated as follows:

$$\frac{\sin(m_k+\frac12)(2^{-(n-2^{l-1})})\pi}{\sin^2\frac12\cdot2^{-(n-2^{l-1})}\pi}\le C(m_k+\tfrac12)2^{n-2^{l-1}}.$$

On the other hand, either $n_l\ge2n_k$ $(l<k)$, or $n_k\ge2n_l$ $(k<l)$. This implies

$$\frac{1}{|n_l-n_k|}+\frac{1}{n_l+n_k+2K}\le3\frac{1}{n_l}.$$

Consequently,

$$\Big|\int_{J_l}T_{n_k,m_k,K}(x)\sin(n_l+K+\tfrac12)\,dx\Big|$$
$$\le C(m_k+\tfrac12)2^{n-2^{l-1}}\frac{1}{n_l}=C(m_k+\tfrac12)2^{-(2^{l-1}-l)},$$

as was stated.

Let us now consider the case

$$(4.6)\qquad (m_k+\tfrac12)2^{-(n-2^{l-1})}\ge\tfrac12.$$

We note that in this case $l>k$.

On the basis of the trigonometric identity

$$\sin(m_k+\tfrac12)x\cos((n_l+K+\tfrac12)\pm(n_k+K+\tfrac12))x$$
$$=\tfrac12(\sin(m_k+\tfrac12+(n_l+K+\tfrac12)\pm(n_k+K+\tfrac12))x$$
$$+\sin(m_k+\tfrac12-(n_l+K+\tfrac12)\mp(n_k+K+\tfrac12))x)$$

the integral in (4.5) can be split into four terms. Indeed,

$$\int_{J_l}T_{n_k,m_k,K}(x)\sin(n_l+K+\tfrac12)x\,dx$$
$$=\frac14\int_{J_l}\frac{1}{\sin^2\frac12x}(-\sin(n_k-n_l-m_k-\tfrac12)x+\sin(n_k-n_l+m_k+\tfrac12)x$$
$$+\sin(n_k+n_l+2K+m_k+\tfrac32)x-\sin(n_k+n_l+2K-m_k+\tfrac12)x)\,dx.$$



Using Lemma 1 four times for the function $f(x)=1/\sin^2\frac12x$, and for the periodic functions of the above expression we obtain

$$\Big|\int_{J_l}T_{n_k,m_k,K}(x)\sin(n_l+K+\tfrac12)x\,dx\Big|$$
$$\le C(2^{n-2^{l-1}})^2\Big(\frac{1}{n_k-n_l-m_k-\frac12}+\frac{1}{n_k-n_l+m_k+\frac12}$$
$$+\frac{1}{n_k+n_l+2K+m_k+\frac32}+\frac{1}{n_k+n_l+2K-m_k+\frac12}\Big).$$

Since $n_l\le\frac12n_k$ and $m_k+\frac12<2^{n-2^k-1}\le\frac14n_k$, we have

$$n_k-n_l-m_k-\tfrac12>\tfrac18n_k.$$

On the other hand, it follows from (4.6) that

$$(2^{n-2^{l-1}})^2\le(2m_k+1)2^{n-2^{l-1}}.$$

Consequently,

$$\Big|\int_{J_l}T_{n_k,m_k,K}(x)\sin(n_l+K+\tfrac12)x\,dx\Big|$$
$$\le C(2m_k+1)2^{n-2^{l-1}}\frac{1}{n_k}=C(2m_k+1)2^{-(2^{l-1}-l)}.$$

There is only one case left. Consider $0<l^*<\log_2n$ for which

$$(m_k+\tfrac12)2^{-(n-2^{l^*-1})}<\tfrac12<(m_k+\tfrac12)2^{-(n-2^{l^*})}.$$

We note that $l^*>k$. Using the first method for the interval $[2^{-(n-2^{l^*-1})}\pi,\ \pi/(2m_k+1)]$ and the second for $[\pi/(2m_k+1),2^{-(n-2^{l^*})}\pi]$ we obtain

$$\Big|\int_{J_{l^*}}T_{n_k,m_k,K}(x)\sin(n_{l^*}+K+\tfrac12)x\,dx\Big|<C(m_k+\tfrac12)2^{-(2^{l^*-1}-l^*)}.$$

The proof of Lemma 2 is complete. ∎

LEMMA 3. *Let $a_k$ $(2\le k<\log_2n)$ be arbitrary real numbers $(4<n\in\mathbb N)$. Using the notation of Lemma 2 set*

$$T_k=T_{n_k,m_k,K}-iT_{n_k,0,K}\qquad(2\le k<\log_2n,\ i=0\text{ or }1).$$

*Then*

$$\int_0^\pi\Big|\sum_{2\le k<\log_2n}a_kT_k(x)\Big|\,dx\ge\sum_{2\le k<\log_2n}|a_k|(2m_k+1-i)(C_1\cdot2^k-C_2).$$

Proof. Denote by $J_l$ $(2 \le l < \log_2 n)$ the same nonoverlapping subintervals of $[0,1)$ as in Lemma 2. Then we have

$$\begin{aligned}
\int_0^\pi \Big| \sum_{2\le k<\log_2 n} a_k T_k(x)\Big|\,dx
&\ge \sum_{2\le l<\log_2 n} \int_{J_l} \sum_{2\le k<\log_2 n} a_k T_k(x)(\operatorname{sign} a_l)\sin(n_l+K+\tfrac12)x\,dx \\
&\ge \sum_{2\le k<\log_2 n} \Big( \int_{J_k} a_k T_k(x)(\operatorname{sign} a_k)\sin(n_k+K+\tfrac12)x\,dx \\
&\quad - \sum_{\substack{2\le l<\log_2 n\\ l\ne k}} \Big|\int_{J_l} \sum_{2\le l<\log_2 n} a_k T_k(x)(\operatorname{sign} a_l)\sin(n_l+K+\tfrac12)x\,dx\Big|\Big).
\end{aligned}$$

Applying Lemma 2 we obtain

$$\begin{aligned}
\int_0^\pi \Big| \sum_{2\le k<\log_2 n} a_k T_k(x)\Big|\,dx
&\ge C_1 \sum_{2\le k<\log_2 n} \Big(|a_k|(2m_k+1-i)2^k \\
&\quad - \sum_{\substack{2\le l<\log_2 n\\ l\ne k}} C_2|a_k|(2m_k+1-i)2^{-(2^{l-1}-2l)}\Big) \\
&\ge \sum_{2\le k<\log_2 n} |a_k|(2m_k+1-i)(C_1\cdot 2^k - C_2). \ \blacksquare
\end{aligned}$$

LEMMA 4. *Let $K, N \in \mathbb{N}$ $(K<N)$. Then*

$$\begin{aligned}
C_1(N-K+1)\Big(1+\log\frac{N+1}{N-K+1}\Big)
&\le \int_0^\pi \Big|\sum_{k=K}^N D_k(x)\Big|\,dx \\
&\le C_2(N-K+1)\Big(1+\log\frac{N+1}{N-K+1}\Big).
\end{aligned}$$

Proof. Since

$$\sum_{k=K}^N D_k(x) = \frac{\sin\frac12(N+K+1)x\cdot\sin\frac12(N-K+1)x}{2\sin^2\frac12 x}$$



we have

$$\begin{aligned}
\int_0^\pi \Big|\sum_{k=K}^N D_k(x)\Big|\,dx &= \Big(\int_0^{\frac{\pi}{2(N+K+1)}} + \int_{\frac{\pi}{2(N+K+1)}}^{\frac{\pi}{2(N-K+1)}} + \int_{\frac{\pi}{2(N-K+1)}}^{\frac{\pi}{2}}\Big) \\
&\qquad \frac{|\sin(N+K+1)x|\cdot|\sin(N-K+1)x|}{\sin^2 x}\,dx \\
&= A_1 + A_2 + A_3 .
\end{aligned}$$

The estimates $(2/\pi)t \le \sin t \le t$ $(0\le t\le \pi/2)$ and $|\sin t|\le 1$ imply

$$|A_1| \le C(N-K+1), \qquad |A_3| \le C(N-K+1),$$

$$|A_2| \le C(N-K+1)\log\frac{N+K+1}{N-K+1}.$$

Adding these estimates we obtain the desired upper estimate. For the lower estimate we note that

$$\begin{aligned}
\int_0^\pi \Big|\sum_{k=K}^N D_k(x)\Big|\,dx
&\ge |A_2| \ge \frac{2}{\pi}(N-K+1)\int_{\frac{\pi}{2(N+K+1)}}^{\frac{\pi}{2(N-K+1)}} \frac{|\sin(N+K+1)x|}{\sin x}\,dx \\
&\ge C(N-K+1)\log\frac{N+K+1}{N-K+1}
\end{aligned}$$

can be proved by standard arguments.

On the other hand, obviously $\int_0^\pi |\sum_{k=K}^N D_k(x)|\,dx \ge A_1 \ge C(N-K+1)$. Lemma 4 is proved. $\blacksquare$

**5. Proofs.** Before starting the proof of the theorems we note that the following inequality can easily be deduced from the paper of Schipp [10].

Let $e_j(x) = \exp(ijx)$ $(j\in\mathbb{N}, i=\sqrt{-1})$ denote the complex trigonometric system with nonnegative indices. If $\sum_{k=K}^{K+2^n-1} c_k = 0$ then

$$\frac{1}{2^n}\int_0^\pi \Big|\sum_{k=K}^{K+2^n-1}\Big(c_k\sum_{j=0}^k e_j(x)\Big)\Big|\,dx \le \|\Gamma_n\|_H \quad (K,n\in\mathbb{N}). \tag{5.1}$$

Recall that $\Gamma_n = \sum_{k=0}^{2^n-1} c_{k+K}\cdot\chi_{[k2^{-n},(k+1)2^{-n})}$.

Proof of Theorem 2. Let $n$ satisfy $N-K+1=2^n+m$ $(0\le m<2^n)$. For the proof of (i) set

$$d_j=\begin{cases} c_j-2^{-(n+1)}\sum_{l=K}^N c_l & \text{if } K\le j\le N,\\ -2^{-(n+1)}\sum_{l=K}^N c_l & \text{if } N<j<K+2^{n+1}.\end{cases}$$

Then

$$\sum_{j=K}^N c_jD_j=\sum_{j=K}^{K+2^{n+1}-1} d_jD_j+\Big(2^{-(n+1)}\sum_{l=K}^N c_l\Big)\sum_{j=K}^{K+2^{n+1}-1} D_j=B_1+B_2.$$

Applying Lemma 4 we have

$$\int_0^\pi |B_2(x)|\,dx\le C\Big(1+\log\frac{N+1}{N-K+1}\Big)\Big|\sum_{j=K}^N c_j\Big|. \tag{5.2}$$

Since $\sum_{j=K}^{K+2^{n+1}-1} d_j=0$, by (5.1) we have

$$\int_0^\pi |B_1(x)|\,dx\le C2^{n+1}\|\varLambda_1\|_H,$$

where $\varLambda_1=\sum_{l=0}^{2^{n+1}-1} d_{l+K}\cdot\chi_{[l2^{-(n+1)},(l+1)2^{-(n+1)})}$. If

$$\varGamma=\sum_{l=0}^{N-K} c_{l+K}\cdot\chi_{[l2^{-(n+1)},(l+1)2^{-(n+1)})},$$

and $\varLambda_2$ denotes the constant function $2^{-(n+1)}\sum_{j=K}^N c_j$ then $\varLambda_1=\varGamma-\varLambda_2$. Obviously, $\|\varLambda_2\|_H=2^{-(n+1)}|\sum_{j=K}^N c_j|\le\|\varGamma\|_1\le\|\varGamma\|_H$. This implies $\|\varLambda_1\|_H\le 2\|\varGamma\|_H$. Consequently,

$$\int_0^\pi |B_1(x)|\,dx\le C2^{n+1}\|\varGamma\|_H,$$

i.e. by (5.2),

$$\int_0^\pi\Big|\sum_{j=K}^N c_jD_j(x)\Big|\,dx\le C\Big(\Big(1+\log\frac{N+1}{N-K+1}\Big)\Big|\sum_{j=K}^N c_j\Big|+\|\varGamma\|_H\Big). \tag{5.3}$$

On the other hand, it is known (see e.g. [11]) that the dyadic Hardy norm can be estimated as follows:

$$\|f\|_H\le\frac{e}{e-1}\Big(\int_0^1 |f(x)|\log^+|f(x)|\,dx+1\Big)\quad(f\in H[0,1]). \tag{5.4}$$



Applying (5.4) to the step function $\varGamma/\|\varGamma\|_1$ we obtain

$$\|\varGamma\|_H\le C\sum_{j=K}^N |c_j|\Big(1+\log^+\frac{|c_j|}{(N-K+1)^{-1}\sum_{l=K}^N|c_l|}\Big). \tag{5.5}$$

Now (i) follows from (5.3) and (5.5).

In the proof of (ii) we may assume that $c_j$'s are ordered as follows:

$$|c_K|\ge\ldots\ge|c_{K+2^n-1}|\ge\ldots\ge|c_N|. \tag{5.6}$$

Since both sides of (ii) are homogeneous with respect to $c_j$'s, we may also assume without loss of generality that

$$\sum_{j=K}^{K+2^n-1}|c_j|=2^n. \tag{5.7}$$

Set

$$M=\max_{p\in P_{N-K}}\int_0^\pi\Big|\sum_{j=0}^{N-K} c_{j+K}D_{p_j+K}(x)\Big|\,dx.$$

Now we give three lower estimates for $M$. Their appropriate convex linear combination leads to (ii). The first estimate is obtained by applying Lemma 4 to the average over the permutations:

$$M\ge\int_0^\pi\Big|\frac{1}{|P_{N-K}|}\sum_{p\in P_{N-K}}\sum_{j=0}^{N-K} c_{j+K}D_{K+p_j}(x)\Big|\,dx \tag{5.8}$$
$$=\frac{|\sum_{j=K}^N c_j|}{N-K+1}\int_0^\pi\Big|\sum_{j=K}^N D_j(x)\Big|\,dx\ge C_1\Big|\sum_{j=K}^N c_j\Big|\log\frac{N+1}{N-K+1}.$$

(Here, $|A|$ stands for the cardinality of the finite set $A$.)

For the next estimate set

$$\mathcal{P}=\{K\le j\le N: c_j\ge0\},\qquad n_1=|\mathcal{P}|,$$
$$\mathcal{N}=\{K\le j\le N: c_j<0\},\qquad n_2=|\mathcal{N}|.$$

Let $p\in P_{N-K}$ such that $p_j\in\mathcal{N}$ $(0\le j<n_2)$. Then the $(K+n_2)$th Fourier coefficient of $\sum_{j=0}^{N-K} c_{p_j+K}D_{j+K}$ is $\sum_{j\in\mathcal{P}} c_j$. Consequently, $M\ge C\sum_{j\in\mathcal{P}} c_j$. Similarly, $M\ge C\sum_{j\in\mathcal{N}}|c_j|$, and so

$$M\ge C_2\sum_{j=K}^N|c_j|. \tag{5.9}$$

Our final inequality reads

$$M\ge C_3\sum_{j=K}^N|c_j|\log^+\frac{|c_j|}{(N-K+1)^{-1}\sum_{l=K}^N|c_l|}-C_4\sum_{j=K}^N|c_j|. \tag{5.10}$$

In order to show (5.10) the set $\{0,\dots,N-K\}$ will be divided into subsets. First for each $k$ $(3\le k<\log_2 n)$ set

$$A_{k,1}=\{0\le j<2^n : 2^k<\log|c_{j+K}|\le 2^{k+1},\ c_{j+K}>0\}\,,$$
$$A_{k,2}=\{0\le j<2^n : 2^k<\log|c_{j+K}|\le 2^{k+1},\ c_{j+K}<0\}\,.$$

Then define

$$A_k=\begin{cases}A_{k,1} & \text{if } \sum_{j\in A_{k,1}}|c_{j+K}|(1+\log^+2|c_{j+K}|)\\ & \qquad\ge\sum_{j\in A_{k,2}}|c_{j+K}|(1+\log^+2|c_{j+K}|)\,,\\ A_{k,2} & \text{otherwise}.\end{cases}$$

Next, let $A_2$ be a set for which

$$A_2\subset\{0\le j<2^n : |c_{j+K}|\le 2\},\qquad |A_2|=2^{n-6}\,.$$

The existence of such a set follows from (5.7). Finally, let

$$A_1=\{0,\dots,N-K\}\setminus\bigcup_{k=2}^{\log_2 n}A_k\,.$$

The sets $A_k$ $(2\le k<\log_2 n)$ are obviously pairwise disjoint. We note that

$$A_1\cup A_2\supset\{0\le j\le 2^n-1 : |c_{j+K}|\le 2\}\,.$$

Consequently, $|A_1|>2^{n-5}$.

The arithmetic mean of $c_j$'s belonging to $A_k$ $(k\ge 3)$ will be denoted by $a_k$, i.e.

$$a_k=\begin{cases}|A_k|^{-1}\sum_{j\in A_k}c_j & \text{if } A_k\ne\emptyset\,,\\ 0 & \text{if } A_k=\emptyset\qquad(3\le k<\log_2 n)\,.\end{cases}$$

Clearly, $|a_k|>2^{2^k}$ if $A_k\ne\emptyset$. Note that $a_1,a_2$ are not yet defined. By the definition of $A_k$, and by (5.6) and (5.7), we have

$$\begin{aligned}(5.11)\qquad \sum_{j=K}^{N}|c_j|\log^+\frac{|c_j|}{(N-K+1)^{-1}\sum_{l=K}^{N}|c_l|}&\le\sum_{j=K}^{N}|c_j|\log^+2|c_j|\\ &=\sum_{3\le k<\log_2 n}\Big(\Big(\sum_{j\in A_{k,1}}+\sum_{j\in A_{k,2}}\Big)|c_{j+K}|\log^+2|c_{j+K}|\Big)\\ &\quad+\sum_{\{0\le j\le N-K:\log_2|c_{j+K}|\le 2^3\}}|c_{j+K}|\log^+2|c_{j+K}|\\ &\le 2\sum_{3\le k<\log_2 n}\Big(\sum_{j\in A_k}|c_j|\log^+2|c_j|\Big)+9\sum_{j=K}^{N}|c_j|\\ &\le 8\sum_{3\le k<\log_2 n}|A_k||a_k|2^k+9\sum_{j=K}^{N}|c_j|\,.\end{aligned}$$



With each $A_k$ we will associate a set $B_k$ of indices. To this end let $n_k$ denote the same natural number as in Lemma 2, and let $m_k$ be the integer part of $|A_k|/2$ $(2\le k<\log_2 n)$. Set

$$B_k=\begin{cases}\emptyset & \text{if } A_k=\emptyset\,,\\ \{n_k\pm j : 0\le j\le m_k/2\} & \text{if } |A_k| \text{ is odd}\,,\\ \{n_k\pm j : 0<j\le m_k/2\} & \text{if } |A_k|>0 \text{ is even}\qquad(2\le k<\log_2 n)\,.\end{cases}$$

If $m_k>0$ then

$$2m_k2^{2^k}\le|A_k|2^{2^k}<\sum_{j\in A_k}|c_{j+K}|\le 2^n\,.$$

Consequently, $m_k<2^{n-2^k-1}$ $(2\le k<\log_2 n)$. This implies $n_k-m_k>n_{k+1}+m_{k+1}$ $(2\le k\le\log_2 n)$, which means that $B_k$'s are pairwise disjoint. Note that using the notation of Lemma 3 we have

$$T_k=T_{n_k,m_k,K}-iT_{n_k,m_k,0}=\sum_{l\in B_k}D_{l+K}\qquad(2\le k<\log_2 n)\,,$$

where $i=1$ if $|A_k|$ is even, and 0 otherwise.

Set $B_1=\{0,\dots,N-K\}\setminus\bigcup_{k=2}^{\log_2 n}B_k$. Obviously, $|A_k|=|B_k|$ $(1\le k<\log_2 n)$. Let $P^*$ denote the collection of permutations of $\{0,\dots,N-K\}$ which map $A_k$ onto $B_k$ for every $3\le k<\log_2 n$.

Recall that $a_1,a_2$ have not been defined yet. Let now

$$a_1=\frac{1}{|A_1|+|A_2|}\sum_{j\in A_1\cup A_2}c_{j+K}\,,\qquad a_2=\frac{1}{|A_2|}\sum_{j\in A_1\cup A_2}c_{j+K}\,.$$

Then

$$\begin{aligned}\frac{1}{|P^*|}\sum_{p\in P^*}\sum_{j=0}^{N-K}c_{j+K}D_{p_j+K}&=\sum_{3\le k<\log_2 n}a_k\sum_{j\in A_k}D_{p_j+K}+a_1\sum_{j\in A_1\cup A_2}D_{p_j+K}\\ &=\sum_{2\le k<\log_2 n}a_k\sum_{l\in B_k}D_{l+K}-a_2\sum_{j\in A_2}D_{p_j+K}+a_1\sum_{j\in A_1\cup A_2}D_{p_j+K}\\ &=\sum_{2\le k<\log_2 n}a_kT_k+\Big((a_1-a_2)\sum_{l\in B_2}D_{l+K}+a_1\sum_{l\in B_1}D_{l+K}\Big)\,.\end{aligned}$$

The first term satisfies the conditions of Lemma 3, therefore

$$\int_0^\pi\Big|\sum_{2\le k<\log_2 n}a_kT_k(x)\Big|\,dx\ge C_1\sum_{2\le k<\log_2 n}|a_k||A_k|2^k-C_2\sum_{2\le k<\log_2 n}|a_k||A_k|\,.$$

By (5.11) we have

$$(5.12)\quad \int_0^{\pi}\Big|\sum_{2\le k<\log_2 n} a_k T_k(x)\Big|\,dx$$
$$\ge C_1\sum_{j=K}^{N}|c_j|\log^+\frac{|c_j|}{(N-K+1)^{-1}\sum_{l=K}^{N}|c_l|}-C_2\sum_{j=K}^{N}|c_j|.$$

The second term will be bounded above by using Theorem 2(i). Observe that by definition

$$(a_1-a_2)|B_2|+a_1|B_1|=0.$$

On the other hand, as was shown above, $|B_2|=2^{n-6}$ and $|B_1|>2^{n-5}$. Thus

$$(N-K+1)^{-1}(|B_2||a_2-a_1|+|B_1||a_1|)$$
$$\ge 2^{-(n+1)}(|B_2||a_2|+(|B_1|-|B_2|)|a_1|)\ge 2^5(|a_1|+|a_2|).$$

On the basis of these observations, by Theorem 2(i) we have

$$\int_0^{\pi}\Big|(a_1-a_2)\sum_{l\in B_2}D_{l+K}(x)+a_1\sum_{l\in B_1}D_{l+K}(x)\Big|\,dx$$
$$\le C\bigg(|B_2|(|a_1|+|a_2|)$$
$$\times\bigg(1+\log^+\frac{|a_1|+|a_2|}{(N-K+1)^{-1}|B_2||a_1-a_2|+|B_1||a_1|}\bigg)$$
$$+|B_1||a_1|\bigg(1+\log^+\frac{|a_1|}{(N-K+1)^{-1}|B_2||a_1-a_2|+(|B_1|+|B_2|)|a_1|}\bigg)\bigg)$$
$$\le C(|B_2||a_2|+|B_1||a_1|)\le C\sum_{j=K}^{N}|c_j|.$$

The combination of this estimate with (5.12) yields (5.10). The proof of Theorem 2 is completed by taking an appropriate convex linear combination of the inequalities (5.8), (5.9), and (5.10). ∎

Proof of Theorem 3. First we give an outline of the proof of the following relation:

$$(5.13)\quad C_1\|f\|_M\le\int_0^1|f(x)|\bigg(1+\log^+\frac{|f(x)|}{\|f\|_1}\bigg)\le C_2\|f\|_M\qquad(f\in L_M).$$

We may suppose that $\|f\|_1=1$. Recall that $k$ is the real number for which

$$\int_0^1 N(p(k|f(x)|))\,dx=1.$$



Divide the integral into the sets $\{k|f(x)|<1\}$ and $\{k|f(x)|\ge1\}$. Using Cauchy's inequality for the first, and an elementary estimate for the second we deduce that $k>1$. In a similar way, it can be shown that $k<3$.

By definition

$$\|f\|_M=\int_{\{k|f(x)|<1\}}k|f(x)|^2\,dx+\int_{\{k|f(x)|\ge1\}}(1+\log k|f(x)|)|f(x)|\,dx.$$

It is not hard to see that the right side is increasing in $k$. Recalling that $\|f\|_1=1$ and $1<k<3$, we have

$$\|f\|_M\ge\int_{\{|f(x)|<1\}}|f(x)|^2\,dx+\int_{\{|f(x)|\ge1\}}(1+\log|f(x)|)|f(x)|\,dx$$
$$\ge\frac34\int_0^1|f(x)|(1+\log^+|f(x)|)\,dx.$$

Similarly,

$$\|f\|_M\le\int_{\{3|f(x)|<1\}}3|f(x)|^2\,dx+\int_{\{3|f(x)|\ge1\}}(1+\log 3|f(x)|)|f(x)|\,dx$$
$$\le 3\int_0^1|f(x)|(1+\log^+|f(x)|)\,dx.$$

Next we will prove $L_M=H^\sharp$. It is clear from (5.4) and (5.13) that $L_M\subset H^\sharp$, and so we only have to prove $H^\sharp\subset L_M$. Let $f\in H^\sharp$. We may suppose that $f$ is decreasing. Denote by $f_+$ and $f_-$ the positive and negative parts of $f$ on $[0,1)$. Then there exist $K,N\in\mathbb{N}$ such that

$$[0,2^{-N})\subseteq\operatorname{supp}f_+\subset[0,2^{-(N-1)}),$$
$$[1-2^{-K},1)\subseteq\operatorname{supp}f_-\subset[1-2^{-(K-1)},1).$$

The monotonicity of $f_+$ and $f_-$ implies

$$f^*(x)\ge f_+^*(x)\quad(x\in[0,2^{-N})),\qquad f^*(x)\ge f_-^*(x)\quad(x\in[1-2^{-K},1)).$$

Furthermore, $f_+^*$ is decreasing and $f_-^*$ is increasing. Thus

$$\|f^*\|_1\ge2^{-N}\|f_+^*\|_1+2^{-K}\|f_-^*\|_1.$$

This means that $f\in H^\sharp$ implies $f_+,f_-\in H$, i.e. $|f|\in H$. It is known (see e.g. [11]) that a nonnegative function $g$ belongs to $H$ if and only if $\int_0^1 g(x)\log^+g(x)\,dx<\infty$. Consequently, by (5.13), $f\in H^\sharp$ implies $f\in L_M$, which was to be proved.

It follows from (2.3), Theorem 2, (5.4), and (5.13) that

$$C_1\|h\|_M\le\|h\|_{H^\sharp}\le C_2\|h\|_M$$

for any dyadic step function $h$. By (5.4) the right inequality holds for any $f \in H^\sharp$. To prove the left side for any $f \in H^\sharp$ we note that the set of dyadic step functions is dense in $L_M$. This follows from the fact that convergence in norm is equivalent to mean convergence since $M$ satisfies the so called $\Delta_2$-condition (see [7]). Then

$$\begin{aligned}\|f\|_{H^\sharp} &\geq \|h\|_{H^\sharp} - \|f-h\|_{H^\sharp} \geq C_1\|h\|_M - C_2\|f-h\|_M \\ &\geq C_1\|f\|_M - (C_1+C_2)\|f-h\|_M,\end{aligned}$$

where $f \in H^\sharp$, and $h$ is an arbitrary dyadic step function. Theorem 3 is proved. ■

Proof of Theorem 4. It is clear that if $(a_k) \in \mathcal{S}$ then $\sum_{k=0}^{\infty} |\Delta a_k| < \infty$. Then the existence of

$$\begin{aligned}f(x) &= \lim_{n\to\infty}\Big(\frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx\Big) = \lim_{n\to\infty}\Big(\sum_{k=0}^{n-1} \Delta a_k D_k(x) + a_n D_n(x)\Big) \\ &= \sum_{k=0}^{\infty} \Delta a_k D_k(x) \quad (0 < x \leq \pi)\end{aligned}$$

follows from $D_k(x) = O(1/x)$ $(x \neq 0)$. Since Theorem 2 and $(a_k) \in \mathcal{S}$ imply

$$\sum_{j=0}^{\infty} \int_0^{\pi} \Big|\sum_{k=N_j}^{N_{j+1}} \Delta a_k D_k(x)\Big|\, dx < \infty, \tag{5.14}$$

we have $f \in L^1[-\pi, \pi]$. Set

$$\widehat{f}(k) = \frac{2}{\pi} \int_0^{\pi} f(x) \cos kx\, dx \quad (k \in \mathbb{N}).$$

It follows from the definition of $f$ that

$$\begin{aligned}|\widehat{f}(k) - a_k + a_{N_{l+1}}| &= \frac{2}{\pi}\Big|\int_0^{\pi}\Big(f(x) - \sum_{j=0}^{l}\sum_{\iota=N_j}^{N_{j+1}-1} \Delta a_\iota D_\iota(x)\Big) \cos kx\, dx\Big| \\ &\leq \sum_{j=l+1}^{\infty} \int_0^{\pi} \Big|\sum_{\iota=N_j}^{N_{j+1}} \Delta a_\iota D_\iota(x)\Big|\, dx\end{aligned}$$

for any $l \in \mathbb{N}$ with $N_{l+1} \geq k$. If $l \to \infty$ then (5.14) yields $|\widehat{f}(k) - a_k| = 0$. Consequently, $a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx$ is the Fourier series of $f$.

In order to prove (iii) let

$$\Gamma_j = \sum_{k=0}^{N_{j+1}-N_j-1} \Delta a_{N_j+k} \chi_{[k2^{-M_j},(k+1)2^{-M_j})} \quad (j \in \mathbb{N}),$$



where $2^{M_j-1} < N_{j+1} - N_j \leq 2^{M_j}$. Furthermore, let

$$\Gamma_{j,l} = \sum_{k=l}^{N_{j+1}-N_j-1} \Delta a_{N_j+k} \chi_{[k2^{-M_j},(k+1)2^{-M_j})} \quad (0 \leq l \leq N_{j+1} - N_j - 1).$$

Then by (5.3) we have

$$\begin{aligned}&\int_0^{\pi} \Big|\sum_{k=N_j+l}^{N_{j+1}-1} \Delta a_k D_k(x)\Big|\, dx \\ &\quad \leq C\bigg(\Big(1 + \log\frac{N_{j+1}+1}{N_{j+1}-N_j+1}\Big)\Big|\sum_{k=N_j+l}^{N_{j+1}-1} \Delta a_k\Big| + \|\Gamma_{j,l}\|_H\bigg).\end{aligned}$$

It follows from $|\Gamma_{j,l}| \leq |\Gamma_j|$ $(0 \leq l \leq N_{j+1} - N_j - 1)$ and from the definition of the Hardy norm that $\|\,|\Gamma_{j,l}|\,\|_H \leq \|\,|\Gamma_j|\,\|_H$.

Consequently, (5.3) and (5.5) yield

$$\begin{aligned}R_{N_n+l} &= \int_0^{\pi} \Big|f(x) - \Big(\frac{a_0}{2} + \sum_{k=1}^{N_n+l} a_k \cos kx\Big) - a_{N_n+l} D_{N_n+l}(x)\Big|\, dx \\ &= \int_0^{\pi} \Big|\sum_{k=N_n+l}^{\infty} \Delta a_k D_k(x)\Big|\, dx \\ &\leq \sum_{j=n+1}^{\infty} \int_0^{\pi} \Big|\sum_{k=N_j}^{N_{j+1}-1} \Delta a_k D_k(x)\Big|\, dx + \int_0^{\pi} \Big|\sum_{k=N_n+l}^{N_{n+1}-1} \Delta a_k D_k(x)\Big|\, dx \\ &\leq C\bigg(\sum_{j=n}^{\infty}\sum_{k=N_j}^{N_{j+1}-1}\Big(1 + \log\frac{N_{j+1}}{N_{j+1}-N_j} \sum_{k=N_j}^{N_{j+1}-1} |\Delta a_k|\Big) + \sum_{j=n}^{\infty} \|\,|\Gamma_j|\,\|_H\bigg) \\ &\leq C\bigg(\sum_{j=n}^{\infty}\sum_{k=N_j}^{N_{j+1}-1}\Big(1 + \log\frac{N_{j+1}}{N_{j+1}-N_j} \sum_{k=N_j}^{N_{j+1}-1} |\Delta a_k|\Big) \\ &\qquad + \sum_{k=N_j}^{N_{j+1}-1} |\Delta a_k|\Big(1 + \log^+ \frac{|\Delta a_k|}{(N_{j+1}-N_j)^{-1}\sum_{j=N_j}^{N_{j+1}-1} |\Delta a_j|}\Big)\bigg).\end{aligned}$$

$\lim_{n\to\infty} R_n = 0$ follows from $(a_k) \in \mathcal{S}^*$. Then $\int_0^{\pi} |D_k(x)|\, dx = O(\log k)$ implies that $a_0/2 + \sum_{k=1}^{n} a_k \cos kx$ tends to $f$ in mean as $n \to \infty$ if and only if $\lim_{n\to\infty} a_n \log n = 0$. This completes the proof of Theorem 4. ■

Proof of Theorem 5. Let $\widetilde{f}$ be the pointwise limit of the conjugate series, i.e.

$$\widetilde{f}(x)=\sum_{k=1}^{\infty} a_k \sin kx=\sum_{k=1}^{\infty} \Delta a_k \widetilde{D}_k(x) \qquad (-\pi \le x \le \pi)\,,$$

where $\widetilde{D}_k(x)=\sum_{j=1}^{k} \sin jx$.

To prove Theorem 5 it is enough to show that $\widetilde{f} \in L^1[-\pi,\pi]$ if and only if $(a_k)$ satisfies (3.5), and then $\sum_{k=1}^{n} a_k \sin kx$ converges to $\widetilde{f}$ in mean if and only if $a_n \log n=o(1)$ as $n$ tends to $\infty$.

The necessity of (3.5) follows from Hardy's inequality. Concerning the sufficiency we first note that (3.5) and $\sum_{k=1}^{\infty}|\Delta a_k|<\infty$ together imply $\sum_{n=0}^{\infty}|a_{2^n}|<\infty$. Indeed, since $a_{2^n}=a_{2^n+k}+\sum_{j=2^n}^{2^n+k-1} \Delta a_j$ $(0<k \le 2^n)$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}|a_{2^n}| &\le \sum_{n=0}^{\infty}\Big(\frac{1}{2^n}\sum_{k=1}^{2^n}\Big|a_{2^n+k}+\sum_{j=2^n}^{2^{n+1}-1}|\Delta a_j|\Big|\Big)\\ &\le \sum_{k=1}^{\infty}|\Delta a_k|+2\sum_{k=1}^{\infty}\frac{|a_k|}{k}<\infty\,.\end{aligned}$$

Then the statements of Theorem 5 can be deduced from the inequality

$$\begin{aligned}(5.15)\qquad &\int_0^{\pi}\Big|\sum_{k=2^n+j}^{\infty}\Delta a_k \widetilde{D}_k(x)\Big|\,dx\\ &\le C\Big(\sum_{l=n}^{\infty}\sum_{k=2^l}^{2^{l+1}}|\Delta a_k|\Big(1+\log^{+}\frac{|\Delta a_k|}{2^{-l}\sum_{m=2^l}^{2^{l+1}-1}|\Delta a_m|}\Big)\\ &\qquad+\sum_{l=n+1}^{\infty}|a_{2^l}|+n|a_{2^n+j}|\Big)\,.\end{aligned}$$

In order to prove (5.15) set

$$d_k=\begin{cases}-(\Delta a_{2^n+j}-\Delta a_{2^{n+1}})/2^n & \text{if } 2^n \le k<2^n+j\,,\\ \Delta a_k-(\Delta a_{2^n+j}-\Delta a_{2^{n+1}})/2^n & \text{if } 2^n+j \le k<2^{n+1}\,,\\ \Delta a_k-(\Delta a_{2^l}-\Delta a_{2^{l+1}})/2^l & \text{if } 2^l \le k<2^{l+1}\ (l>n)\,.\end{cases}$$

Then

$$\begin{aligned}\sum_{l=2^n+j}^{\infty}\Delta a_l \widetilde{D}_l(x) &=\Big(\sum_{l=n}^{\infty}\sum_{k=2^l}^{2^{l+1}-1} d_k \widetilde{D}_k(x)\Big)+\Big(a_{2^n+j}2^{-n}\sum_{k=2^n}^{2^{n+1}}\widetilde{D}_k(x)\Big)\\ &\quad+\Big(\sum_{l=n+1}^{\infty}a_{2^l}\Big(2^{-l}\sum_{k=2^l}^{2^{l+1}-1}\widetilde{D}_k(x)-2^{l-1}\sum_{k=2^{l-1}}^{2^l-1}\widetilde{D}_k(x)\Big)\Big)\\ &=A_1(x)+A_2(x)+A_3(x)\,.\end{aligned}$$



Using the same considerations as in the proof of Theorem 2(i) and Theorem 4(iii), we see by (5.1) that

$$\int_0^{\pi}|A_1(x)|\,dx \le C\sum_{l=n}^{\infty}\sum_{k=2^l}^{2^{l+1}}|\Delta a_k|\Big(1+\log^{+}\frac{|\Delta a_k|}{2^{-l}\sum_{m=2^l}^{2^{l+1}-1}|\Delta a_m|}\Big)$$

and

$$\int_0^{\pi}|A_3(x)|\,dx \le C\sum_{l=n+1}^{\infty}|a_{2^l}|\,.$$

Obviously,

$$\int_0^{\pi}|A_2(x)|\,dx \le Cn|a_{2^n+j}|\,.$$

(5.15) follows from the above estimates. The proof of Theorem 5 is complete. ∎

## References


[1] R. Bojanic and Č. Stanojević, *A class of $L^1$-convergence*, Trans. Amer. Math. Soc. 269 (1982), 677–683.

[2] M. Buntinas and N. Tanović-Miller, *New integrability and $L^1$-convergence classes for even trigonometric series*, Rad. Mat. 6 (1990), 149–170.

[3] —, —, *New integrability and $L^1$-convergence classes for even trigonometric series II*, in: Approximation Theory, Kecskemét 1990, Colloq. Math. Soc. János Bolyai 58, North-Holland, 1991, 103–125.

[4] G. A. Fomin, *A class of trigonometric series*, Mat. Zametki 23 (1978), 213–222 (in Russian); English transl.: Math. Notes 23 (1978), 117–123.

[5] B. S. Kashin and A. A. Saakyan, *Orthogonal Series*, Nauka, Moscow 1984.

[6] A. N. Kolmogorov, *Sur l'ordre de grandeur des coefficients de la série de Fourier–Lebesgue*, Bull. Internat. Acad. Polon. Sci. Lettres Sér. (A) Sci. Math. 1923, 83–86.

[7] M. A. Krasnosel'skiĭ and Ya. B. Rutickiĭ, *Convex Functions and Orlicz Spaces*, Noordhoff, Groningen 1961.

[8] F. Móricz, *Sidon type inequalities*, Publ. Inst. Math. (Beograd) 48 (62) (1990), 101–109.

[9] —, *On the integrability and $L^1$-convergence of sine series*, Studia Math. 92 (1989), 187–200.

[10] F. Schipp, *Sidon-type inequalities*, in: Approximation Theory, Lecture Notes in Pure and Appl. Math. 138, Dekker, New York 1991, 421–436.

[11] F. Schipp, W. R. Wade and P. Simon (with assistance from J. Pál), *Walsh Series*, Hilger, Bristol 1990.

[12] S. Sidon, *Hinreichende Bedingungen für den Fourier-Charakter einer trigonometrischen Reihe*, J. London Math. Soc. 14 (1939), 158–160.

[13] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, Princeton, N.J., 1971.

[14] N. Tanović-Miller, *On integrability and $L^1$ convergence of cosine series*, Boll. Un. Mat. Ital. (7) 4-B (1990), 499–516.

[15] S. A. Telyakovskiĭ, *Concerning a sufficient condition of Sidon for the integrability of trigonometric series*, Mat. Zametki 14 (1973), 317–328 (in Russian); English transl.: Math. Notes 14 (1973), 742–748.
[16] —, *On the integrability of sine series*, Trudy Mat. Inst. Steklov. 163 (1984), 229–233 (in Russian); English transl.: Proc. Steklov Inst. Mat. 4 (1985), 269–273.
[17] W. H. Young, *On the Fourier series of bounded functions*, Proc. London Math. Soc. (2) 12 (1913), 41–70.

DEPARTMENT OF NUMERICAL ANALYSIS
L. EÖTVÖS UNIVERSITY
BOGDÁNFY U. 10/B
H-1117 BUDAPEST, HUNGARY
E-mail: FRIDLI@LUDENS.ELTE.HU

*Received December 14, 1992* (3037)