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On the ramification set of a positive quadratic form over an algebraic number field

by

MARTIN EPKENHANS (Paderborn)

1. Introduction and notation. Let $\mathcal{A}$ be a finite-dimensional commutative and étale algebra over K, i.e. a finite product of separable and finite field extensions of K. With it we associate the *trace form* which is the following non-degenerate quadratic form over K:

$$\mathcal{A} \to K, \quad x \mapsto \operatorname{tr}_{\mathcal{A}/K}(x^2).$$

It is denoted by $\langle \mathcal{A} \rangle$. By a quadratic form over K we always mean a non-degenerate quadratic form. We know that a quadratic form ψ over an algebraic number field K of dimension $m \geq 4$ is isometric to a trace form of a field extension of K if and only if the signatures of ψ are non-negative for all real orderings of K (see [9]). Following P. E. Conner and R. Perlis [4] we call a Witt class X of the Witt ring $W(K)$ *algebraic* if X contains a trace form of a field extension of K. Let K be an algebraic number field. The *ramification set* $\operatorname{Ram}(X)$ of an algebraic Witt class X consists of those finite spots $\mathfrak{p}$ of K which are ramified in every field extension L/K with $\langle L \rangle \in X$ ([4], p. 166). Let $\mathfrak{p}$ be a finite spot of K and let $\kappa_{\mathfrak{p}}$ be the residue class field of K at $\mathfrak{p}$. Consider the second residue class homomorphism $\partial_{\mathfrak{p}} : W(K) \to W(\kappa_{\mathfrak{p}})$ (see [22], 6.2.5). The investigation of trace forms over local fields gives $\partial_{\mathfrak{p}}\langle L \rangle = 0$ for all finite spots $\mathfrak{p}$ of K which are unramified in L/K. In [5] P. E. Conner and N. Yui conjectured that for an algebraic class $X \in W(\mathbb{Q})$ we get

$$\operatorname{Ram}(X) = \{\mathfrak{p} \mid \mathfrak{p} \text{ is finite and } \partial_{\mathfrak{p}} X \neq 0\}.$$

Our main result implies the validity of this conjecture. Let Ω_K be the set of spots of K.

DEFINITION 1. Let ψ be a quadratic form over the algebraic number field K with non-negative signatures. The *ramification set* $\operatorname{Ram}(\psi)$ of ψ is defined by

$$\operatorname{Ram}(\psi) = \{\mathfrak{p} \in \Omega_K \mid \mathfrak{p} \text{ is finite and } \mathfrak{p} \text{ is ramified in every extension } L/K \text{ with } \psi \simeq_K \langle L \rangle\}.$$

Here $\simeq_K$ denotes the isometry of quadratic forms over K. We call ψ a *positive* form if all signatures of ψ are non-negative. In this paper we determine the ramification set of a positive form. In particular, we prove the following. Let ψ be a quadratic form with non-negative signatures and let $\mathcal{T} \subset \Omega_K$ be a finite set of finite spots with $\mathcal{T} \cap \mathrm{Ram}(\psi) = \emptyset$. Then there is a field extension L/K with $\psi \simeq_K \langle L \rangle$ and all $\mathfrak{p} \in \mathcal{T}$ are unramified in L/K.

The proof of this result is organized as follows. We start with forms of dimension $n = 4$. Next suppose $n = 2^l \geq 8$. Then we can choose a quadratic field extension F/K such that all $\mathfrak{p} \in \mathcal{T}$ are quadratically unramified in F/K and such that there is a positive form φ over F with $\psi \simeq_K \mathrm{tr}_{F/K}(\varphi)$ and $\{\mathfrak{P} \in \Omega_F \mid \mathfrak{P} \cap \mathfrak{o}_K = \mathfrak{p} \in \mathcal{T}\} \cap \mathrm{Ram}(\varphi) = \emptyset$. Hence by induction we get the result for forms of dimension 2^l. As usual, we write $\mathfrak{P} \mid \mathfrak{p}$ to indicate that $\mathfrak{P} \in \Omega_F$ is a spot lying above $\mathfrak{p} \in \Omega_K$, and $\mathrm{tr}_{F/K}(\varphi)$ is the "Scharlau transfer" of the form φ (see [22], p. 47). We treat forms of arbitrary even dimension in a similar way. Next we consider forms of odd dimension. We use Mestre's deformation process. We can choose trace forms ψ_i of dimension 1, 2 or 4 with $\mathrm{Ram}(\psi_i) \cap \mathcal{T} = \emptyset$ and $\psi \simeq_K \perp \psi_i$. Hence ψ is isometric to the trace form of some étale algebra $\mathcal{A} = K_1 \times \ldots \times K_\nu$ and all $\mathfrak{p} \in \mathcal{T}$ are unramified in every field extension K_i/K. Then we prove that there is a deformation of the algebra $\mathcal{A}$ leaving the trace form intact and preserving the decomposition structure of all spots $\mathfrak{p} \in \mathcal{T}$.

We call ψ a *normal* (*abelian*, *cyclic*) *trace form* if there is a normal (abelian, cyclic) field extension L/K with $\psi \simeq_K \langle L \rangle$. In [7] we determined all normal (abelian, cyclic) trace forms of an algebraic number field. In this paper we investigate the *Galois ramification set* $\mathrm{GRam}(\psi)$ of a normal trace form ψ, i.e. the set of all finite spots which are ramified in every Galois extension L/K with $\psi \simeq_K \langle L \rangle$. In general, $\mathrm{Ram}(\psi)$ and $\mathrm{GRam}(\psi)$ coincide if ψ is a normal trace form.

We begin by fixing our notations. Let K be an algebraic number field. Then $\mathfrak{o}_K$ is the ring of integers of K. Let $\mathfrak{p} \in \Omega_K$ be a spot. Then $K_\mathfrak{p}$ is a completion of K at $\mathfrak{p}$. If $\mathfrak{p}$ is a finite spot, then $v_\mathfrak{p} : K \to \mathbb{Z}$ denotes the normalized valuation of K defined by $\mathfrak{p}$. $\Delta_\mathfrak{p} \in \mathfrak{o}_K$ is an element which is a non-square unit at $\mathfrak{p}$ such that $K_\mathfrak{p}(\sqrt{\Delta_\mathfrak{p}})/K_\mathfrak{p}$ is unramified. Let L/K be a finite field extension and let $\mathfrak{p} \in \Omega_K$, $\mathfrak{P} \in \Omega_L$ be spots with $\mathfrak{P} \mid \mathfrak{p}$. The inertia degree of $\mathfrak{P} \mid \mathfrak{p}$ is denoted $f(\mathfrak{P}/\mathfrak{p})$. If L/K is a Galois extension, then we also write $f_\mathfrak{p}(L/K)$ and we set $n_\mathfrak{p}(L/K) = [L_\mathfrak{P} : K_\mathfrak{p}]$. If L/K is any finite field extension, then $\Lambda_{L/K} = N_{L/K}(L^*) \cdot K^{*2}$.

$\langle a_1, \ldots, a_n \rangle$ denotes the diagonal form $a_1 t_1^2 + \ldots + a_n t_n^2$. Let ψ be a quadratic form over K. Then $\dim_K \psi$ is the dimension of ψ, $\det_K \psi \in K^*$ is its determinant. Let $a, b \in K^*$. Then $(a, b)_K$ denotes the generalized quaternion algebra generated over K by i, j and satisfying $i^2 = a$, $j^2 = b$,

$ij = -ji$. The class of $(a,b)_K$ in the Brauer group $\mathrm{Br}(K)$ of K is also denoted by $(a,b)_K$. Let $\psi \simeq_K \langle a_1,\ldots,a_n\rangle$ be a diagonalization of ψ. The Hasse invariant $H_K\psi$ of ψ is defined by

$$H_K\psi = \bigotimes_{i<j}(a_i,a_j)_K \in \mathrm{Br}(K)\,.$$

$I^n(K)$ is the nth power of the fundamental ideal of $W(K)$. The Scharlau transfer of the one-dimensional form $\langle\lambda\rangle$, $\lambda \in L^*$, is called the *scaled trace form*, and denoted by $\langle L\rangle_\lambda$.

Again, let K be an algebraic number field and let $\mathfrak{p} \in \Omega_K$ be a finite spot. $H_\mathfrak{p}\psi \in \{-1,1\}$ is the local Hasse invariant of ψ and $(a,b)_\mathfrak{p}$ denotes the local Hilbert symbol. Let $\mathfrak{p} \in \Omega_K$ be a real spot. Then $\mathrm{sign}_\mathfrak{p}\,\psi$ denotes the signature of ψ with respect to the ordering induced by $\mathfrak{p}$.

2. The main results. We get the following well-known result from local trace form considerations (see [4], I.5, II.5 or [11]). This gives the necessary condition for $\mathfrak{p} \notin \mathrm{Ram}(\psi)$. For the convenience of the reader we sketch a proof.

PROPOSITION 1. *Let L/K be a finite extension of algebraic number fields.*

(1) *Let $\mathfrak{p} \in \Omega_K$ be a finite spot. If $\mathfrak{p}$ is unramified in L/K, then $\mathfrak{p}$ is unramified in $K(\sqrt{\det_K\langle L\rangle})/K$ and $H_\mathfrak{p}\langle L\rangle = (2,\det_K\langle L\rangle)_\mathfrak{p}$.*

(2) *Let $\mathfrak{p} \in \Omega_K$ be a real spot. Then $[L_\mathfrak{P} : K_\mathfrak{p}] = 1$ for all spots $\mathfrak{P} \in \Omega_L$ lying above $\mathfrak{p}$ if and only if $\mathrm{sign}_\mathfrak{p}\langle L\rangle = [L:K]$.*

Proof. (1) Let $\mathfrak{p} \in \Omega_K$. We know

$$\langle L \otimes_K K_\mathfrak{p}\rangle \simeq_{K_\mathfrak{p}} \perp_{\mathfrak{P}|\mathfrak{p}}\langle L_\mathfrak{P}\rangle$$

(see [4], I.5.1). If $\mathfrak{p}$ is unramified in L/K, then the local extension $L_\mathfrak{P}/K_\mathfrak{p}$ is unramified for any $\mathfrak{P} \in \Omega_L$ lying above $\mathfrak{p}$. The trace form of an unramified local extension is first determined in [11]. Let $L_\mathfrak{P}/K_\mathfrak{p}$ be an unramified local extension of degree f. Then $\langle L_\mathfrak{P}\rangle \simeq_{K_\mathfrak{p}} f\cdot\langle 1\rangle$ if f is odd.

Let f be even. Then $K_\mathfrak{p}(\sqrt{\det_{K_\mathfrak{p}}\langle L_\mathfrak{P}\rangle})/K_\mathfrak{p}$ is the unique unramified extension of degree 2 and we get $H_\mathfrak{p}\langle L_\mathfrak{P}\rangle = (2,\det_{K_\mathfrak{p}}\langle L_\mathfrak{P}\rangle)_\mathfrak{p}$ (see also [8], Theorem 1).

(2) Let $\mathfrak{p}$ be a real spot of K. By a classical result of Sylvester we know that the signature $\mathrm{sign}_\mathfrak{p}\langle L\rangle$ equals the number of spots $\mathfrak{P} \in \Omega_L$ lying above $\mathfrak{p}$ and such that the local degree is 1 ([22], 3.2.6 or [23]). ∎

We now state the main results of this paper.

THEOREM 1. *Let K be an algebraic number field. Let ψ be a positive quadratic form over K of dimension ≥ 4 or let $\psi \simeq_K \langle 2,2D\rangle$, $D \notin K^{*2}$ or $\psi \simeq_K \langle 1,2,D\rangle$, $D \in K^*$. Let $\mathcal{T} \subset \Omega_K$ be a finite set of finite spots with $\mathfrak{p}$ unramified in $K(\sqrt{\det_K\psi})/K$ and $H_\mathfrak{p}\psi = (2,\det_K\psi)_\mathfrak{p}$. Then there* *is a field extension* L/K *with* $\psi \simeq_K \langle L\rangle$ *and such that all spots* $\mathfrak{p} \in \mathcal{T}$ *are unramified in* L/K. *If* $n = \dim_K \psi$ *is even, we can choose* L/K *such that* $f(\mathfrak{P}/\mathfrak{p}) \in \{n, n/2\}$ *for all* $\mathfrak{P} \cap \mathfrak{o}_K = \mathfrak{p} \in \mathcal{T}$. *In particular*,

$$\operatorname{Ram}(\psi) = \{\mathfrak{p} \in \Omega_K \mid \mathfrak{p} \text{ ramifies in } K(\sqrt{\det_K \psi})/K \text{ or } H_{\mathfrak{p}}\psi \neq (2, \det_K \psi)_{\mathfrak{p}}\}.$$

Corollary 1. *The conjecture of Conner and Yui holds true.*

Proof. If $\mathfrak{p}$ is a non-dyadic spot, then $\mathfrak{p}$ is unramified in $K(\sqrt{\det_K\langle L\rangle})/K$ and $H_{\mathfrak{p}}\langle L\rangle = (2, \det_K\langle L\rangle)_{\mathfrak{p}}$ is equivalent to $\partial_{\mathfrak{p}}\langle L\rangle = 0$. Hence by Theorem 1 it remains to consider the prime 2. Let $X \in W(\mathbb{Q})$ be a Witt class with $\partial_2 X = 0$, i.e. $\operatorname{ord}_2 \operatorname{dis} X$ is even. Choose a quadratic form $\psi \in X$ such that $\det_{\mathbb{Q}} \psi \equiv 1, 5 \bmod 8$. Then $H_2\psi \neq H_2(\psi \perp 2 \cdot \langle 1, -1\rangle)$ and $\psi \perp 2 \cdot \langle 1, -1\rangle \in X$. Hence we can choose $\psi \in X$ such that 2 is unramified in $\mathbb{Q}(\sqrt{\det_{\mathbb{Q}} \psi})$ and $H_2\psi = (2, \det_{\mathbb{Q}} \psi)_2$. By Theorem 1 there is a field extension $L/\mathbb{Q}$ such that 2 is unramified in $L/\mathbb{Q}$ and $\langle L\rangle \simeq_{\mathbb{Q}} \psi \in X$. ∎

In the next theorem we consider the Galois ramification set of a normal trace form. Let μ_d be the group of dth roots of unity.

Theorem 2. *Let* K *be an algebraic number field and let* ψ *be a quadratic form of dimension* $n = 2^l m$, m *odd, over* K. *Let* $D \in K^*$ *with* $\det_K \psi \equiv D \bmod K^{*2}$.

(1) *Let* n *be odd. Then* ψ *is a normal trace form iff* ψ *is a cyclic trace form iff* $\psi \simeq_K n \cdot \langle 1\rangle$. *Then* $\operatorname{GRam}(\psi) = \operatorname{Ram}(\psi) = \emptyset$.

(2) *Let* $n = 2m \equiv 2 \bmod 4$. *Then* ψ *is a normal trace form iff* ψ *is a cyclic trace form iff* $\psi \simeq_K m \cdot \langle 2, 2D\rangle$ *and* $D \notin K^{*2}$. *Then* $\operatorname{GRam}(\psi) = \operatorname{Ram}(\psi)$.

(3) *Let* $n = 2^l m \equiv 0 \bmod 4$ *and* $D \in K^{*2}$. *Then* ψ *is a normal trace form iff* ψ *is an abelian trace form iff* $\operatorname{sign}_{\mathfrak{p}} \psi \in \{0, n\}$ *for all real spots* $\mathfrak{p} \in \Omega_K$. *But* ψ *is not a cyclic trace form. Then* $\operatorname{GRam}(\psi) = \operatorname{Ram}(\psi) = \{\mathfrak{p} \in \Omega_K \mid H_{\mathfrak{p}}\psi = -1\}$.

(4) *Let* $n = 4m \equiv 4 \bmod 8$ *and* $D \notin K^{*2}$. *Then* ψ *is a normal trace form iff* ψ *is a cyclic trace form iff* $D = a^2 + b^2$ *with* $a, b \in K$ *and* $\psi \simeq_K m \cdot \langle 1, D, c, c\rangle$ *for some* $c \in K^*$. *Then* $\operatorname{GRam}(\psi) = \operatorname{Ram}(\psi)$.

(5) *Let* $n = 2^l m \equiv 0 \bmod 8$ *and* $D \notin K^{*2}$. *Then* ψ *is a normal trace form iff* ψ *is a cyclic trace form iff* $H_K\psi = (2, D)_K$, $\operatorname{sign}_{\mathfrak{p}} \psi \in \{0, n\}$ *for all real spots* $\mathfrak{p} \in \Omega_K$ *and* $K(\sqrt{D})/K$ *is contained in a cyclic extension of degree* 2^l. *Then* $H_{\mathfrak{p}}\psi = 1$ *for all non-dyadic spots and for all infinite spots* $\mathfrak{p} \in \Omega_K$. *Set* $\mathcal{T}_l := \{\mathfrak{p} \in \Omega_K, \mathfrak{p} \mid 2$ *and* $\mathfrak{p}$ *is completely non-split in* $K(\mu_{2^l})/K\}$. *Then either* $\operatorname{GRam}(\psi) = \operatorname{Ram}(\psi)$, *or* $K(\mu_{2^l})/K$ *is not cyclic*, $\mathcal{T}_l = \{\mathfrak{p}_0\}$ *and* $\mathfrak{p}_0$ *is unramified in* $K(\sqrt{D})/K$. *Then* $\operatorname{Ram}(\psi) \subset \operatorname{GRam}(\psi) \subset \operatorname{Ram}(\psi) \cup \{\mathfrak{p}_0\}$.

(6) *Let* ψ *be a normal* (*cyclic*) *trace form and let* $\mathcal{T} \subset \Omega_K$ *be a finite set of finite spots with* $\mathcal{T} \cap \operatorname{GRam}(\psi) = \emptyset$. *If* $\det_K \psi \notin K^{*2}$ *and* $n \equiv 0 \bmod 8$

suppose $K(\mu_{2^l})/K$ *is cyclic or* $\mathcal{T}_l = \emptyset$ *or* $\mathcal{T}_l \not\subset \mathcal{T}$. *Then there is an abelian* (*cyclic*) *field extension* L/K *with* $\psi \simeq_K \langle L\rangle$ *and all* $\mathfrak{p} \in \mathcal{T}$ *are unramified in* L/K.

3. Positive forms of even dimension. We start with forms of dimension 4.

LEMMA 1. *Let* K *be a field of* $\operatorname{char}(K) \neq 2$ *and let* $f(X) = X^4 - 2aX^2 + b \in K[X]$ *be an irreducible and separable polynomial. Set* $L = K[X]/(f(X))$. *Then*

$$\langle L\rangle \simeq_K \langle 1, a^2 - b, ab, a(a^2 - b)\rangle .$$

Hence $\det_K\langle L\rangle \equiv b \bmod K^{*2}$ *and*

$$H_K\langle L\rangle = (a, -b(a^2 - b))_K \otimes ((a^2 - b), -1)_K .$$

For a proof see [4], Theorem I.10.1.

LEMMA 2. *Theorem* 1 *holds for quadratic forms of dimension* 4.

Proof. Let $\psi \simeq_K \langle 1, u, v, uvD\rangle$ and suppose $\mathcal{T} \neq \emptyset$. The equation

$$ux_1^2 + vx_2^2 + uvDx_3^2 = D(Dx_4^2 - 1)$$

has a solution in K since ψ is a positive form of dimension 4. The set

$$\mathcal{S} = \{\mathfrak{p} \in \Omega_K \mid \mathfrak{p} \text{ is real or } \mathfrak{p} \notin \mathcal{T} \text{ with } H_{\mathfrak{p}}\psi \neq (-D, -1)_{\mathfrak{p}}\}$$

is finite and disjoint from $\mathcal{T}$. Let $\tau \in K$ be an element with

(1) $(D\tau^2 - 1)(Dx_4^2 - 1) \in K_{\mathfrak{p}}^{*2}$ for all $\mathfrak{p} \in \mathcal{S}$ and

(2) $D(D\tau^2 - 1)\Delta \in K_{\mathfrak{p}}^{*2}$ for all $\mathfrak{p} \in \mathcal{T}$, where $\Delta \in K^*$ is a non-square unit at all $\mathfrak{p} \in \mathcal{T}$ such that $K_{\mathfrak{p}}(\sqrt{\Delta_{\mathfrak{p}}})/K_{\mathfrak{p}}$ is unramified.

Set $g(X) = X^2 + 2D\tau^2X + D\tau^2$. The discriminant of $g(X)$ satisfies $\operatorname{dis}(g(X)) = 4D\tau^2(D\tau^2 - 1) \equiv \Delta \bmod K_{\mathfrak{p}}^{*2}$ for all $\mathfrak{p} \in \mathcal{T}$. Since $\mathcal{T} \neq \emptyset$, the polynomial $g(X)$ is irreducible. Let $g(\beta) = 0$ and set $F = K(\beta)$. By the Hasse–Minkowski Local Global Principle $\langle u, v, uvD\rangle$ represents $D(D\tau^2 - 1)$. We can choose $w \in K^*$ such that $-w\beta \notin F^{*2}$ and

$$\psi \simeq_K \langle 1, D(D\tau^2 - 1), w, w(D\tau^2 - 1)\rangle .$$

Hence

$$h(X) = g(-X^2w^{-1})w^2 = X^4 - 2Dw\tau^2X^2 + Dw^2\tau^2 \in K[X]$$

is irreducible. Set $M = K[X]/(h(X))$. From Lemma 1 we know $\psi \simeq_K \langle M\rangle$. The extension $K_{\mathfrak{p}}(\sqrt{D(D\tau^2 - 1)})/K_{\mathfrak{p}}$ is quadratically unramified for all $\mathfrak{p} \in \mathcal{T}$ since $D(D\tau^2 - 1)\Delta \in K_{\mathfrak{p}}^{*2}$.

Now let $\mathfrak{p} \in \mathcal{T}$ be a non-dyadic spot which ramifies in M/K, hence $\mathfrak{p} = \mathfrak{P}^2$ with $f(\mathfrak{P}/\mathfrak{p}) = 2$. From [6], Satz 5.5(3), we know $1 = H_{\mathfrak{p}}\langle M\rangle = -(\pi, -D)_{\mathfrak{p}}$, where $\pi \in K$ is a prime at $\mathfrak{p}$. Hence $-D\Delta \in K_{\mathfrak{p}}^{*2}$ for these spots, which gives $-(D\tau^2-1) \in K_{\mathfrak{p}}^{*2}$. Therefore the form $\langle 1, (D\tau^2-1)\rangle \simeq_{K_{\mathfrak{p}}} \langle 1,-1\rangle$ is isotropic over $K_{\mathfrak{p}}$.

Next let $\mathfrak{p} \in \mathcal{T}$ be a dyadic spot and let $\mathfrak{P} \in \Omega_F$, $\widetilde{\mathfrak{P}} \in \Omega_M$ be spots with $\widetilde{\mathfrak{P}} \mid \mathfrak{P}$ and $\mathfrak{P} \mid \mathfrak{p}$. Suppose that $D \in K_{\mathfrak{p}}^{*2}$ and $-w\beta \notin F_{\mathfrak{P}}^{*2}$. Then $M_{\widetilde{\mathfrak{P}}} = K_{\mathfrak{p}}(\sqrt{\Delta}, \sqrt{z})$ for some $z \in K_{\mathfrak{p}}^*$ with $-w\beta z \in F_{\mathfrak{P}}^{*2}$. We further get $1 = H_{\mathfrak{p}}\psi = H_{\mathfrak{p}}\langle M\rangle = (-\Delta, z)_{\mathfrak{p}}$. Thus $\langle 1, D\tau^2-1\rangle \simeq_{K_{\mathfrak{p}}} \langle 1, \Delta\rangle$ represents $z \in K_{\mathfrak{p}}^*$ over $K_{\mathfrak{p}}$.

Let $\mathfrak{p} \in \mathcal{T}$ be a dyadic spot with $D\Delta \in K_{\mathfrak{p}}^{*2}$. Then $N_{F/K}(-w\beta) \equiv N_{F/K}(\beta) = g(0) \equiv \Delta \bmod K_{\mathfrak{p}}^{*2}$. Thus $[M_{\widetilde{\mathfrak{P}}} : K_{\mathfrak{p}}] = 4$ and $M_{\widetilde{\mathfrak{P}}}/K_{\mathfrak{p}}$ is a cyclic extension (see [6], Satz 2.2(3)(b)). Every square class in the kernel of the map $N_{F_{\mathfrak{P}}/K_{\mathfrak{p}}} : F_{\mathfrak{P}}^* \to K_{\mathfrak{p}}^*$ contains some $z \in K_{\mathfrak{p}}^*$. Hence there is some $z \in K_{\mathfrak{p}}^*$ with $-w\beta z \equiv \Delta_{\mathfrak{P}} \bmod F_{\mathfrak{P}}^{*2}$. We get

$$\begin{aligned}(z,-1)_{\mathfrak{p}} &= H_{\mathfrak{p}}\langle F_{\mathfrak{P}}\rangle_{-w\beta z} \cdot H_{\mathfrak{p}}\langle F_{\mathfrak{P}}\rangle_{-w\beta} = H_{\mathfrak{p}}\langle F_{\mathfrak{P}}\rangle_{2\Delta_{\mathfrak{P}}} \cdot H_{\mathfrak{p}}\langle F_{\mathfrak{P}}\rangle_{-2w\beta} \\ &= H_{\mathfrak{p}}(\operatorname{tr}_{F_{\mathfrak{P}}/K_{\mathfrak{p}}}(\langle 2, 2\Delta_{\mathfrak{P}}\rangle)) \cdot H_{\mathfrak{p}}\langle M_{\widetilde{\mathfrak{P}}}\rangle = 1\,,\end{aligned}$$

since $M = F(\sqrt{-w\beta})$ and $\operatorname{tr}_{F_{\mathfrak{P}}/K_{\mathfrak{p}}}(\langle 2, 2\Delta_{\mathfrak{P}}\rangle)$ is the trace form of the unique unramified extension of $K_{\mathfrak{p}}$ having degree 4. Thus $\langle 1, D\tau^2-1\rangle \simeq_{K_{\mathfrak{p}}} \langle 1, 1\rangle$ represents $z \in K_{\mathfrak{p}}^*$ over $K_{\mathfrak{p}}$.

Hence $\langle 1, D\tau^2-1\rangle$ represents some $z \in K^*$ with

(1) $v_{\mathfrak{p}}(z)$ is odd, if $\mathfrak{p} \in \mathcal{T}$ is a non-dyadic spot which ramifies in M/K,

(2) $z \in K_{\mathfrak{p}}^{*2}$ if $\mathfrak{p} \in \mathcal{T}$ is unramified in M/K,

(3) $-w\beta z \in F_{\mathfrak{P}}^{*2}$, if $\mathfrak{p}$ is a dyadic spot with $D \in K_{\mathfrak{p}}^{*2}$, $-w\beta \notin F_{\mathfrak{P}}^{*2}$ and $\mathfrak{p}$ ramifies in M/K and

(4) $F_{\mathfrak{P}}(\sqrt{-w\beta z}\,)/K_{\mathfrak{p}}$ is unramified of degree 4 if $D\Delta \in K_{\mathfrak{p}}^{*2}$ and $\mathfrak{p} \in \mathcal{T}$ is a dyadic spot which ramifies in M/K.

Hence $\langle w, w(D\tau^2-1)\rangle \simeq_K \langle zw, zw(D\tau^2-1)\rangle$. Now

$$f(X) = X^4 - 2Dwz\tau^2X^2 + Dw^2z^2\tau^2 \in K[X]$$

defines the desired field extension of degree 4. ∎

Let F/K be a finite field extension of algebraic number fields with $[F : K] = m$. M. Krüskemper [16] investigated the transfer of quadratic forms. He gave sufficient conditions for a positive quadratic form ψ of dimension nm, $n \geq 3$, to be the transfer of a positive form φ over F. The proof of the next result follows the lines of [16], Lemma 7, and [15], Lemma 1, where a similar result is proven without taking care of the ramification of primes. We construct the field extensions with the help of Grunwald's Theorem. This simplifies some of Krüskemper's original proofs.

PROPOSITION 2. *Let ψ be a positive quadratic form of dimension mn, $n \equiv 0 \bmod 4$, over the algebraic number field K. Let $\mathcal{T} \subset \Omega_K$ be a finite set* *of finite spots which are unramified in* $K(\sqrt{\det_K \psi})/K$ *and such that* $H_{\mathfrak{p}}\psi = (2, \det_K \psi)_{\mathfrak{p}}$. *Let* F/K *be a Galois extension of degree* m *with* $\det_K \psi \in \Lambda_{F/K}$, $\mathrm{sign}_{\mathfrak{p}}\langle F\rangle = m$ *for all real spots* $\mathfrak{p} \in \Omega_K$ *and* $f_{\mathfrak{p}}(F/K) = m$ *for all* $\mathfrak{p} \in \mathcal{T}$. *Suppose, further, that* $\mathcal{T}$ *contains no dyadic spot if* m *is even and* $m \neq 2$. *Then there is a positive quadratic form* φ *over* F *with*

(1) $\psi \simeq_K \mathrm{tr}_{F/K}(\varphi)$ *and*

(2) *all* $\mathfrak{P} \in \Omega_F$ *with* $\mathfrak{P} \cap \mathfrak{o}_K = \mathfrak{p} \in \mathcal{T}$ *are unramified in* $F(\sqrt{\det_F \varphi})/F$ *and we get* $H_{\mathfrak{P}}\varphi = (2, \det_F \varphi)_{\mathfrak{P}}$ *for these spots.*

Proof. Let $\mathcal{T}_F$ be the set of spots $\mathfrak{P}$ of F for which $\mathfrak{P} \cap \mathfrak{o}_K = \mathfrak{p} \in \mathcal{T}$. First let φ be an arbitrary quadratic form of even dimension over F such that all $\mathfrak{P} \in \mathcal{T}_F$ are unramified in $F(\sqrt{\det_F \varphi})/F$. A manipulation with Hasse invariants and Hilbert symbols implies $H_{\mathfrak{p}}\psi = H_{\mathfrak{P}}\varphi$ for all $\mathfrak{p} \in \mathcal{T}$, $\mathfrak{P} \in \mathcal{T}_F$, $\mathfrak{P} \cap \mathfrak{o}_K = \mathfrak{p}$ (use [6], Satz 0.6 and Satz 3.9). Thus we only have to prove that there is a positive form φ over F with $\psi \simeq_K \mathrm{tr}_{F/K}(\varphi)$ and all $\mathfrak{P} \in \mathcal{T}_F$ are unramified in $F(\sqrt{\det_F \varphi})/F$.

First let ψ be a torsion form. Let m be even. Then $\det_K \psi$ is totally positive, hence a sum of squares. Since F/K is a Galois extension with $\det_K \psi \in \Lambda_{F/K}$, there is a totally positive element $\lambda' \in F$ with $N_{F/K}(\lambda') \equiv \det_K \psi \bmod K^{*2}$ (apply [16], Proposition 7(b)). We can choose some totally positive $z \in K$ such that $z\lambda'\Delta_{\mathfrak{P}} \in F_{\mathfrak{P}}^{*2}$ or $z\lambda' \in F_{\mathfrak{P}}^{*2}$ for all $\mathfrak{P} \in \mathcal{T}_F$. Note that $\mathcal{T}$ contains no dyadic spots if $m \neq 2$. Set $\lambda = z \cdot \lambda'$. If m is odd, set $\lambda = \det_K \psi$. Hence $N_{F/K}(\lambda) \equiv \det_K \psi \bmod K^{*2}$, λ is totally positive and all $\mathfrak{P} \in \mathcal{T}_F$ are unramified in $F(\sqrt{\lambda})/F$. Now $\psi - \mathrm{tr}_{F/K}(\langle 1, -\lambda\rangle)$ is a torsion form in $I^2(K)$. By a result of Leep and Wadsworth (see [18], Theorem 1.11, resp. [14], Theorem 1.2) there is a torsion form $\varrho \in I^2(F)$ with

$$\mathrm{tr}_{F/K}(\varrho) = \psi - \mathrm{tr}_{F/K}(\langle 1, -\lambda\rangle)\,.$$

Of course, the torsion form $\varrho \perp \langle 1, -\lambda\rangle$ is a positive form with

$$\det_F(\varrho \perp \langle 1, -\lambda\rangle) \equiv \lambda \equiv 1,\ \Delta_{\mathfrak{P}} \bmod F_{\mathfrak{P}}^{*2}$$

for all $\mathfrak{P} \in \mathcal{T}_F$.

Finally, let ψ be an arbitrary form for which the condition of the proposition holds. We can choose a form ϱ over F such that

(1) $\dim_F \varrho \equiv 0 \bmod 4$,

(2) $0 \leq \mathrm{sign}_{\mathfrak{P}}\, \varrho \leq n$ for all real spots $\mathfrak{P} \in \Omega_F$,

(3) $\sum_{\mathfrak{P}|\mathfrak{p}} \mathrm{sign}_{\mathfrak{P}}\, \varrho = \mathrm{sign}_{\mathfrak{p}}\, \psi$ for all real spots $\mathfrak{p} \in \Omega_K$,

(4) all $\mathfrak{P} \in \mathcal{T}_F$ are unramified in $F(\sqrt{\det_F \varrho})/F$ and $H_{\mathfrak{P}}\varrho = (2, \det_F \varrho)_{\mathfrak{P}}$, where $\det_F \varrho \in F_{\mathfrak{P}}^{*2}$ iff $\det_K \psi \in K_{\mathfrak{p}}^{*2}$, $\mathfrak{P} \mid \mathfrak{p}$.

Thus by [22], 3.4.5, $\mathrm{tr}_{F/K}(\varrho) - \psi$ is a torsion form with

$$\det_K(\mathrm{tr}_{F/K}(\varrho) - \psi) \equiv N_{F/K}(\det_F \varrho) \cdot \det_K \psi \equiv 1 \bmod K_{\mathfrak{p}}^{*2}$$

and

$$H_{\mathfrak{p}}(\mathrm{tr}_{F/K}(\varrho)-\psi) = H_{\mathfrak{p}}(\mathrm{tr}_{F/K}(\varrho))\cdot H_{\mathfrak{p}}(-\psi)\cdot(N_{F/K}(\det_F\varrho),\det_K\psi)_{\mathfrak{p}}$$
$$= H_{\mathfrak{P}}\varrho\cdot H_{\mathfrak{p}}\psi = (2,\det_F\varrho)_{\mathfrak{P}}\cdot(2,\det_K\psi)_{\mathfrak{p}} = 1$$

for all $\mathfrak{p}\in\mathcal{T}$. We get $\det_K(\mathrm{tr}_{F/K}(\varrho)-\psi)\in\Lambda_{F/K}$. By the above, there is a torsion form τ over F with $\mathrm{tr}_{F/K}(\tau)=\mathrm{tr}_{F/K}(\varrho)-\psi$ and all $\mathfrak{P}\in\mathcal{T}_F$ are unramified in $F(\sqrt{\det_F\tau})/F$ and $H_{\mathfrak{P}}\tau=(2,\det_F\tau)_{\mathfrak{P}}$ for these spots. The Witt class of $\varrho-\tau$ can be represented by a form φ of dimension n (see [22], 6.6.6). Hence $\psi\simeq_K \mathrm{tr}_{F/K}(\varphi)$. It follows that $\det_F\varphi\equiv\det_F\varrho\cdot\det_F\tau \bmod F^{*2}$ and $H_{\mathfrak{P}}\varphi = H_{\mathfrak{P}}(\varrho-\tau) = H_{\mathfrak{P}}\varrho\cdot H_{\mathfrak{P}}(-\tau)\cdot(\det_F\varrho,\det_F(-\tau))_{\mathfrak{P}} = (2,\det_F\varphi)_{\mathfrak{P}}$ for all $\mathfrak{P}\in\mathcal{T}_F$. ∎

Lemma 3. *Theorem* 1 *holds for positive forms of dimension* $n=2^l m$, m *odd*, $l\ge 2$.

Proof. First let $m=1$. We proceed by induction on l. If $l=2$, use Lemma 2. Let $l\ge 3$. We can choose some totally positive $P\in\mathfrak{o}_K$ such that

(1) $K_{\mathfrak{p}}(\sqrt{P})/K_{\mathfrak{p}}$ is quadratic unramified for all $\mathfrak{p}\in\mathcal{T}$ and

(2) $(\det_K\psi,P)_K=0$, hence $\det_K\psi\in\Lambda_{F/K}$, where $F=K(\sqrt{P})$.

Now apply Proposition 2. Next let m be an arbitrary odd number. By the Theorem of Grunwald–Hasse–Wang [21], Korollar 6.9, we can choose some cyclic field extension F/K of degree m with $f_{\mathfrak{p}}(F/K)=m$ for all $\mathfrak{p}\in\mathcal{T}$. Now apply Proposition 2 again. ∎

We have to consider forms of dimension $n\equiv 2 \bmod 4$ separately since forms of dimension 2 are somewhat exceptional. A binary quadratic form ψ is a trace form iff $\psi\simeq_K\langle 2,2D\rangle$ with $D\notin K^{*2}$. Based on a result of E. Bender [1], M. Krüskemper gave a local global principle for scaled trace forms of odd dimension over an algebraic number field (see [16], Theorem 1). We give a stronger version of this result.

Proposition 3. *Let F/K be an extension of algebraic number fields of odd degree m. Let ψ be a quadratic form of dimension m over K with $|\mathrm{sign}_{\mathfrak{p}}\psi|\le\mathrm{sign}_{\mathfrak{p}}\langle F\rangle$ for all real spots $\mathfrak{p}\in\Omega_K$ and $H_{\mathfrak{p}}\psi=(\det_K\psi,-1)_{\mathfrak{p}}^{(m+1)/2}$ for all non-dyadic spots $\mathfrak{p}\in\Omega_K$ for which there is only one spot of F lying above $\mathfrak{p}$. Let $\mathcal{S}$ be a finite set of finite spots containing all dyadic spots and all non-dyadic spots which ramify in F/K or in $K(\sqrt{\det_K\psi})/K$ or for which $H_{\mathfrak{p}}\psi\ne(\det_K\psi,-1)_{\mathfrak{p}}^{(m+1)/2}$.*

(1) *Then for every $\mathfrak{p}\in\mathcal{S}$ there is some $\lambda_{\mathfrak{p}}\in F^*$ with $\langle F\rangle_{\lambda_{\mathfrak{p}}}\simeq_{K_{\mathfrak{p}}}\psi$.*

(2) *Suppose that for every $\mathfrak{p}\in\mathcal{S}$ there is some $\lambda_{\mathfrak{p}}\in K^*$ with $\langle F\rangle_{\lambda_{\mathfrak{p}}}\simeq_{K_{\mathfrak{p}}}\psi$. Then there is some $\lambda\in F^*$ with $\psi\simeq_K\langle F\rangle_\lambda$ and $\lambda\cdot\lambda_{\mathfrak{p}}\in F_{\mathfrak{P}}^{*2}$ for all $\mathfrak{P}\in\Omega_F$ with $\mathfrak{P}\cap\mathfrak{o}_K=\mathfrak{p}\in\mathcal{S}$.*

Proof. (1) Let $F_{\mathfrak{P}}/K_{\mathfrak{p}}$ be an extension of non-dyadic fields with $[F_{\mathfrak{P}} : K_{\mathfrak{p}}] = m$. Then $H_{\mathfrak{p}}\langle F_{\mathfrak{P}}\rangle_\lambda = (\det_{K_{\mathfrak{p}}}\langle F_{\mathfrak{P}}\rangle_\lambda, -1)_{\mathfrak{p}}^{(m+1)/2}$ (see [6], Satz 4.2). If $\mathfrak{p}$ is a finite spot which splits over F, then use [16], Lemma 6. If $\mathfrak{p}$ does not split over F, apply [2], Lemma 3, for dyadic spots, and note that $\langle F\rangle_\lambda$, $\lambda = \det_K\langle F\rangle \cdot \det_K \psi$ and ψ have the same determinant.

(2) See [16], Proofs of Proposition 1 and Theorem 1. ∎

LEMMA 4. *Theorem* 1 *holds for quadratic forms* ψ *of dimension* $n = 2m \equiv 2 \bmod 4$.

Proof. By the above we can assume $n \neq 2$. Assume further $\mathcal{T} \neq \emptyset$. Choose some non-dyadic spot $\mathfrak{p}_0 \in \Omega_K$ with $\partial_{\mathfrak{p}_0}\psi = 0$, $\mathfrak{p}_0 \notin \mathcal{T}$ and $-1 \in K_{\mathfrak{p}_0}^{*2}$. Let $\mathcal{S} \subset \Omega_K$ be the set of non-dyadic spots with $\partial_{\mathfrak{p}}\psi \neq 0$. Then $\mathcal{S}$ is a finite set with $\mathcal{S} \cap \mathcal{T} = \emptyset$. Because of the Theorem of Grunwald–Hasse–Wang [21], Korollar 6.9, there is a cyclic field extension F/K of degree m such that

(1) $f_{\mathfrak{p}}(F/K) = m$ for all $\mathfrak{p} \in \mathcal{T}$,

(2) $n_{\mathfrak{p}}(F/K) \neq m$ for all $\mathfrak{p} \in \mathcal{S}$,

(3) $n_{\mathfrak{p}_0}(F/K) = 1$.

Let $\mathcal{T}_F$ be the set of spots $\mathfrak{P}$ of F with $\mathfrak{P} \cap \mathfrak{o}_K = \mathfrak{p} \in \mathcal{T}$. Then $\operatorname{sign}_{\mathfrak{p}}(\langle 2\rangle \otimes \psi - \langle F\rangle) = \operatorname{sign}_{\mathfrak{p}} \psi - m$ and $\operatorname{sign}_{\mathfrak{p}} \psi \geq 0$ gives $|\operatorname{sign}_{\mathfrak{p}}(\langle 2\rangle \otimes \psi - \langle F\rangle)| \leq m$ for all real $\mathfrak{p} \in \Omega_K$. Therefore there is a form φ of dimension m over K which is Witt-equivalent to $\langle 2\rangle \otimes \psi - \langle F\rangle$ (see [22], 6.6.6). Thus $\psi \simeq_K \langle 2\rangle \otimes \varphi \perp \langle F\rangle_2$. Let $\mathfrak{p}$ be a non-dyadic spot with $(\det_K\varphi, -1)_{\mathfrak{p}}^{(m+1)/2} \neq H_{\mathfrak{p}}\varphi$. Then $v_{\mathfrak{p}}(\det_K \varphi) \equiv 1 \bmod 2$ or $H_{\mathfrak{p}}\varphi = -1$. We know $H_{\mathfrak{p}}\varphi = H_{\mathfrak{p}}(\langle 2\rangle \otimes \psi) = (2, \det_K\psi)_{\mathfrak{p}} \cdot H_{\mathfrak{p}}\psi$. Hence $\mathfrak{p} \in \mathcal{S}$. But $n_{\mathfrak{p}}(F/K) \neq m$ for these spots. Therefore we can choose some $\lambda_{\mathfrak{p}} \in F^*$ with $\langle F\rangle_{\lambda_{\mathfrak{p}}} \simeq_{K_{\mathfrak{p}}} \varphi$ for these spots.

Let $\mathfrak{p} \in \mathcal{T}$. Set $\lambda_{\mathfrak{p}} = 1$ if $\det_K \psi \in K_{\mathfrak{p}}^{*2}$ and $\lambda_{\mathfrak{p}} = \Delta_{\mathfrak{P}}$ if $\det_K \psi \notin K_{\mathfrak{p}}^{*2}$, where $\mathfrak{P} \in \mathcal{T}_F$ is the unique spot lying above $\mathfrak{p} \in \mathcal{T}$. Then $\langle F\rangle_{\lambda_{\mathfrak{p}}} \simeq_{K_{\mathfrak{p}}} \varphi$ (use [6], Satz 3.9(3)).

Fix some $\mathfrak{P}_0 \in \Omega_F$ lying above $\mathfrak{p}_0$. If $\psi \simeq_K n \cdot \langle 1\rangle$, let $a \in F^*$ be an element with $v_{\mathfrak{P}_0}(a) \equiv 1 \bmod 2$ and $a \in F_{\mathfrak{P}}^{*2}$ for all $\mathfrak{P} \neq \mathfrak{P}_0$, $\mathfrak{P} \mid \mathfrak{p}_0$. Set $\lambda_{\mathfrak{p}_0} = a \cdot \sigma(a)$ with $\langle\sigma\rangle = G(F/K)$. Then $\lambda_{\mathfrak{p}_0} \notin F^{*2}$ and $\langle F\rangle_{\lambda_{\mathfrak{p}_0}} \simeq_{K_{\mathfrak{p}_0}} m \cdot \langle 1\rangle$ (see proof of [15], Proposition 2).

By Proposition 3 there is some $\lambda \in F^*$ with $\varphi \simeq_K \langle F\rangle_\lambda$ and $\lambda \in F_{\mathfrak{P}}^{*2}$ if $\det_K \psi \in K_{\mathfrak{p}}^{*2}$, resp. $\lambda \cdot \Delta_{\mathfrak{P}} \in F_{\mathfrak{P}}^{*2}$ if $\det_K \psi \notin K_{\mathfrak{p}}^{*2}$ for all $\mathfrak{p} \in \mathcal{T}$ and $\lambda \cdot \lambda_{\mathfrak{p}_0} \in F_{\mathfrak{P}_0}^{*2}$ if $\psi \simeq_K n \cdot \langle 1\rangle$. Thus $\psi \simeq_K \operatorname{tr}_{F/K}(\langle 2, 2\lambda\rangle)$.

Now $\lambda \in F^{*2}$ gives $\psi \simeq_K \operatorname{tr}_{F/K}(\langle 2, 2\rangle) \simeq_K n \cdot \langle 1\rangle$, which contradicts $\lambda \equiv \lambda_{\mathfrak{p}_0} \not\equiv 1 \bmod F_{\mathfrak{P}_0}^{*2}$. Set $L = F(\sqrt{\lambda})$. Then all $\mathfrak{P} \in \mathcal{T}_F$ are unramified in L/F. ∎

4. Positive forms of odd dimension. Now we treat positive forms of odd dimension. We modify our original proof of [9], resp. [10], which is based on a deformation process of Mestre [20]. We first recall this result.

PROPOSITION 4. *Let K be an algebraic number field. Let $f_1(X),\dots,f_s(X) \in \mathfrak{o}_K[X]$ be monic polynomials such that $f(X) = f_1(X)\dots f_s(X)$ has odd degree $m \ge 3$. Then there are monic polynomials $p_1(X),\dots,p_s(X) \in \mathfrak{o}_K[X]$ and a polynomial $q(X) \in \mathfrak{o}_K[X]$ such that*

(1) *$K[X]/(f_i(X)) \simeq K[X]/(p_i(X))$ for $i = 1,\dots,s$.*
(2) *$\deg(q(X)) < \deg(f(X))$.*
(3) *$p(X) = p_1(X)\dots p_s(X)$ and $q(X)$ are relatively prime. Hence*

$$F(T,X) = p(X) - Tq(X) \in \mathfrak{o}_K[T,X]$$

is irreducible.

(4) *For every $\tau \in K$ the trace forms of $K[X]/(f(X))$ and $K[X]/(F(\tau,X))$ over K are isometric.*

Proof. See [20], Proposition (1) and (2), and [9], Theorem 2. ∎

We need this result in the following version.

PROPOSITION 5. *Let K be an algebraic number field. Let $f(X) \in K[X]$ be a monic separable polynomial of odd degree $m \ge 3$. Let $\mathcal{T} \subset \Omega_K$ be a finite set of finite spots. There is a polynomial $F(T,X) \in \mathfrak{o}_K[T,X]$ and there are infinitely many elements $\tau \in K$ such that $F(\tau,X) \in \mathfrak{o}_K[X]$ is a monic irreducible polynomial with the following properties*:

(1) *$\langle K[X]/(f(X))\rangle \simeq_K \langle K[X]/(F(\tau,X))\rangle$.*

(2) *Let $\mathfrak{p} \in \mathcal{T}$ and let $f(X) = f_1(X)\dots f_r(X)$ be the decomposition of $f(X)$ into monic prime factors in $K_{\mathfrak{p}}[X]$. Then $F(\tau,X)$ factors in $K_{\mathfrak{p}}[X]$ as $F(\tau,X) = F_1(X)\dots F_r(X)$ and*

$$K_{\mathfrak{p}}[X]/(f_i(X)) \simeq K_{\mathfrak{p}}[X]/(F_i(X)) \quad \text{for } i = 1,\dots,r\,;$$

i.e. $f(X)$ and $F(\tau,X)$ have the same ramification structure for all $\mathfrak{p} \in \mathcal{T}$.

Proof. Let $p(X), q(X) \in \mathfrak{o}_K[X]$ be as in Proposition 4. Obviously, the ramification structures of $f(X)$ and of $p(X)$ coincide for all spots $\mathfrak{p} \in \Omega_K$. Let $\pi \in \mathfrak{o}_K$ be an element with $v_{\mathfrak{p}}(\pi) > 0$ for all $\mathfrak{p} \in \mathcal{T}$. We can choose some $s \in \mathbb{N}$ such that $F(\pi^s T, X)$ has the following property.

For every $\tau \in \mathfrak{o}_K$ the polynomials $F(\pi^s\tau, X)$ and $p(X)$ (hence $F(\pi^s\tau, X)$ and $f(X)$) have the same ramification structure for all $\mathfrak{p} \in \mathcal{T}$. Use [3], IV § 3 Satz 1 and Bemerkung and [17], Proposition 4, or apply [19], Exercise 24.22. Then use Hilbert's Irreducibility Theorem. ∎

LEMMA 5. *Let K be an algebraic number field. Let ψ be a positive quadratic form of dimension $m \ge 5$ over K. Let $\mathcal{T} \subset \Omega_K$ be a finite set* *of finite spots of* K. *There are elements* $a_1, \ldots, a_s \in \mathfrak{o}_K$, $s = [(m-5)/2]$, *and there is a positive quadratic form* φ *of dimension* 4 *over* K *such that*

(1) $\psi \simeq_K \varphi \perp \langle 2, 2a_1 \rangle \perp \ldots \perp \langle 2, 2a_s \rangle$ *if* m *is even and*

(2) $\psi \simeq_K \varphi \perp \langle 1 \rangle \perp \langle 2, 2a_1 \rangle \perp \ldots \perp \langle 2, 2a_s \rangle$ *if* m *is odd*,

and for all $\mathfrak{p} \in \mathcal{T}$ *we get* $a_i \in K_{\mathfrak{p}}^{*2}$, $\det_K \psi \cdot \det_K \varphi \in K_{\mathfrak{p}}^{*2}$ *and* $H_{\mathfrak{p}}\psi = H_{\mathfrak{p}}\varphi$.

Proof. If m is odd, then $\psi \simeq_K \widetilde{\psi} \perp \langle 1 \rangle$ with some positive form $\widetilde{\psi}$. Hence assume m is even. By the Approximation Theorem we can choose some $a \in K^*$ such that

(1) $a \in K_{\mathfrak{p}}^{*2}$ if $\mathfrak{p} \in \mathcal{T}$.

(2) Let $\mathfrak{p} \in \Omega_K$ be a real spot. Then a is negative at $\mathfrak{p}$ iff $\operatorname{sign}_{\mathfrak{p}} \psi \neq \dim_K \psi$.

The Hasse–Minkowski Local Global Principle gives $\psi \simeq_K \psi_1 \perp \langle a \rangle$ with $\operatorname{sign}_{\mathfrak{p}} \psi_1 = \operatorname{sign}_{\mathfrak{p}} \psi - \operatorname{sign}_{\mathfrak{p}} a \geq 0$, since $\dim_K \psi \geq 4$. By induction we get $\psi \simeq_K \widetilde{\psi} \perp \langle a_1, \ldots, a_s \rangle$, where $a_1, \ldots, a_s \in K^*$ have the properties (1) and (2). Let $\mathfrak{p} \in \Omega_K$ be a real spot. Then $a_i \in -\mathfrak{p}$ implies $a_j \in -\mathfrak{p}$ for $m \geq j \geq i$. We further get $\operatorname{sign}_{\mathfrak{p}}(\widetilde{\psi} \perp s \cdot \langle -2 \rangle) \in \{0, 2, 4\}$. Hence $\widetilde{\psi} \simeq_K \varphi \perp s \cdot \langle 2 \rangle$ with $\dim_K \varphi = 4$ and φ is a positive form. ∎

Proof of Theorem 1. The trace forms of dimension ≤ 3 are $\langle 1 \rangle$, $\langle 2, 2D \rangle$ with $D \notin K^{*2}$ and $\langle 1, 2, D \rangle$ with $D \in K^*$ (see [4], III 3.6). By Lemmas 3 and 4 it remains to consider positive forms of odd dimension. Then use Lemmas 5 and 2 and Proposition 5.

5. Proof of Theorem 2. In [7], Theorem 1, we classified all normal, abelian and cyclic trace forms of an algebraic number field. Hence in view of Proposition 1 we only have to prove (6).

(1) By the Very Weak Existence Theorem of Grunwald [12] there is a cyclic field extension L/K of degree n with $n_{\mathfrak{p}}(L/K) = 1$ for all $\mathfrak{p} \in \mathcal{T}$. Hence all $\mathfrak{p} \in \mathcal{T}$ are unramified in L/K.

Since the compositum of unramified field extensions is an unramified field extension we can assume $n = 2^l \geq 2$. Hence the proof of (2) is obvious.

(3) Because of the proof of Lemma 2 we can assume $n = 2^l \geq 8$.

(a) $\psi \in I^3(K)$. By the Theorem of Grunwald–Hasse–Wang [21], Korollar 6.9, there is a Galois extension L/K with $G(L/K) \simeq (\mathbb{Z}_2)^l$ and such that every $\mathfrak{p} \in \mathcal{T}$ is unramified in L/K and, for a real spot, $n_{\mathfrak{p}}(L/K) = 2$ iff $\operatorname{sign}_{\mathfrak{p}} \psi = 0$.

(b) $\psi \notin I^3(K)$. We use the Very Weak Existence Theorem of Grunwald [12]. There is a cyclic field extension F/K of degree 2^{l-1} with

(1) $n_{\mathfrak{p}}(F/K) = 2^{l-1}$ if $H_{\mathfrak{p}}\psi = -1$,

(2) $n_{\mathfrak{p}}(F/K) = 1$ if $\mathfrak{p}$ is a real spot or $\mathfrak{p} \in \mathcal{T}$.

Hence $\det_K\langle F\rangle \notin K_{\mathfrak p}^{*2}$ if $H_{\mathfrak p}\psi = -1$. Thus there is some $a \in K^*$ with $H_K\psi = (\det_K\langle F\rangle, a)_K$ and a is negative at $\mathfrak p$ iff $\operatorname{sign}_{\mathfrak p}\psi = 0$ (see proof of Proposition 3 in [7]). Choose some totally positive $b \in K^*$ with $(b, \det_K\langle F\rangle)_K = 0$ and $ab \in K_{\mathfrak p}^{*2}$ for all $\mathfrak p \in \mathcal T$. Set $L = F(\sqrt{ab})$.

(4) Since $H_K\psi = (c, -1)_K$, we can define local extensions $L(\mathfrak p)/K_{\mathfrak p}$ as follows (see [6], Satz 3.14, 3.16):

(1) $L(\mathfrak p) = \mathbb C$ if $\operatorname{sign}_{\mathfrak p}\psi = 0$ and $\mathfrak p$ is a real spot; otherwise let $L(\mathfrak p) = \mathbb R$,
(2) $L(\mathfrak p)/K_{\mathfrak p}$ is unramified of degree 4 if $D\Delta \in K_{\mathfrak p}^{*2}$ and $\mathfrak p \in \mathcal T$,
(3) $L(\mathfrak p) = K_{\mathfrak p}$ if $D \in K_{\mathfrak p}^{*2}$ and $\mathfrak p \in \mathcal T$,
(4) $\psi \simeq_{K_{\mathfrak p}} \langle L(\mathfrak p)\rangle$ where $L(\mathfrak p)/K_{\mathfrak p}$ is cyclic of degree 4 if $D \notin K_{\mathfrak p}^{*2}$ and either $\mathfrak p \in \operatorname{Ram}(\psi)$ or $\mathfrak p$ is dyadic,
(5) $L(\mathfrak p) = K_{\mathfrak p}(\sqrt c)$ if $D \in K_{\mathfrak p}^{*2}$ and either $\mathfrak p \in \operatorname{Ram}(\psi)$ or $\mathfrak p$ is dyadic.

These are finitely many local conditions. By [13], Korollar zu Satz 8, the quadratic extension $K(\sqrt D)/K$ is contained in a cyclic field extension M/K of degree 4 which has the given completion at the above spots, i.e. $M_{\mathfrak P} \simeq L(\mathfrak p)$, $\mathfrak P \mid \mathfrak p$. Now choose a totally positive element $t \in K^*$ as follows:

(1) $t \in K_{\mathfrak p}^{*2}$, if $\mathfrak p \in \mathcal T \cup \operatorname{Ram}(\psi)$ or $\mathfrak p$ is dyadic,
(2) $v_{\mathfrak p}(t) \equiv 1 \bmod 2$, if $H_{\mathfrak p}\psi \ne H_{\mathfrak p}\langle M\rangle$,
(3) $v_{\mathfrak p}(t) \equiv 0 \bmod 2$ for all other spots except maybe one non-dyadic spot $\mathfrak p_0 \notin \mathcal T \cup \operatorname{Ram}(\psi)$ with $H_{\mathfrak p}\psi = H_{\mathfrak p}\langle M\rangle$.

Set $F = K(\sqrt D)$, $M = F(\sqrt{x + y\sqrt D})$. Then $L = F(\sqrt{tx + ty\sqrt D})$ defines the desired field extension.

(5) Use [13], Korollar zu Satz 8. ∎

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FB MATHEMATIK
UNIVERSITÄT-GESAMTHOCHSCHULE
D-33095 PADERBORN, GERMANY
E-mail: MARTINE@UNI-PADERBORN.DE

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