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Upper bounds for class numbers of real quadratic fields

by

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1. Introduction. Let $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Q}$ denote the sets of integers, positive integers and rational numbers respectively. Let $D \in \mathbb{N}$ be square free, and let Δ, h, ε denote the discriminant, the class number and the fundamental unit of the real quadratic field $K = \mathbb{Q}(\sqrt{D})$ respectively. Then

$$\Delta = \begin{cases} D & \text{if } D \equiv 1 \pmod 4, \\ 4D & \text{if } D \not\equiv 1 \pmod 4. \end{cases}$$

For the case that D is an odd prime, Gut [3] proved that if $D \equiv 1 \pmod 4$, then $h < D/4$. Newman [6] proved that $h < 2\sqrt{D}/3$. Agoh [1] proved that if $\nu > 1/2$ and $D \equiv 1 \pmod 4$, then $h < \nu\sqrt{D}$ except for a finite number of D. In this paper, we prove a general result as follows.

THEOREM. (a) *For any square free $D \in \mathbb{N}$, we have $h \le [\sqrt{\Delta}/2]$.*

(b) *Moreover, if $D \equiv 3 \pmod 4$ is an odd prime, then*

$$h \le \begin{cases} [\sqrt{D}/3] + 1 & \text{if } D = 36k^2 + 36k + 7,\ k \in \mathbb{Z},\ k \ge 0, \\ [\sqrt{D}/4] + 1 & \text{otherwise}, \end{cases}$$

where $[x]$ is the greatest integer less than or equal to x.

2. Preliminaries. Here and below, let χ be the non-trivial Dirichlet character of K, and let $L(s,\chi)$ denote the L-function attached to χ. Then χ is an even quadratic character of conductor Δ. The two lemmas below follow immediately from [5, Theorem] and [8, p. 531] respectively.

LEMMA 1. *Let γ be Euler's constant. We have*

$$|L(1,\chi)| \le \begin{cases} \frac{1}{4}(\log \Delta + 2 + \gamma - \log \pi) & \text{if } 2 \mid \Delta, \\ \frac{1}{2}(\log \Delta + 2 + \gamma - \log 4\pi) & \text{otherwise}. \end{cases}$$

LEMMA 2. *If $D > 1500$ and $D \equiv 5 \pmod 8$, then*

$$|L(1,\chi)| < \tfrac{1}{6}(\log D + 5.16).$$

By much the same argument as in the proof of [4, Theorem A], we can prove the following lemma.

LEMMA 3. *If* $\chi(2) = 0$ *and* $\chi(3) = -1$, *then*

$$|L(1,\chi)| \le \tfrac{1}{8}(\log \varDelta + 3\log 6 + 8).$$

LEMMA 4. *For any square free* $D \in \mathbb{N}$, *we have*

$$\varepsilon^2 > \begin{cases} D-3 & \text{if } D = a^2 \pm 4,\ a \in \mathbb{N},\\ 4D-3 & \text{otherwise.}\end{cases} \tag{1}$$

Moreover, *if* D *is a prime with* $D \equiv 3 \pmod 4$, *then*

$$\varepsilon > \begin{cases} 2D-3 & \text{if } D = a^2 \pm 2,\ a \in \mathbb{N},\\ 18D-3 & \text{otherwise.}\end{cases} \tag{2}$$

Proof. Since ε is equal to the fundamental solution $(u_1 + v_1\sqrt{D})/2$ of the equation

$$u^2 - Dv^2 = \pm 4, \qquad u, v \in \mathbb{Z},$$

we have

$$\begin{aligned}\varepsilon^2 &= \tfrac{1}{4}(u_1 + v_1\sqrt{D})^2 \ge \tfrac{1}{4}(\sqrt{Dv_1^2 - 4} + v_1\sqrt{D})^2\\ &> Dv_1^2 - 3 \ge \begin{cases} D-3 & \text{if } v_1 = 1,\\ 4D-3 & \text{if } v_1 > 1,\end{cases}\end{aligned}$$

and (1) follows.

By [7], if D is a prime with $D \equiv 3 \pmod 4$, then the equation

$$U^2 - DV^2 = \pm 2, \qquad U, V \in \mathbb{N}, \tag{3}$$

has solutions (U,V) and $\varepsilon = (U_1 + V_1\sqrt{D})^2/2$, where (U_1, V_1) is the least solution of (3). So we have

$$\varepsilon \ge \tfrac{1}{2}(\sqrt{DV_1^2 - 2} + V_1\sqrt{D})^2 > 2DV_1^2 - 3. \tag{4}$$

Since $2 \nmid V_1$, we see from (4) that

$$\varepsilon > \begin{cases} 2D-3 & \text{if } V_1 = 1,\\ 18D-3 & \text{if } V_1 > 1,\end{cases}$$

and (2) follows. The lemma is proved.

3. Proof of Theorem. By the numerical results of [2], it suffices to prove the Theorem for $\varDelta > 24572$. By the class number formula, we have

$$h = \frac{\sqrt{\varDelta}}{2\log\varepsilon}|L(1,\chi)|. \tag{5}$$

First, we consider the case $D \equiv 1 \pmod 4$. Then $\varDelta = D$ and $D > 24572$. If $D = a^2 \pm 4$ with $a \in \mathbb{N}$, then $D \equiv 5 \pmod 8$. On applying Lemmas 2 and

4 with (5), we get

$$h < \frac{\sqrt{D}}{2}\left(\frac{\log D + 5.16}{3\log(D-3)}\right) < \frac{\sqrt{D}}{2}, \qquad D \geq 18. \tag{6}$$

On the other hand, by Lemmas 1 and 4, if $D \neq a^2 \pm 4$, then

$$h < \frac{\sqrt{D}}{2}\left(\frac{\log D + 0.046}{\log(4D-3)}\right) < \frac{\sqrt{D}}{2}, \qquad D \geq 2. \tag{7}$$

Since $h \in \mathbb{N}$, we see from (6) and (7) that $h \leq [\sqrt{D}/2]$ for $D \equiv 1 \pmod 4$.

Second, we consider the case $D \not\equiv 1 \pmod 4$. Then $\Delta = 4D$, $D > 6143$ and $\chi(2) = 0$. By Lemmas 1 and 4, we get

$$h < \sqrt{D}\left(\frac{\log 4D + 1.433}{2\log(4D-3)}\right) < \sqrt{D}, \qquad D \geq 3.$$

It implies that $h \leq [\sqrt{D}]$ for $D \not\equiv 1 \pmod 4$. Up to now, we obtain $h \leq [\sqrt{\Delta}/2]$.

Finally, we consider the case where $D \equiv 3 \pmod 4$ is an odd prime. If $D = a^2 - 2$, $a \in \mathbb{N}$ and $3 \mid a$, then $D = 36k^2 + 36k + 7$, where $k \in \mathbb{Z}$ with $k \geq 0$. On applying Lemmas 1 and 4 with (5), we get

$$h < \frac{\sqrt{D}}{4}\left(\frac{\log 4D + 1.433}{\log(2D-3)}\right) < \frac{\sqrt{D}}{3}, \qquad D \geq 250.$$

If $D = a^2 + 2$ or $a^2 - 2$ and $3 \nmid a$, then $D \equiv 2 \pmod 3$ and $\chi(3) = -1$. By Lemmas 3 and 4, we get

$$h < \frac{\sqrt{D}}{4}\left(\frac{\log 4D + 13.38}{2\log(2D-3)}\right) < \frac{\sqrt{D}}{4}, \qquad D \geq 7 \cdot 10^5.$$

Furthermore, using the methods of [9], we can check that $h < \sqrt{D}/4$ for $79 < D < 10^6$. Similarly, by Lemmas 1 and 4, if $D \equiv 3 \pmod 4$ and $D \neq a^2 \pm 2$, then

$$h < \frac{\sqrt{D}}{4}\left(\frac{\log 4D + 1.433}{\log(18D-3)}\right) < \frac{\sqrt{D}}{4}, \qquad D \geq 3.$$

All cases are considered and the Theorem is proved.

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