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Extremal selections of multifunctions generating a continuous flow

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Abstract. Let $F : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a continuous multifunction with compact, not necessarily convex values. In this paper, we prove that, if F satisfies the following Lipschitz Selection Property:

(LSP) *For every t, x, every $y \in \overline{\text{co}}F(t,x)$ and $\varepsilon > 0$, there exists a Lipschitz selection ϕ of $\overline{\text{co}}F$, defined on a neighborhood of (t,x), with $|\phi(t,x) - y| < \varepsilon$,*

then there exists a measurable selection f of $\text{ext}\, F$ such that, for every x_0, the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

has a unique Carathéodory solution, depending continuously on x_0.

We remark that every Lipschitz multifunction with compact values satisfies (LSP). Another interesting class for which (LSP) holds consists of those continuous multifunctions F whose values are compact and have convex closure with nonempty interior.

1. Introduction. Let $F : [0,T] \times \mathbb{R}^n \to 2^{\mathbb{R}^n}$ be a continuous multifunction with compact, not necessarily convex values. If F is Lipschitz continuous, it was shown in [5] that there exists a measurable selection f of F such that, for every x_0, the Cauchy problem

$$\dot{x}(t) = f(t, x(t)), \quad x(0) = x_0,$$

has a unique Carathéodory solution, depending continuously on x_0.

In this paper, we prove that the above selection f can be chosen so that $f(t,x) \in \text{ext}\, F(t,x)$ for all t, x. More generally, the result remains valid if F satisfies the following Lipschitz Selection Property:

(LSP) *For every t, x, every $y \in \overline{\text{co}}F(t,x)$ and $\varepsilon > 0$, there exists a Lipschitz selection ϕ of $\overline{\text{co}}F$, defined on a neighborhood of (t,x), with $|\phi(t,x) - y| < \varepsilon$.*

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We remark that, by [10, 12], every Lipschitz multifunction with compact values satisfies (LSP). Another interesting class for which (LSP) holds consists of those continuous multifunctions F whose values are compact and have convex closure with nonempty interior. Indeed, for any given t, x, y, ε, choosing $y' \in \operatorname{int} \overline{\mathrm{co}} F(t,x)$ with $|y'-y| < \varepsilon$, the constant function $\phi \equiv y'$ is a local selection from $\overline{\mathrm{co}} F$ satisfying the requirements.

In the following, $\Omega \subseteq \mathbb{R}^n$ is an open set, $\overline{B}(0,M)$ is the closed ball centered at the origin with radius M, $\overline{B}(D,MT)$ is the closed neighborhood of radius MT around the set D, while $\mathcal{AC}$ is the Sobolev space of all absolutely continuous functions $u : [0,T] \to \mathbb{R}^n$, with norm $\|u\|_{\mathcal{AC}} = \int_0^T (|u(t)| + |\dot{u}(t)|)\,dt$.

THEOREM 1. *Let* $F : [0,T] \times \Omega \to 2^{\mathbb{R}^n}$ *be a bounded continuous multifunction with compact values, satisfying* (LSP). *Assume that* $F(t,x) \subseteq \overline{B}(0,M)$ *for all* t, x *and let* D *be a compact set such that* $\overline{B}(D,MT) \subset \Omega$. *Then there exists a measurable function* f *with*

$$f(t,x) \in \operatorname{ext} F(t,x) \qquad \forall t, x, \tag{1.1}$$

such that, for every $(t_0, x_0) \in [0,T] \times D$, *the Cauchy problem*

$$\dot{x}(t) = f(t, x(t)), \qquad x(t_0) = x_0 \tag{1.2}$$

has a unique Carathéodory solution $x(\cdot) = x(\cdot, t_0, x_0)$ *on* $[0,T]$, *depending continuously on* t_0, x_0 *in the norm of* $\mathcal{AC}$.

Moreover, if $\varepsilon_0 > 0$ *and a Lipschitz continuous selection* f_0 *of* $\overline{\mathrm{co}} F$ *are given, then one can construct* f *with the following additional property: Denoting by* $y(\cdot, t_0, x_0)$ *the unique solution of*

$$\dot{y}(t) = f_0(t, y(t)), \qquad y(t_0) = x_0, \tag{1.3}$$

for every $(t_0, x_0) \in [0,T] \times D$ *one has*

$$|y(t, t_0, x_0) - x(t, t_0, x_0)| \le \varepsilon_0 \qquad \forall t \in [0,T]. \tag{1.4}$$

The proof of the above theorem, given in Section 3, starts with the construction of a sequence f_n of directionally continuous selections from $\overline{\mathrm{co}} F$ which are piecewise Lipschitz continuous in the (t,x)-space. For every $u : [0,T] \to \mathbb{R}^n$ in a class of Lipschitz continuous functions, we then show that the composed maps $t \to f_n(t, u(t))$ form a Cauchy sequence in $\mathcal{L}^1([0,T]; \mathbb{R}^n)$ converging pointwise almost everywhere to a map of the form $f(\cdot, u(\cdot))$, taking values within the extreme points of F. This convergence is obtained through an argument which is considerably different from previous works. Indeed, it relies on a careful use of the likelihood functional introduced in [4], interpreted here as a measure of "oscillatory nonconvergence" of a set of derivatives.

Among various corollaries, Theorem 1 yields an extension, valid for the wider class of multifunctions with the property (LSP), of the following results, proved in [7], [5] and [8], respectively.

(i) Existence of selections from the solution set of a differential inclusion, depending continuously on the initial data.

(ii) Existence of selections from a multifunction, which generate a continuous flow.

(iii) Contractibility of the solution sets of $\dot{x} \in F(t,x)$ and $\dot{x} \in \mathrm{ext}\, F(t,x)$.

These consequences, together with an application to bang-bang feedback controls, are described in Section 4. Topological properties of the set of solutions of nonconvex differential inclusions have been studied in [3, 6] with the technique of directionally continuous selections and in [8, 9, 13] using the method of Baire category.

2. Preliminaries. As customary, $\overline{A}$ and $\overline{\mathrm{co}}\, A$ denote here the closure and the closed convex hull of A respectively, while $A \backslash B$ indicates a set-theoretic difference. The Lebesgue measure of a set $J \subset \mathbb{R}$ is $m(J)$. The characteristic function of a set A is written as χ_A.

In the following, $\mathcal{K}_n$ denotes the family of all nonempty compact convex subsets of $\mathbb{R}^n$, endowed with Hausdorff metric. A key technical tool used in our proofs will be the function $h : \mathbb{R}^n \times \mathcal{K}_n \to \mathbb{R} \cup \{-\infty\}$ defined by

$$h(y,K) \doteq \sup\Big\{\Big(\int_0^1 |w(\xi) - y|^2\, d\xi\Big)^{1/2};\, w : [0,1] \to K,\ \int_0^1 w(\xi)\, d\xi = y\Big\} \tag{2.1}$$

with the understanding that $h(y,K) = -\infty$ if $y \notin K$. Observe that $h^2(y,K)$ can be interpreted as the maximum variance among all random variables supported inside K whose mean value is y. The following results were proved in [4]:

LEMMA 1. *The map $(y,K) \mapsto h(y,K)$ is upper semicontinuous in both variables; for each fixed $K \in \mathcal{K}_n$ the function $y \mapsto h(y,K)$ is strictly concave down on K. Moreover, one has*

$$h(y,K) = 0 \quad \textit{if and only if} \quad y \in \mathrm{ext}\, K, \tag{2.2}$$

$$h^2(y,K) \le r^2(K) - |y - c(K)|^2, \tag{2.3}$$

where $c(K)$ and $r(K)$ denote the Chebyshev center and the Chebyshev radius of K, respectively.

Remark 1. By the above lemma, the function h has all the qualitative properties of the Choquet function d_F considered, for example, in [9, Proposition 2.6]. It could thus be used within any argument based on Baire category. Moreover, the likelihood functional

$$L(u) \doteq \Big(\int_0^T h^2(\dot u(t), F(t,u(t)))\,dt\Big)^{1/2}$$

provides an upper bound to the distance $\|\dot v - \dot u\|_{\mathcal{L}^2}$ between derivatives, for solutions of $\dot v \in F(t,v)$ which remain close to u uniformly on $[0,T]$. This additional quantitative property of the function h will be a crucial ingredient in our proof.

For the basic theory of multifunctions and differential inclusions we refer to [1]. As in [2], given a map $g : [0,T]\times\Omega \to \mathbb{R}^n$, we say that g is *directionally continuous* along the directions of the cone $\Gamma^N = \{(s,y)\ ;\ |y| \le Ns\}$ if

$$g(t,x) = \lim_{k\to\infty} g(t_k,x_k)$$

for every (t,x) and every sequence (t_k,x_k) in the domain of g such that $t_k \to t$ and $|x_k - x| \le N(t_k - t)$ for every k. Equivalently, g is Γ^N-continuous iff it is continuous w.r.t. the topology generated by the family of all half-open cones of the form

$$\{(s,y)\ ;\ \widehat t \le s < \widehat t + \varepsilon,\ |y-\widehat x| \le N(s-t)\} \tag{2.4}$$

with $(\widehat t, \widehat x) \in \mathbb{R}\times\mathbb{R}^n$, $\varepsilon > 0$. A set of the form (2.4) will be called an N-*cone*.

Under the assumptions on Ω, D made in Theorem 1, consider the set of Lipschitzean functions

$$Y \doteq \{u : [0,T] \to \overline{B}(D, MT)\ ;\ |u(t) - u(s)| \le M|t-s|\ \forall t,s\}.$$

The Picard operator of a map $g : [0,T]\times\Omega\to\mathbb{R}^n$ is defined as

$$\mathcal{P}^g(u)(t) \doteq \int_0^t g(s,u(s))\,ds, \qquad u \in Y.$$

The distance between two Picard operators will be measured by

$$\begin{aligned}\|\mathcal{P}^f - \mathcal{P}^g\| \\ &= \sup\Big\{\Big|\int_0^t [f(s,u(s)) - g(s,u(s))]\,ds\Big|\ ;\ t\in[0,T],\ u \in Y\Big\}.\end{aligned} \tag{2.5}$$

The next lemma will be useful in order to prove the uniqueness of solutions of the Cauchy problems (1.2).

Lemma 2. *Let f be a measurable map from $[0,T]\times\Omega$ into $\overline{B}(0,M)$, with $\mathcal{P}^f$ continuous on Y. Let D be compact, with $\overline{B}(D,MT)\subset\Omega$, and assume that the Cauchy problem*

$$\dot x(t) = f(t,x(t)), \qquad x(t_0) = x_0, \qquad t\in[0,T], \tag{2.6}$$

has a unique solution, for each $(t_0,x_0) \in [0,T]\times D$.

Then, for every $\varepsilon > 0$, there exists $\delta > 0$ with the following property. If $g : [0,T] \times \Omega \to \overline{B}(0,M)$ satisfies $\|\mathcal{P}^g - \mathcal{P}^f\| \le \delta$, then for every $(t_0, x_0) \in [0,T] \times D$, any solution of the Cauchy problem

$$\dot{y}(t) = g(t, y(t)), \qquad y(t_0) = x_0, \qquad t \in [0,T], \tag{2.7}$$

has distance $< \varepsilon$ from the corresponding solution of (2.6). *In particular, the solution set of* (2.7) *has diameter $\le 2\varepsilon$ in $\mathcal{C}^0([0,T];\mathbb{R}^n)$.*

Proof. If the conclusion fails, then there exist sequences of times t_ν, t'_ν, maps g_ν with $\|\mathcal{P}^{g_\nu} - \mathcal{P}^f\| \to 0$, and couples of solutions $x_\nu, y_\nu : [0,T] \to \overline{B}(D, MT)$ of

$$\dot{x}_\nu(t) = f(t, x_\nu(t)), \qquad \dot{y}_\nu(t) = g_\nu(t, y_\nu(t)), \qquad t \in [0,T], \tag{2.8}$$

with

$$x_\nu(t_\nu) = y_\nu(t_\nu) \in D, \qquad |x_\nu(t'_\nu) - y_\nu(t'_\nu)| \ge \varepsilon \qquad \forall \nu. \tag{2.9}$$

By taking subsequences, we can assume that $t_\nu \to t_0$, $t'_\nu \to \tau$, $x_\nu(t_0) \to x_0$, while $x_\nu \to x$ and $y_\nu \to y$ uniformly on $[0,T]$. From (2.8) it follows that

$$\begin{aligned} \Big| y(t) - x_0 - \int_{t_0}^{t} f(s, y(s))\, ds \Big| &\le |y(t) - y_\nu(t)| + |x_0 - y_\nu(t_0)| \\ &+ \Big| \int_{t_0}^{t} [f(s, y(s)) - f(s, y_\nu(s))]\, ds \Big| + \Big| \int_{t_0}^{t} [f(s, y_\nu(s)) - g_\nu(s, y_\nu(s))]\, ds \Big|. \end{aligned} \tag{2.10}$$

As $\nu \to \infty$, the right hand side of (2.10) tends to zero, showing that $y(\cdot)$ is a solution of (2.6). By the continuity of $\mathcal{P}^f$, $x(\cdot)$ is also a solution of (2.6), distinct from $y(\cdot)$ because

$$|x(\tau) - y(\tau)| = \lim_{\nu \to \infty} |x_\nu(\tau) - y_\nu(\tau)| = \lim_{\nu \to \infty} |x_\nu(t'_\nu) - y_\nu(t'_\nu)| \ge \varepsilon.$$

This contradicts the uniqueness assumption, proving the lemma.

3. Proof of the main theorem. Observing that $\operatorname{ext} F(t,x) = \operatorname{ext} \overline{\operatorname{co}} F(t,x)$ for every compact set $F(t,x)$, it is clearly not restrictive to prove Theorem 1 under the additional assumption that all values of F are convex. Moreover, the bounds on F and D imply that no solution of the Cauchy problem

$$\dot{x}(t) \in F(t, x(t)), \qquad x(t_0) = x_0, \qquad t \in [0,T],$$

with $x_0 \in D$, can escape from the set $\overline{B}(D, MT)$. Therefore, it suffices to construct the selection f on the compact set $\Omega^\dagger \doteq [0,T] \times \overline{B}(D, MT)$. Finally, since every convex-valued multifunction satisfying (LSP) admits a globally defined Lipschitz selection, it suffices to prove the second part of the theorem, with f_0 and $\varepsilon_0 > 0$ assigned.

We shall define a sequence of directionally continuous selections of F, converging a.e. to a selection from ext F. The basic step of our constructive procedure will be provided by the next lemma.

LEMMA 3. *Fix any $\varepsilon > 0$. Let S be a compact subset of $[0,T] \times \Omega$ and let $\phi : S \to \mathbb{R}^n$ be a continuous selection of F such that*

$$h(\phi(t,x), F(t,x)) < \eta \quad \forall (t,x) \in S, \tag{3.1}$$

with h as in (2.1). *Then there exists a piecewise Lipschitz selection $g : S \to \mathbb{R}^n$ of F with the following properties*:

(i) *There exists a finite covering $\{\Gamma_i\}_{i=1,\ldots,\nu}$, consisting of Γ^{M+1}-cones, such that, if we define the pairwise disjoint sets $\Delta^i \doteq \Gamma_i \setminus \bigcup_{l<i} \Gamma_l$, then on each Δ^i the following holds*:

  (a) *There exist Lipschitzean selections $\psi^i_j : \overline{\Delta^i} \to \mathbb{R}^n$, $j = 0, \ldots, n$, such that*

$$g|_{\Delta^i} = \sum_{j=0}^{n} \psi^i_j \chi_{A^i_j}, \tag{3.2}$$

  where each A^i_j is a finite union of strips of the form $([t', t'') \times \mathbb{R}^n) \cap \Delta^i$.

  (b) *For every $j = 0, \ldots, n$ there exists an affine map $\varphi^i_j(\cdot) = \langle a^i_j, \cdot\rangle + b^i_j$ such that*

$$\varphi^i_j(\psi^i_j(t,x)) \le \varepsilon, \ \varphi^i_j(z) \ge h(z, F(t,x)), \quad \forall (t,x) \in \overline{\Delta^i}, \ z \in F(t,x). \tag{3.3}$$

(ii) *For every $u \in Y$ and every interval $[\tau, \tau']$ such that $(s, u(s)) \in S$ for $\tau \le s < \tau'$, the following estimates hold*:

$$\Big| \int_{\tau}^{\tau'} [\phi(s,u(s)) - g(s,u(s))]\, ds \Big| \le \varepsilon, \tag{3.4}$$

$$\int_{\tau}^{\tau'} |\phi(s,u(s)) - g(s,u(s))|\, ds \le \varepsilon + \eta(\tau' - \tau). \tag{3.5}$$

Remark 2. Thinking of $h(y,K)$ as a measure for the distance of y from the extreme points of K, the above lemma can be interpreted as follows. Given any selection ϕ of F, one can find a Γ^{M+1}-continuous selection g whose values lie close to the extreme points of F and whose Picard operator $\mathcal{P}^g$, by (3.4), is close to $\mathcal{P}^\phi$. Moreover, if the values of ϕ are near the extreme points of F, i.e. if η in (3.1) is small, then g can be chosen close to ϕ. The estimate (3.5) will be a direct consequence of the definition (2.1) of h and of Hölder's inequality.

Remark 3. Since h is only upper semicontinuous, the two assumptions $y_\nu \to y$ and $h(y_\nu, K) \to 0$ do not necessarily imply $h(y, K) = 0$. As a consequence, the a.e. limit of a convergent sequence of approximately extremal selections f_ν of F need not take values inside $\operatorname{ext} F$. To overcome this difficulty, the estimates in (3.3) provide upper bounds for h in terms of the affine maps φ^i_j. Since each φ^i_j is continuous, limits of the form $\varphi^i_j(y_\nu) \to \varphi^i_j(y)$ will be straightforward.

Proof of Lemma 3. For every $(t,x) \in S$ there exist values $y_j(t,x) \in F(t,x)$ and coefficients $\theta_j(t,x) \ge 0$ with

$$\phi(t,x) = \sum_{j=0}^{n} \theta_j(t,x) y_j(t,x), \qquad \sum_{j=0}^{n} \theta_j(t,x) = 1,$$
$$h(y_j(t,x), F(t,x)) < \varepsilon/2.$$

By the concavity and the upper semicontinuity of h, for every $j = 0, \ldots, n$ there exists an affine function $\varphi_j^{(t,x)}(\cdot) = \langle a_j^{(t,x)}, \cdot \rangle + b_j^{(t,x)}$ such that

$$\varphi_j^{(t,x)}(y_j(t,x)) < h(y_j(t,x), F(t,x)) + \varepsilon/2 < \varepsilon,$$
$$\varphi_j^{(t,x)}(z) > h(z, F(t,x)) \qquad \forall z \in F(t,x).$$

By (LSP) and the continuity of each $\varphi_j^{(t,x)}$, there exists a neighborhood $\mathcal{U}$ of (t,x) together with Lipschitzean selections $\psi_j^{(t,x)} : \mathcal{U} \to \mathbb{R}^n$ such that, for every j and every $(s,y) \in \mathcal{U}$,

$$|\psi_j^{(t,x)}(s,y) - y_j(t,x)| < \frac{\varepsilon}{4T}, \tag{3.6}$$

$$\varphi_j^{(t,x)}(\psi_j^{(t,x)}(s,y)) < \varepsilon. \tag{3.7}$$

Using again the upper semicontinuity of h, we can find a neighborhood $\mathcal{U}'$ of (t,x) such that

$$\varphi_j^{(t,x)}(z) \ge h(z, F(s,y)) \qquad \forall z \in F(s,y),\ (s,y) \in \mathcal{U}',\ j = 0, \ldots, n. \tag{3.8}$$

Choose a neighborhood $\Gamma_{t,x}$ of (t,x), contained in $\mathcal{U} \cap \mathcal{U}'$, such that, for every point (s,y) in the closure $\overline{\Gamma}_{t,x}$, one has

$$|\phi(s,y) - \phi(t,x)| < \frac{\varepsilon}{4T}. \tag{3.9}$$

It is not restrictive to assume that $\Gamma_{t,x}$ is an $(M+1)$-cone, i.e. it has the form (2.4) with $N = M+1$. By the compactness of S we can extract a finite subcovering $\{\Gamma^i\ ;\ 1 \le i \le \nu\}$, with $\Gamma_i \doteq \Gamma_{t_i,x_i}$. Define $\Delta^i \doteq \Gamma_i \setminus \bigcup_{j<i} \Gamma_j$ and set $\theta^i_j = \theta_j(t_i,x_i)$, $y^i_j = y_j(t_i,x_i)$, $\psi^i_j = \psi_j^{(t_i,x_i)}$, $\varphi^i_j = {\varphi_j}^{(t_i,x_i)}$. Choose

an integer N such that

$$N > \frac{8M\nu^2 T}{\varepsilon} \tag{3.10}$$

and divide $[0,T]$ into N equal subintervals $J_1, \dots, J_N$, with

$$J_k = [t_{k-1}, t_k), \qquad t_k = \frac{kT}{N}. \tag{3.11}$$

For each i,k such that $(J_k \times \mathbb{R}^n) \cap \Delta^i \neq \emptyset$, we then split J_k into $n+1$ subintervals $J^i_{k,0}, \dots, J^i_{k,n}$ with lengths proportional to $\theta^i_0, \dots, \theta^i_n$, by setting

$$J^i_{k,j} = [t_{k,j-1}, t_{k,j}), \qquad t_{k,j} = \frac{T}{N}\Big(k + \sum_{l=0}^{j} \theta^i_l\Big), \qquad t_{k,-1} = \frac{Tk}{N}.$$

For any point $(t,x) \in \overline{\Delta^i}$ we now set

$$\begin{cases} g^i(t,x) \doteq \psi^i_j(t,x) \\ \overline{g}^i(t,x) = y^i_j \end{cases} \qquad \text{if } t \in \bigcup_{k=1}^{N} J^i_{k,j}. \tag{3.12}$$

The piecewise Lipschitz selection g and a piecewise constant approximation $\overline{g}$ of g can now be defined as

$$g = \sum_{i=1}^{\nu} g^i \chi_{\Delta^i}, \qquad \overline{g} = \sum_{i=1}^{\nu} \overline{g}^i \chi_{\Delta^i}. \tag{3.13}$$

By construction, recalling (3.7) and (3.8), the conditions (a), (b) in (i) clearly hold.

It remains to show that the estimates in (ii) hold as well. Let $\tau, \tau' \in [0,T]$ and $u \in Y$ be such that $(t, u(t)) \in S$ for every $t \in [\tau, \tau']$, and define

$$E^i = \{t \in I \ ; \ (t, u(t)) \in \Delta^i\}, \qquad i = 1, \dots, \nu.$$

From our previous definition $\Delta^i \doteq \Gamma_i \setminus \bigcup_{j<i} \Gamma_j$, where each Γ_j is an $(M+1)$-cone, it follows that every E^i is the union of at most i disjoint intervals. We can thus write

$$E^i = \Big(\bigcup_{J_k \subset E^i} J_k\Big) \cup \widehat{E}^i,$$

with J_k given by (3.11) and

$$m(\widehat{E}^i) \le \frac{2iT}{N} \le \frac{2\nu T}{N}. \tag{3.14}$$

Since

$$\phi(t_i, x_i) = \sum_{j=0}^{n} \theta^i_j y^i_j, \tag{3.15}$$

the definition of $\overline{g}$ in (3.12), (3.13) implies

$$\int\limits_{J_k} [\phi(t_i, x_i) - \overline{g}(s, u(s))]\, ds = m(J_k)\Big[\phi(t_i, x_i) - \sum_{j=0}^{n} \theta_j^i y_j^i\Big] = 0.$$

Therefore, from (3.9) and (3.6) it follows that

$$\begin{aligned}
&\Big| \int\limits_{J_k} [\phi(s, u(s)) - g(s, u(s))]\, ds\Big| \\
&\le \Big| \int\limits_{J_k} [\phi(s, u(s)) - \phi(t_i, x_i)]\, ds\Big| \\
&\quad + \Big| \int\limits_{J_k} [\phi(t_i, x_i) - \overline{g}(s, u(s))]\, ds\Big| + \Big| \int\limits_{J_k} [\overline{g}(s, u(s)) - g(s, u(s))]\, ds\Big| \\
&\le m(J_k)\bigg[\frac{\varepsilon}{4T} + 0 + \frac{\varepsilon}{4T}\bigg] = m(J_k)\frac{\varepsilon}{2T}.
\end{aligned}$$

The choice of N in (3.10) and the bound (3.14) thus imply

$$\begin{aligned}
\Big|\int\limits_{\tau}^{\tau'} [\phi(s, u(s)) - g(s, u(s))]\, ds\Big| &\le 2Mm\Big(\bigcup_{i=1}^{\nu} \widehat{E}^i\Big) + (\tau' - \tau)\frac{\varepsilon}{2T} \\
&\le 2M\nu\frac{2\nu T}{N} + \frac{\varepsilon}{2} \le \varepsilon,
\end{aligned}$$

proving (3.4).

We next consider (3.5). For a fixed $i \in \{1, \ldots, \nu\}$, let E^i be as before and define

$$\xi_{-1} = 0, \qquad \xi_j = \sum_{l=0}^{j} \theta_l^i, \qquad w^i(\xi) = \sum_{j=0}^{n} y_j^i \chi_{[\xi_{j-1}, \xi_j]}.$$

Recalling (3.15), the definition of h at (2.1) and Hölder's inequality together imply

$$\begin{aligned}
h(\phi(t_i, x_i), F(t_i, x_i)) &\ge \Big(\int\limits_0^1 |\phi(t_i, x_i) - w^i(\xi)|^2\, d\xi\Big)^{1/2} \\
&\ge \int\limits_0^1 |\phi(t_i, x_i) - w^i(\xi)|\, d\xi \\
&= \sum_{j=0}^{n} \theta_j^i |\phi(t_i, x_i) - y_j^i|.
\end{aligned}$$

Using this inequality we obtain

$$\int\limits_{J_k} |\phi(t_i,x_i) - \overline{g}(s,u(s))|\,ds = m(J_k)\sum_{j=0}^{n}\theta_j^i|\phi(t_i,x_i) - y_j^i|$$
$$\le m(J_k)\cdot h(\phi(t_i,x_i),F(t_i,x_i)) \le \eta m(J_k),$$

and therefore, by (3.9) and (3.6),

$$\int\limits_{J_k} |\phi(s,u(s)) - g(s,u(s))|\,ds$$
$$\le \int\limits_{J_k} |\phi(s,u(s)) - \phi(t_i,x_i)|\,ds + \int\limits_{J_k} |\overline{g}(s,u(s)) - g(s,u(s))|\,ds$$
$$+ \int\limits_{J_k} |\phi(t_i,x_i) - \overline{g}(s,u(s))|,$$
$$\le m(J_k)\left[\frac{\varepsilon}{4T} + \frac{\varepsilon}{4T} + \eta\right] = m(J_k)\left(\frac{\varepsilon}{2T} + \eta\right).$$

Using again (3.14) and (3.10), we conclude that

$$\int\limits_{\tau}^{\tau'} |\phi(s,u(s)) - g(s,u(s))|\,ds \le (\tau'-\tau)\left(\frac{\varepsilon}{2T} + \eta\right) + 2M\nu\frac{2\nu T}{N}$$
$$\le \varepsilon + (\tau'-\tau)\eta.$$

which finishes the proof of Lemma 3.

Using Lemma 3, given any continuous selection $\widetilde{f}$ of F on $\Omega^\dagger$, and any sequence $(\varepsilon_k)_{k\ge1}$ of strictly positive numbers, we can generate a sequence $(f_k)_{k\ge1}$ of selections from F as follows.

To construct f_1, we apply the lemma with $S = \Omega^\dagger$, $\phi = f_0$, $\varepsilon = \varepsilon_1$. This yields a partition $\{A_1^i;\ i = 1,\ldots,\nu_1\}$ of $\Omega^\dagger$ and a piecewise Lipschitz selection f_1 of F of the form

$$f_1 = \sum_{i=1}^{\nu_1} f_1^i \chi_{A_1^i}.$$

In general, at the beginning of the kth step we are given a partition of $\Omega^\dagger$, say $\{A_k^i\ ;\ i = 1,\ldots,\nu_k\}$, and a selection

$$f_k = \sum_{i=1}^{\nu_k} f_k^i \chi_{A_k^i},$$

where each f_k^i is Lipschitz continuous and satisfies

$$h(f_k(t,x),F(t,x)) \le \varepsilon_k \qquad \forall (t,x) \in \overline{A_k^i}.$$

We then apply Lemma 3 separately to each A_k^i, choosing $S = \overline{A_k^i}$, $\varepsilon = \varepsilon_k$, $\phi = f_k^i$. This yields a partition $\{A_{k+1}^i;\ i = 1, \ldots, \nu_{k+1}\}$ of $\Omega^\dagger$ and functions of the form

$$f_{k+1} = \sum_{i=1}^{\nu_{k+1}} f_{k+1}^i \chi_{A_{k+1}^i}, \qquad \varphi_{k+1}^i(\cdot) = \langle a_{k+1}^i, \cdot\rangle + b_{k+1}^i,$$

where each $f_{k+1}^i : \overline{A_{k+1}^i} \to \mathbb{R}^n$ is a Lipschitz continuous selection from F, satisfying the following estimates:

$$\varphi_{k+1}^i(z) > h(z, F(t,x)) \quad \forall (t,x) \in A_{k+1}^i, \tag{3.16}$$

$$\varphi_{k+1}^i(f_{k+1}^i(t,x)) \le \varepsilon_{k+1} \quad \forall (t,x) \in A_{k+1}^i, \tag{3.17}$$

$$\Big|\int_\tau^{\tau'} [f_{k+1}(s,u(s)) - f_k(s,u(s))]\,ds\Big| \le \varepsilon_{k+1}, \tag{3.18}$$

$$\int_\tau^{\tau'} |f_{k+1}(s,u(s)) - f_k(s,u(s))|\,ds \le \varepsilon_{k+1} + \varepsilon_k(\tau' - \tau), \tag{3.19}$$

for every $u \in Y$ and every τ, τ', as long as the values $(s, u(s))$ remain inside a single set A_k^i, for $s \in [\tau, \tau')$.

Observe that, according to Lemma 3, each A_k^i is closed-open in the finer topology generated by all $(M+1)$-cones. Therefore, each f_k is Γ^{M+1}-continuous. By Theorem 2 in [2], the substitution operator $\mathcal{S}^{f_k} : u(\cdot) \mapsto f_k(\cdot, u(\cdot))$ is continuous from the set Y defined in (2.5) into $\mathcal{L}^1([0,T];\ \mathbb{R}^n)$. The Picard map $\mathcal{P}^{f_k}$ is thus continuous as well.

Furthermore, there exists an integer N_k with the following property. Given any $u \in Y$, there exists a finite partition of $[0,T]$ with nodes $0 = \tau_0 < \tau_1 < \ldots < \tau_{n(u)} = T$, with $n(u) \le N_k$, such that, as t ranges in any $[\tau_{l-1},\ \tau_l)$, the point $(t, u(t))$ remains inside one single set A_k^i. Otherwise stated, the number of times the curve $t \mapsto (t, u(t))$ crosses a boundary between two distinct sets A_k^i, A_k^j is smaller than N_k, for every $u \in Y$. The construction of the A_k^i in terms of $(M+1)$-cones implies that all these crossings are transversal. Since the restriction of f_k to each A_k^i is Lipschitz continuous, it is clear that every Cauchy problem

$$\dot{x}(t) = f_k(t, x(t)), \qquad x(t_0) = x_0,$$

has a unique solution, depending continuously on the initial data $(t_0, x_0) \in [0,T] \times D$.

From (3.18), (3.19) and the property of N_k it follows that

$$\begin{aligned} \Big|\int_0^t [f_{k+1}(s,u(s)) - f_k(s,u(s))]\,ds\Big| \\ \le \sum_{l=1}^{L} \Big|\int_{\tau_{l-1}}^{\tau_l} [f_{k+1}(s,u(s)) - f_k(s,u(s))]\,ds\Big| \le N_k \varepsilon_{k+1}, \end{aligned} \tag{3.20}$$

where $0 = \tau_0 < \tau_1 < \ldots < \tau_L = t$ are the times at which the map $s \mapsto (s, u(s))$ crosses a boundary between two distinct sets A_k^i, A_k^j. Since (3.20) holds for every $t \in [0,T]$, we conclude that

$$\|\mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k}\| \le N_k \varepsilon_{k+1}. \tag{3.21}$$

Similarly, for every $u \in Y$ one has

$$\begin{aligned} \Big\| f_{k+1}(\cdot, u(\cdot)) - f_k(\cdot, u(\cdot)) \Big\|_{\mathcal{L}^1([0,T];\mathbb{R}^n)} &\le \sum_{l=1}^{n(u)} \int_{\tau_{l-1}}^{\tau_l} |f_{k+1}(s,u(s)) - f_k(s,u(s))|\, ds \\ &\le \sum_{l=1}^{n(u)} [\varepsilon_{k+1} + \varepsilon_k(\tau_l - \tau_{l-1})] \le N_k\varepsilon_{k+1} + \varepsilon_k T. \end{aligned} \tag{3.22}$$

Now consider the functions $\varphi_k : \mathbb{R}^n \times \Omega^\dagger \to \mathbb{R}$ with

$$\varphi_k(y,t,x) \doteq \langle a_k^i, y\rangle + b_k^i \quad \text{if } (t,x) \in A_k^i. \tag{3.23}$$

From (3.16), (3.17) it follows that

$$\varphi_k(y,t,x) \ge h(y, F(t,x)) \quad \forall (t,x) \in \Omega^\dagger,\ y \in F(t,x), \tag{3.24}$$

$$\varphi_k(f_k(t,x),t,x) \le \varepsilon_k \quad \forall (t,x) \in \Omega^\dagger. \tag{3.25}$$

For every $u \in Y$, (3.18) and the linearity of φ_k in y imply

$$\begin{aligned} &\Big| \int_0^T [\varphi_k(f_{k+1}(s,u(s)), s, u(s)) - \varphi_k(f_k(s,u(s)), s, u(s))]\, ds \Big| \\ &\quad \le \sum_{l=1}^{n(u)} \max\{|a_k^1|, \ldots, |a_k^{\nu_k}|\} \Big| \int_{\tau_{l-1}}^{\tau_l} [f_{k+1}(s,u(s)) - f_k(s,u(s))]\, ds \Big| \\ &\quad \le N_k \max\{|a_k^1|, \ldots, |a_k^{\nu_k}|\} \varepsilon_{k+1}. \end{aligned} \tag{3.26}$$

Moreover, for every $l \ge k$, from (3.19) it follows that

$$\begin{aligned} &\int_0^T \Big| \varphi_k(f_{l+1}(s,u(s)), s, u(s)) - \varphi_k(f_l(s,u(s)), s, u(s)) \Big|\, ds \\ &\quad \le \max\{|a_k^1|, \ldots, |a_k^{\nu_k}|\} \int_0^T |f_{l+1}(s,u(s)) - f_l(s,u(s))|\, ds \\ &\quad \le \max\{|a_k^1|, \ldots, |a_k^{\nu_k}|\} \cdot (N_l \varepsilon_{l+1} + \varepsilon_l T). \end{aligned} \tag{3.27}$$

Observe that all of the above estimates hold regardless of the choice of the ε_k. We now introduce an inductive procedure for choosing the constants ε_k, which will yield the convergence of the sequence f_k to a function f with the desired properties.

Given f_0 and ε_0, by Lemma 2 there exists $\delta_0 > 0$ such that, if $g : \Omega^\dagger \to \overline{B}(0, M)$ and $\|\mathcal{P}^g - \mathcal{P}^{f_0}\| \leq \delta_0$, then, for each $(t_0, x_0) \in [0, T] \times D$, every solution of (2.7) remains ε_0-close to the unique solution of (1.3). We then choose $\varepsilon_1 = \delta_0/2$.

By induction on k, assume that the functions $f_1, \dots, f_k$ have been constructed, together with the linear functions $\varphi_l^i(\cdot) = \langle a_l^i, \cdot\rangle + b_l^i$ and the integers $N_l,\ l = 1, \dots, k$. Let the values $\delta_0, \delta_1, \dots, \delta_k > 0$ be inductively chosen, satisfying

$$\delta_l \leq \delta_{l-1}/2, \qquad l = 1, \dots, k, \tag{3.28}$$

and such that $\|\mathcal{P}^g - \mathcal{P}^{f_l}\| \leq \delta_l$ implies that for every $(t_0, x_0) \in [0, T] \times D$ the solution set of (2.7) has diameter $\leq 2^{-l}$, for $l = 1, \dots, k$. This is possible again because of Lemma 2. For $k \geq 1$ we then choose

$$\varepsilon_{k+1} \doteq \min\left\{\frac{\delta_k}{2N_k},\ \frac{2^{-k}}{N_k},\ \frac{2^{-k}}{N_k \max\{|a_l^i|; 1 \leq l \leq k,\ 1 \leq i \leq \nu_l\}}\right\}. \tag{3.29}$$

Using (3.28), (3.29) in (3.21), with $N_0 \doteq 1$, we now obtain

$$\sum_{k=p}^{\infty} \|\mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k}\| \leq \sum_{k=p}^{\infty} N_k \frac{\delta_k}{2N_k} \leq \sum_{k=p}^{\infty} \frac{2^{p-k}\delta_p}{2} \leq \delta_p \tag{3.30}$$

for every $p \geq 0$. From (3.22) and (3.29) we further obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \|f_{k+1}(\cdot, u(\cdot)) - f_k(\cdot, u(\cdot))\|_{\mathcal{L}^1} &\leq \sum_{k=1}^{\infty} \left(N_k \frac{2^{-k}}{N_k} + \frac{2^{1-k}T}{N_k}\right) \\ &\leq \sum_{k=1}^{\infty} (2^{-k} + 2^{1-k}T) \leq 1 + 2T. \end{aligned} \tag{3.31}$$

Define

$$f(t, x) \doteq \lim_{k\to\infty} f_k(t, x) \tag{3.32}$$

for all $(t, x) \in \Omega^\dagger$ at which the sequence f_k converges. By (3.31), for every $u \in Y$ the sequence $f_k(\cdot, u(\cdot))$ converges in $\mathcal{L}^1([0, T]; \mathbb{R}^n)$ and a.e. on $[0, T]$. In particular, considering the constant functions $u \equiv x \in \overline{B}(D, MT)$, by Fubini's theorem we conclude that f is defined a.e. on $\Omega^\dagger$. Moreover, the substitution operators $\mathcal{S}^{f_k} : u(\cdot) \mapsto f_k(\cdot, u(\cdot))$ converge to the operator $\mathcal{S}^f : u(\cdot) \mapsto f(\cdot, u(\cdot))$ uniformly on Y. Since each $\mathcal{S}^{f_k}$ is continuous, $\mathcal{S}^f$ is also continuous. Clearly, the Picard map $\mathcal{P}^f$ is continuous as well. By (3.30) we have

$$\|\mathcal{P}^f - \mathcal{P}^{f_k}\| \le \sum_{k=p}^{\infty} \|\mathcal{P}^{f_{k+1}} - \mathcal{P}^{f_k}\| \le \delta_p \qquad \forall p \ge 1.$$

Recalling the property of δ_p, this implies that, for every p, the solution set of (2.7) has diameter $\le 2^{-p}$. Since p is arbitrary, for every $(t_0, x_0) \in [0,T] \times D$ the Cauchy problem can have at most one solution. On the other hand, the existence of such a solution is guaranteed by Schauder's theorem. The continuous dependence of this solution on the initial data t_0, x_0, in the norm of $\mathcal{AC}$, is now an immediate consequence of uniqueness and of the continuity of the operators $\mathcal{S}^f$, $\mathcal{P}^f$. Furthermore, for $p = 0$, (3.30) yields $\|\mathcal{P}^f - \mathcal{P}^{f_0}\| \le \delta_0$. The choice of δ_0 thus implies (1.4).

It now remains to prove (1.1). Since every set $F(t,x)$ is closed, it is clear that $f(t,x) \in F(t,x)$. For every $u \in Y$ and $k \ge 1$, by (3.24)–(3.27) the choices of ε_k at (3.29) yield

$$\begin{aligned}
(3.33)\quad & \int_0^T h(f(s,u(s)), F(s,u(s)))\, ds \\
&\le \int_0^T \varphi_k(f(s,u(s)), s, u(s))\, ds \\
&\le \int_0^T \varphi_k(f_k(s,u(s)), s, u(s))\, ds \\
&\quad + \Big| \int_0^T [\varphi_k(f_{k+1}(s,u(s)), s, u(s)) - \varphi_k(f_k(s,u(s)), s, u(s))]\, ds \Big| \\
&\quad + \sum_{l=k+1}^{\infty} \int_0^T |\varphi_k(f_{l+1}((s,u(s)), s, u(s)) - \varphi_k(f_l(s,u(s)), s, u(s))|\, ds \\
&\le 2^{1-k}T + 2^{-k} + \sum_{l=k+1}^{\infty} (2^{-l} + 2^{1-l}T).
\end{aligned}$$

Observing that the right hand side of (3.33) approaches zero as $k \to \infty$, we conclude that

$$\int_0^T h(f(t,u(t)), F(t,u(t)))\, dt = 0.$$

By (2.2), given any $u \in Y$, this implies $f(t,u(t)) \in \operatorname{ext} F(t,u(t))$ for almost every $t \in [0,T]$. By possibly redefining f on a set of measure zero, this yields (1.1).

4. Applications. Throughout this section we make the following assumptions:

(H) $F : [0,T] \times \Omega \to \overline{B}(0,M)$ is a bounded continuous multifunction with compact values satisfying (LSP), while D is a compact set such that $\overline{B}(D, MT) \subset \Omega$.

An immediate consequence of Theorem 1 is

COROLLARY 1. *Let the hypotheses* (H) *hold. Then there exists a continuous map* $(t_0, x_0) \mapsto x(\cdot, t_0, x_0)$ *from* $[0,T] \times D$ *into* $\mathcal{AC}$ *such that*

$$\begin{cases} \dot{x}(t, t_0, x_0) \in \operatorname{ext} F(t, x(t, t_0, x_0)) & \forall t \in [0,T], \\ x(t_0, t_0, x_0) = x_0 & \forall t_0, x_0. \end{cases}$$

Another consequence of Theorem 1 is the contractibility of the sets of solutions of certain differential inclusions. We recall here that a metric space X is contractible if there exist a point $\widetilde{u} \in X$ and a continuous mapping $\Phi : X \times [0,1] \to X$ such that

$$\Phi(v, 0) = \widetilde{u}, \quad \Phi(v, 1) = v, \quad \forall v \in X.$$

The map Φ is then called a *null homotopy* of X.

COROLLARY 2. *Let the assumptions* (H) *hold. Then*, *for any* $\overline{x} \in D$, *the sets* $\mathcal{M}$, $\mathcal{M}^{\mathrm{ext}}$ *of solutions of*

$$x(0) = \overline{x}, \quad \dot{x}(t) \in F(t, x(t)), \quad t \in [0,T],$$
$$x(0) = \overline{x}, \quad \dot{x} \in \operatorname{ext} F(t, x(t)), \quad t \in [0,T],$$

are both contractible in $\mathcal{AC}$.

Proof. Let f be a selection from $\operatorname{ext} F$ with the properties stated in Theorem 1. As usual, we denote by $x(\cdot, t_0, x_0)$ the unique solution of the Cauchy problem (1.2). Define the null homotopy $\Phi : \mathcal{M} \times [0,1] \to \mathcal{M}$ by

$$\Phi(v, \lambda)(t) \doteq \begin{cases} v(t) & \text{if } t \in [0, \lambda T], \\ x(t, \lambda T, v(\lambda T)) & \text{if } t \in [\lambda T, T]. \end{cases}$$

By Theorem 1, Φ is continuous. Moreover, setting $\widetilde{u}(\cdot) \doteq u(\cdot, 0, \overline{x})$, we obtain

$$\Phi(v, 0) = \widetilde{u}, \quad \Phi(v, 1) = v, \quad \Phi(v, \lambda) \in \mathcal{M} \quad \forall v \in \mathcal{M},$$

proving that $\mathcal{M}$ is contractible. We now observe that, if $v \in \mathcal{M}^{\mathrm{ext}}$, then $\Phi(v, \lambda) \in \mathcal{M}^{\mathrm{ext}}$ for every λ. Therefore, $\mathcal{M}^{\mathrm{ext}}$ is contractible as well.

Our last application is concerned with feedback controls. Let $\Omega \subseteq \mathbb{R}^n$ be open, $U \subset \mathbb{R}^m$ compact, and let $g : [0,T] \times \Omega \times U \to \mathbb{R}^n$ be a continuous function. By a well-known theorem of Filippov [11], the solutions of the control system

$$\dot{x} = g(t, x, u), \quad u \in U, \tag{4.1}$$

correspond to the trajectories of the differential inclusion

$$\dot{x} \in F(t,x) \doteq \{g(t,x,\omega);\ \omega \in U\}. \tag{4.2}$$

In connection with (4.1), one can consider the "relaxed" system

$$\dot{x} = g^{\#}(t,x,u^{\#}), \qquad u^{\#} \in U^{\#}, \tag{4.3}$$

whose trajectories are precisely those of the differential inclusion

$$\dot{x} \in F^{\#}(t,x) \doteq \overline{\mathrm{co}} F(t,x).$$

The control system (4.3) is obtained by defining the compact set

$$U^{\#} \doteq U \times \ldots \times U \times \Delta_n = U^{n+1} \times \Delta_n,$$

where

$$\Delta_n \doteq \Big\{\theta = (\theta_0,\ldots,\theta_n)\ ; \sum_{i=0}^{n} \theta_i = 1,\ \theta_i \ge 0\ \forall i\Big\}$$

is the standard simplex in $\mathbb{R}^{n+1}$, and by setting

$$g^{\#}(t,x,u^{\#}) = g^{\#}(t,x,(u_0,\ldots,u_n,(\theta_0,\ldots,\theta_n))) \doteq \sum_{i=0}^{n} \theta_i f(t,x,u_i).$$

Generalized controls of the form $u^{\#} = (u_0,\ldots,u_n,\theta)$ taking values in the set $U^{n+1} \times \Delta_n$ are called *chattering controls*.

COROLLARY 3. *Consider the control system* (4.1), *with* $g : [0,T] \times \Omega \times U \to \overline{B}(0,M)$ *Lipschitz continuous. Let* D *be a compact set with* $\overline{B}(D;MT) \subset \Omega$. *Let* $u^{\#}(t,x) \in U^{\#}$ *be a chattering feedback control such that the mapping*

$$(t,x) \mapsto g^{\#}(t,x,u^{\#}(t,x)) \doteq f_0(t,x)$$

is Lipschitz continuous.

Then for every $\varepsilon_0 > 0$ *there exists a measurable feedback control* $\overline{u} = \overline{u}(t,x)$ *with the following properties*:

(a) *for every* (t,x), *one has* $g(t,x,\overline{u}(t,x)) \in \operatorname{ext} F(t,x)$, *with* F *as in* (4.2),

(b) *for every* $(t_0,x_0) \in [0,T] \times D$, *the Cauchy problem*

$$\dot{x}(t) = g(t,x(t),\overline{u}(t,x(t))), \qquad x(t_0) = x_0,$$

has a unique solution $x(\cdot,t_0,x_0)$,

(c) *if* $y(\cdot,t_0,x_0)$ *denotes the* (*unique*) *solution of the Cauchy problem*

$$\dot{y} = f_0(t,y(t)), \qquad y(t_0) = x_0,$$

then for every (t_0,x_0) *one has*

$$|x(t,t_0,x_0) - y(t,t_0,x_0)| < \varepsilon_0 \qquad \forall t \in [0,T].$$

Proof. The Lipschitz continuity of g implies that the multifunction F in (4.2) is Lipschitz continuous in the Hausdorff metric, hence it satisfies

(LSP). We can thus apply Theorem 1, and obtain a suitable selection f of $\operatorname{ext} F$, in connection with f_0, ε_0. For every (t,x), the set

$$W(t,x) \doteq \{\omega \in U; g(t,x,\omega) = f(t,x)\} \subset \mathbb{R}^m$$

is a compact nonempty subset of U. Let $\overline{u}(t,x) \in W(t,x)$ be the lexicographic selection. Then the feedback control $\overline{u}$ is measurable, and it is trivial to check that $\overline{u}$ has all the required properties.

References

[1] J. P. Aubin and A. Cellina, *Differential Inclusions*, Springer, Berlin, 1984.

[2] A. Bressan, *Directionally continuous selections and differential inclusions*, Funkcial. Ekvac. 31 (1988), 459–470.

[3] —, *On the qualitative theory of lower semicontinuous differential inclusions*, J. Differential Equations 77 (1989), 379–391.

[4] —, *The most likely path of a differential inclusion*, ibid. 88 (1990), 155–174.

[5] —, *Selections of Lipschitz multifunctions generating a continuous flow*, Differential Integral Equations 4 (1991), 483–490.

[6] A. Bressan and G. Colombo, *Boundary value problems for lower semicontinuous differential inclusions*, Funkcial. Ekvac. 36 (1993), 359–373.

[7] A. Cellina, *On the set of solutions to Lipschitzean differential inclusions*, Differential Integral Equations 1 (1988), 495–500.

[8] F. S. De Blasi and G. Pianigiani, *On the solution set of nonconvex differential inclusions*, J. Differential Equations, to appear.

[9] —, —, *Topological properties of nonconvex differential inclusions*, Nonlinear Anal. 20 (1993), 871–894.

[10] A. LeDonne and M. V. Marchi, *Representation of Lipschitz compact convex valued mappings*, Atti Accad. Naz. Lincei Rend. 68 (1980), 278–280.

[11] A. F. Filippov, *On certain questions in the theory of optimal control*, SIAM J. Control Optim. 1 (1962), 76–84.

[12] A. Ornelas, *Parametrization of Carathéodory multifunctions*, Rend. Sem. Mat. Univ. Padova 83 (1990), 33–44.

[13] A. A. Tolstonogov, *Extreme continuous selectors of multivalued maps and their applications*, preprint, 1992.

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