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Spectrum preserving linear mappings in Banach algebras

by

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Abstract. Let A and B be two unitary Banach algebras. We study linear mappings from A into B which preserve the polynomially convex hull of the spectrum. In particular, we give conditions under which such surjective linear mappings are Jordan morphisms.

1. Introduction. The theory of spectrum preserving linear mappings originates from Hua's theorem on fields which has very interesting geometrical applications. This theorem says that an additive mapping $\sigma : K_1 \to K_2$, where K_1, K_2 are two fields, such that $\sigma(1) = 1$, and $\sigma(x^{-1}) = \sigma(x)^{-1}$ for $x \neq 0$, is an isomorphism or an anti-isomorphism. If ϕ is a linear mapping from a Banach algebra A_1 into another one A_2 such that $\phi(1) = 1$ and $\phi(x)^{-1} = \phi(x^{-1})$ for x invertible, then using exponentials it is easy to prove that ϕ is a Jordan morphism, that is, $\phi(x^2) = \phi(x)^2$ for every x in A. In the situation of Banach algebras the problem was enlarged by I. Kaplansky [5] to the following one: if ϕ is linear, satisfies $\phi(1) = 1$ and ϕ maps invertible elements into invertible elements, is it true that ϕ is a Jordan morphism? By Lemma 4, page 30 of [1], this question is equivalent to the study of linear mappings which preserve the spectrum.

Almost at the same time, in 1967–1968, A. Gleason, J.-P. Kahane and W. Żelazko proved that if A and B are Banach algebras, with B commutative and semisimple and if $\phi : A \to B$ is a linear mapping that satisfies $\phi(1) = 1$ and x invertible in A implies $\phi(x)$ invertible in B, then ϕ is a homomorphism (see [2], pp. 69–70, for the simple and elegant proof given by M. Roitman and Y. Sternfeld).

In the case of matrices the general problem is justified by a result of M. Marcus and R. Purves [6] which says that if $\phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a linear mapping which preserves eigenvalues and their multiplicity then ϕ is either of the form $\phi(T) = ATA^{-1}$ or $\phi(T) = AT^tA^{-1}$ (incidentally, the same conclusion is true if ϕ preserves only the greatest eigenvalue).

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Unfortunately, Kaplansky's problem is too general to have an affirmative answer, as the following example (taken from [1]) shows. Let A be the subalgebra of $M_4(\mathbb{C})$ built up with matrices of the form

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$$

with $a, b, c \in M_2(\mathbb{C})$ and define a linear mapping ϕ from A onto A by

$$\phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = \begin{pmatrix} a & b \\ 0 & c^t \end{pmatrix}.$$

Then ϕ is bijective, $\phi(1) = 1$ and ϕ maps invertible elements onto invertible elements. However,

$$\phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}^2\right) - \phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right)^2$$

is in general not zero but just in the radical of A. So the natural question is the following: if A and B are two semisimple Banach algebras and if $T : A \to B$ is a surjective spectrum preserving linear mapping, is T Jordan?

In this direction, A. A. Jafarian and A. R. Sourour [4] generalized Marcus–Purves theorem proving the following result. If $\phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ is a surjective spectrum preserving linear mapping then either

1) there exists a bounded invertible linear operator A from X into Y such that $\phi(T) = ATA^{-1}$ for every $T \in \mathcal{B}(X)$, or

2) there exists a bounded invertible operator B from the dual X^* into Y such that $\phi(T) = BT^*B^{-1}$ for every $T \in \mathcal{B}(X)$.

Denoting by σ the full spectrum, that is, the polynomially convex hull of the spectrum, in this paper we shall study a slightly more general problem: if A and B are two semisimple Banach algebras and if T is a surjective linear mapping with the property that $\sigma(Ta) = \sigma(a)$ for every a in A, is T Jordan?

In Section 3 we solve this problem for two extremal classes of Banach algebras, first the primitive algebras with minimal ideals (this class contains $\mathcal{B}(X)$, and consequently we get the Jafarian–Sourour result as a corollary), second the scattered algebras for which every element has a countable spectrum. Unfortunately, the general problem is still unsolved even for the class of C^*-algebras.

Section 2 contains spectral characterizations of the rank one elements, the socle and the kernel of the hull of the socle in a Banach algebra. These are the essential tools needed in Section 3.

2. Spectral characterizations of $\mathcal{F}_1(A)$, $\operatorname{soc} A$ and $\operatorname{kh}(\operatorname{soc} A)$. We assume throughout this section that A is a semisimple Banach algebra. The socle of A, denoted by $\operatorname{soc} A$, is the sum of all minimal left ideals of A. It is well-known that the socle of A is a two-sided ideal of A and that all its elements are algebraic. If I is a two-sided ideal of A, we define the set $\operatorname{kh}(I)$ by

$$\operatorname{kh}(I) = \{a \in A : \overline{a} \in \operatorname{Rad}(A/\overline{I})\}.$$

It is easy to see that $\operatorname{kh}(\operatorname{soc} A)$ is a closed two-sided ideal of A and (by [2], Corollary 5.7.7) the spectrum of every element of $\operatorname{kh}(\operatorname{soc} A)$ has at most 0 as a limit point.

Let K be an analytic multifunction from $U \subseteq \mathbb{C}$ into $\mathbb{C}$. We say that $z_0 \in K(\lambda_0)$ is a *good isolated point* of $K(\lambda_0)$ if there exist $r, s > 0$ such that $\overline{B}(z_0, s) \cap K(\lambda_0) = \{z_0\}$ and $B(z_0, s) \cap K(\lambda)$ is finite for $|\lambda - \lambda_0| < r$. We denote by $DK(\lambda)$ the set of points of $K(\lambda)$ which are not good isolated points, while $K(\lambda)'$ denotes the set of limit points of $K(\lambda)$ (see [2], VII, §2, for more details).

THEOREM 2.1 (Multiplicative characterization of $\operatorname{soc} A$ and $\operatorname{kh}(\operatorname{soc} A)$). *Let $a \in A$. Then we have:*

(1) *$a \in \operatorname{soc} A$ if and only if $\operatorname{Sp}(xa)$ is finite for all $x \in A$,*

(2) *$a \in \operatorname{kh}(\operatorname{soc} A)$ if and only if $\operatorname{Sp}(xa)$ has at most 0 as a limit point for every $x \in A$.*

Proof. If $a \in \operatorname{soc} A$, then a is algebraic, so $\operatorname{Sp}(a)$ is finite. If $a \in \operatorname{kh}(\operatorname{soc} A)$, then $\operatorname{Sp}(a)$ has at most 0 as a limit point. So the necessary conditions are obvious. Suppose now that $\operatorname{Sp}(xa)$ is finite for every $x \in A$. In particular, $\operatorname{Sp}(a)$ is finite. If p is the spectral idempotent associated with some nonzero isolated spectral value of a, then there exists $c \in A$ such that $p = ac = ca$ ([7], Proposition 2.4) so $\operatorname{Sp}(pxp)$ is finite for all $x \in A$. This implies that the semisimple Banach algebra pAp, with identity p, is finite-dimensional and hence $p \in \operatorname{soc} A$. Because a may be written as $\sum \lambda_i p_i$ it follows that $a \in \operatorname{soc} A$.

Suppose that $\operatorname{Sp}(xa)' \subset \{0\}$ for all $x \in A$. The same argument shows that there exists a sequence $a_n \in \operatorname{soc} A$ such that $\varrho(a - a_n) \to 0$ where ϱ denotes the spectral radius. Arguing in $A/\operatorname{kh}(\operatorname{soc} A)$ we have $\varrho(\overline{a}) = 0$, so $\varrho(\overline{Aa}) = 0$ and $a \in \operatorname{kh}(\operatorname{soc} A)$. ∎

We define the *rank one elements* of A as the set $\mathcal{F}_1(A) = \{a \in A : \operatorname{Sp}(xa)$ contains at most one nonzero point for every $x \in A\}$.

Clearly the set $\mathcal{F}_1(A)$ is closed under multiplication by elements of A and by Theorem 2.1(1) we have $\mathcal{F}_1(A) \subset \operatorname{soc} A$. Examples of rank one elements are the minimal idempotents of A. Furthermore, every minimal left ideal of A is of the form Ap, where p is a minimal idempotent and hence $\operatorname{soc} A$ is equal to the set of all finite sums of rank one elements of A.

THEOREM 2.2 (Additive characterization of $\mathcal{F}_1(A)$, $\operatorname{soc} A$ and $\operatorname{kh}(\operatorname{soc} A)$). *Let $a \in A$. Then we have:*

(1) *$a \in \mathcal{F}_1(A)$ if and only if for every $x \in A$ and for every two-element subset $F \subset \mathbb{C}\backslash\{0\}$ we have*

$$\bigcap_{t\in F} \sigma(x+ta) \subset \sigma(x),$$

(2) *$a \in \operatorname{soc} A$ if and only if there exists an integer $n \geq 1$ such that*

$$\bigcap_{t\in F} \sigma(x+ta) \subset \sigma(x)$$

for every $x \in A$ and every n-element subset $F \subset \mathbb{C}\backslash\{0\}$,

(3) *$a \in \operatorname{kh}(\operatorname{soc} A)$ if and only if for every $x \in A$ and for every subset $F \subset \mathbb{C}\backslash\{0\}$ having only a nonzero limit point we have*

$$\bigcap_{t\in F} \sigma(x+ta) \subset \sigma(x).$$

Proof. (1)($\Rightarrow$) Let $x \in A$ and $\mu \notin \sigma(x)$. If F is a two-element subset of $\mathbb{C}\backslash\{0\}$, then

$$\{1/t : t \in F\} \not\subset \operatorname{Sp}((\mu-x)^{-1}a)$$

because $\operatorname{Sp}((\mu-x)^{-1}a)$ contains at most one nonzero point. Hence there exists $t_0 \in F$ such that $1/t_0 - (\mu-x)^{-1}a$ is invertible. But the relation

$$(*) \qquad \mu-(x+ta) = (\mu-x)[1-t(\mu-x)^{-1}a]$$

implies that $\mu-(x+t_0a)$ is invertible and consequently $\mu \notin \operatorname{Sp}(x+t_0a)$. Suppose that $\mu \in \sigma(x+t_0a)$. Then μ belongs to a hole of $\operatorname{Sp}(x+t_0a)$. But (by [2], Theorem 5.7.4(ii)) $\sigma(x+t_0a)$ and $\sigma(x)$ differ at most by isolated points, contradicting the fact that $\mu \notin \sigma(x)$. So $\mu \notin \bigcap_{t\in F}\sigma(x+ta)$ and the inclusion is proved.

($\Leftarrow$) Suppose the inclusion is true for every two-element subset F of $\mathbb{C}\backslash\{0\}$. Let $\mu \notin \sigma(x)$. Then $\mu - x$ is invertible and $\mu \notin \sigma(x+t_1a)$ for some $t_1 \in F$. Consequently, $1/t_1 \notin \operatorname{Sp}((\mu-x)^{-1}a)$ by $(*)$. This says that every two-element subset of $\mathbb{C}\backslash\{0\}$ meets $\mathbb{C}\backslash\operatorname{Sp}((\mu-x)^{-1}a)$, so $\operatorname{Sp}((\mu-x)^{-1}a)$ contains at most one nonzero point. Consider the following two cases:

(i) If a is invertible then $\operatorname{Sp}((\mu-x)^{-1}a)$ consists of a single nonzero point. Let $y \in A$ be arbitrary and $|\lambda| > 2\varrho(y)$. Then $y-\lambda = (\mu-x)^{-1}$, where

$$x = \frac{y}{\lambda^2}\left(1+\frac{y}{\lambda}+\frac{y^2}{\lambda^2}+\ldots\right) \quad \text{and} \quad \mu = -\frac{1}{\lambda}.$$

Furthermore, $\varrho(x) < |\mu|$ so $\#\operatorname{Sp}((\lambda-y)a) = 1$. If we apply the scarcity lemma ([2], Theorem 7.1.7) to the analytic multifunction $\lambda \mapsto \operatorname{Sp}((\lambda-y)a)$ we conclude that $\#\operatorname{Sp}(ya) = 1$ for all $y \in A$. Hence $a \in \mathcal{F}_1(A)$.

(ii) If a is not invertible then $Aa \neq A$ or $aA \neq A$. We only consider the first case. Since $Aa \neq A$, no element of Aa is invertible and so $0 \in \operatorname{Sp}((\mu-x)^{-1}a)$. As in case (i) we conclude that $\#\operatorname{Sp}(ya) \leq 2$ for all $y \in A$

and since 0 is in $\operatorname{Sp}(ya)$, $\operatorname{Sp}(ya)$ consists of 0 and possibly one other point. Hence $a \in \mathcal{F}_1(A)$.

(2)($\Rightarrow$) Let $a \in \operatorname{soc} A$. For every $\mu \notin \sigma(x)$ the analytic multifunction $\mu \mapsto \operatorname{Sp}((\mu-x)^{-1}a)$ is finite, so by the scarcity theorem, there exists an integer $n \geq 1$ such that $\#\operatorname{Sp}((\mu-x)^{-1}a) \leq n$ for every $\mu \notin \sigma(x)$. Let F be a subset of $\mathbb{C}\backslash\{0\}$ having $n+1$ points. Then

$$\{1/t : t \in F\} \not\subset \operatorname{Sp}((\mu-x)^{-1}a)$$

because $\operatorname{Sp}((\mu-x)^{-1}a)$ contains at most n points. Hence there exists $t_1 \in F$ such that $1/t_1 - (\mu-x)^{-1}a$ is invertible and so $\mu \notin \operatorname{Sp}(x+t_1a)$. As in (1) we conclude that $\mu \notin \bigcap_{t\in F}\sigma(x+ta)$ and the inclusion is proved.

($\Leftarrow$) Suppose the inclusion is true for every n-element subset of $\mathbb{C}\backslash\{0\}$. Let $\mu \notin \sigma(x)$. Then $\mu - x$ is invertible and $\mu \notin \sigma(x+t_2a)$ for some $t_2 \in F$. Consequently, $1/t_2 \notin \operatorname{Sp}((\mu-x)^{-1}a)$ by $(*)$. This says that every n-element subset of $\mathbb{C}\backslash\{0\}$ must meet $\mathbb{C}\backslash\operatorname{Sp}((\mu-x)^{-1}a)$, so

$$\#\operatorname{Sp}((\mu-x)^{-1}a) \leq n+1.$$

As in case (1), $\#\operatorname{Sp}(ya) \leq n+1$ for all $y \in A$, so $a \in \operatorname{soc} A$.

(3)($\Rightarrow$) The proof is very similar to that of (1) and (2). For $\mu \notin \sigma(x)$ we have $\operatorname{Sp}((\mu-x)^{-1}a)' \subset \{0\}$. Let $F \subset \mathbb{C}\backslash\{0\}$ have a nonzero limit point. Then $\{1/t : t \in F\}$ has a nonzero limit point, so it is not contained in $\operatorname{Sp}((\mu-x)^{-1}a)$ and we finish as previously.

($\Leftarrow$) Suppose the inclusion is true for every subset F of $\mathbb{C}\backslash\{0\}$ having only a nonzero limit point. Let $\mu \notin \sigma(x)$. Then $\mu - x$ is invertible and $\mu \notin \sigma(x+t_3a)$ for some $t_3 \in F$ and by $(*)$ $1/t_3 \notin \operatorname{Sp}((\mu-x)^{-1}a)$. We conclude as previously that

$$\operatorname{Sp}((\mu-x)^{-1}a)' \subset \{0\},$$

and as previously

$$\operatorname{Sp}((\mu-y)a)' \subset \{0\}$$

for all $|\lambda| > 2\varrho(y)$. Consequently,

$$D\operatorname{Sp}((\lambda-y)a) \subset \{0\}$$

for $|\lambda| > 2\varrho(y)$. But by the Oka–Nishino theorem ([2], Theorem 7.2.4) either $D\operatorname{Sp}((\lambda-y)a) = \emptyset$ for all $\lambda \in \mathbb{C}$ or the multifunction $\lambda \mapsto D\operatorname{Sp}((\lambda-y)a)$ is analytic. In both cases $D\operatorname{Sp}((\lambda-y)a) \subset \{0\}$ for all $\lambda \in \mathbb{C}$. Hence $\operatorname{Sp}(ay)' \subset \{0\}$ and the result follows from Theorem 2.1(2). ■

Remark. Note that the proof of case 1 of Theorem 2.2 also shows that A contains invertible elements of rank one if and only if $A \simeq \mathbb{C}$.

If A is not isomorphic to $\mathbb{C}$ and $a \in \mathcal{F}_1(A)$ then $\operatorname{Sp}(a)$ consists of 0 and possibly one other point. We define a map $t : \mathcal{F}_1(A) \to \mathbb{C}$ by $\operatorname{Sp}(a) = \{0, t(a)\}$.

LEMMA 2.3. *Let $a, b \in \mathcal{F}_1(A)$ such that $a + \lambda b \in \mathcal{F}_1(A)$ for all $\lambda \in \mathbb{C}$. Then $t(a+b) = t(a) + t(b)$.*

Proof. By [2], Theorem 3.4.17, the map $h : \mathbb{C} \to \mathbb{C}$, $h(\lambda) = t(a + \lambda b)$, is entire and

$$\lim_{|\lambda| \to \infty} \frac{h(\lambda)}{|\lambda|} = \lim_{|\lambda| \to \infty} t\left(\frac{a}{\lambda} + b\right) = t(b),$$

by Newburgh's theorem (see [2], Corollary 3.4.5). Hence, by Liouville's theorem we have $t(a + \lambda b) = t(a) + \lambda t(b)$ and the result follows. ■

The condition $a + \lambda b \in \mathcal{F}_1(A)$ will be automatically satisfied if a and b are left multiples of the same element in the socle. We shall use this fact in the proof of the next theorem.

3. Linear mappings which preserve the full spectrum and the Jordan property. Throughout this section we assume that A and B are semisimple Banach algebras and that $T : A \to B$ is a surjective linear mapping with the property that $\sigma(Ta) = \sigma(a)$ for every $a \in A$. We start by giving a few elementary properties of these mappings.

THEOREM 3.1. *With the above hypotheses we have:*

(1) *T is injective,*
(2) *$T1 = 1$,*
(3) *$T(\mathcal{F}_1(A)) = \mathcal{F}_1(B)$,*
(4) *$T(\operatorname{soc} A) = \operatorname{soc} B$,*
(5) *$T(\operatorname{kh}(\operatorname{soc} A)) = \operatorname{kh}(\operatorname{soc} B)$.*

Proof. Let $a \in T^{-1}(0)$. Then $\sigma(a) = \{0\}$ and if q is any quasinilpotent element of A, then $\sigma(a + q) = \{0\}$. By Zemánek's characterization of the radical ([2], Theorem 5.3.1), $a = 0$. A similar argument shows that $T1 = 1$, while (3), (4) and (5) follow from Theorem 2.2. ■

Remark. In fact the same argument shows that $T(\operatorname{Rad} A) = \operatorname{Rad} B$ if A and B are not semisimple.

THEOREM 3.2. *With the above hypotheses, for every $x \in \operatorname{soc} B$ and $a \in A$ we have $(Ta^2 - (Ta)^2)x = 0$.*

Proof. By [2], Theorem 5.5.2, T is continuous. Let $b \in \mathcal{F}_1(A)$ and let $c \in A$ be such that $0 \notin \sigma(c)$. Then (by using [2], Theorem 5.7.4(ii)) it is easy to see that $0 \in \sigma(c + b)$ is equivalent to saying that $c + b$ is not invertible. Furthermore,

$$\begin{aligned} c + b \text{ not invertible } &\Leftrightarrow 1 + c^{-1}b \text{ not invertible} \\ &\Leftrightarrow -1 \in \operatorname{Sp}(c^{-1}b) \\ &\Leftrightarrow t(c^{-1}b) = -1, \end{aligned}$$

and since $0 \in \sigma(c + b) \Leftrightarrow 0 \in \sigma(Tc + Tb)$, it follows that

$$\begin{aligned} t(c^{-1}b) = \alpha \neq 0 &\Leftrightarrow t((-\alpha c)^{-1}b) = -1 \\ &\Leftrightarrow t((-\alpha Tc)^{-1}Tb) = -1 \\ &\Leftrightarrow t((Tc)^{-1}Tb) = \alpha. \end{aligned}$$

This shows that $t(c^{-1}b) = t((Tc)^{-1}Tb)$ for all $b \in \mathcal{F}_1(A)$ and $c \in A$ satisfying $0 \notin \sigma(c)$. Let $a \in A$ and $b \in \mathcal{F}_1(A)$. For all $\lambda \in \mathbb{C}$ with $|\lambda| > \varrho(a)$, we have $0 \notin \sigma(\lambda - a)$ and hence

$$t(T((\lambda - a)^{-1}b)) = t((\lambda - Ta)^{-1}Tb).$$

Expanding both sides of this relation, we get

$$t\left(\frac{Tb}{\lambda} + \frac{T(ab)}{\lambda^2} + \frac{T(a^2b)}{\lambda^3} + \ldots\right) = t\left(\frac{Tb}{\lambda} + \frac{TaTb}{\lambda^2} + \frac{(Ta)^2Tb}{\lambda^3} + \ldots\right).$$

So applying Lemma 2.3 and comparing coefficients, we get

$$t(T(ab)) = t(TaTb) \tag{1}$$

and

$$t(T(a^2b)) = t((Ta)^2Tb). \tag{2}$$

From (1) it follows that $t(T(a^2b)) = t(Ta^2Tb)$, while by (2) and Lemma 2.3 we have $t((Ta^2 - (Ta)^2)Tb) = 0$. Let $u = Ta^2 - (Ta)^2$. Then $t(ud) = 0$ for every $d \in \mathcal{F}_1(B)$. Suppose that $ud \neq 0$ for some $d \in \mathcal{F}_1(B)$. Since B is semisimple, there exists an $x \in B$ such that $\operatorname{Sp}(udx) \neq \{0\}$. But $dx \in \mathcal{F}_1(B)$ and $t(udx) = 0$. This shows that $ud = 0$ for every $d \in \mathcal{F}_1(B)$ and the result follows from the fact that $\operatorname{soc} B$ is the set of all finite sums of elements of $\mathcal{F}_1(B)$. ■

COROLLARY 3.3. *If B has the property that $b \operatorname{soc} B = \{0\}$ implies $b = 0$, then T is Jordan.*

A Banach algebra A is said to be *prime* if $aAb = \{0\}$ implies $a = 0$ or $b = 0$. By Jacobson's density theorem it can easily be seen that every primitive Banach algebra is prime. In [3], pp. 47–51, it is shown that if A and B are prime rings, then every Jordan morphism $T : A \to B$ is either a homomorphism or an antimorphism. Furthermore, if A is a primitive Banach algebra with minimal ideals, then A has the property that $a \operatorname{soc} A = \{0\}$ implies $a = 0$ (see [8], p. 73). Hence we have the following result.

COROLLARY 3.4. *If B is primitive Banach algebra with minimal ideals then T is either a homomorphism or an antimorphism.*

The following is an improvement on the result of Jafarian and Sourour [4].

COROLLARY 3.5. *Let $\phi : B(X) \to B(Y)$ be a surjective linear mapping such that $\sigma(\phi(T)) = \sigma(T)$ for every $T \in B(X)$. Then either*

(1) *there is a bounded invertible operator* $A : X \to Y$ *such that* $\phi(T) = ATA^{-1}$, *or*

(2) *there is a bounded invertible operator* $B : X^* \to Y$ *such that* $\phi(T) = BT^*B^{-1}$.

Proof. By Corollary 3.4, ϕ is either a homomorphism or an antimorphism. If ϕ is a homomorphism, then (1) follows from the fundamental isomorphism theorem ([8], Theorem 2.5.19, p. 76). If ϕ is an antimorphism, then (2) follows from the fundamental isomorphism theorem and the fact that the Banach algebra $C = \{T^* : T \in B(X)\}$ is a strictly dense subalgebra of $B(X')$ which is anti-isomorphic to $B(X)$ under the map $T \mapsto T^*$. ■

LEMMA 3.6. *If* $a \in A$ *and* $u = Ta^2 - (Ta)^2 \in \mathrm{kh}(\mathrm{soc}\,B)$, *then* $u = 0$.

Proof. Suppose that $u \neq 0$. Since B is semisimple there exists an $x \in B$ such that $\mathrm{Sp}(ux) \neq \{0\}$. Obviously $ux \in \mathrm{kh}(\mathrm{soc}\,B)$. Let λ be a nonzero isolated point of $\mathrm{Sp}(ux)$ and let p be the spectral idempotent associated with λ. By [2], Lemma 5.7.1, $p \in \mathrm{soc}\,B$ and $\mathrm{Sp}(uxp) = \{0, \lambda\}$, which means that $uxp \neq 0$. But $xp \in \mathrm{soc}\,B$, contradicting Theorem 3.2. ■

If B is a modular annihilator algebra (see [1], p. 82) then $B = \mathrm{kh}(\mathrm{soc}\,B)$ so, by the above lemma, T is Jordan. In this situation the spectrum of every element has at most zero as a limit point. This result can be extended to the more general situation of *scattered Banach algebras*, that is, Banach algebras for which the spectrum of every element is finite or countable. By Barnes's theorem their socle is nonzero and they have a very particular algebraic structure (see [2], Theorems 5.7.8 and 5.7.9).

THEOREM 3.7. *If* B *is a separable scattered Banach algebra, then* T *is Jordan.*

Proof. Let $I_1 = \mathrm{kh}(\mathrm{soc}\,A)$ and $J_1 = \mathrm{kh}(\mathrm{soc}\,B)$. By Theorem 3.1(5), $T(I_1) = J_1$. Define semisimple Banach algebras A_1 and B_1 by $A_1 = A/I_1$, $B_1 = B/J_1$ and let π_1 and γ_1 denote the corresponding canonical maps onto A_1 and B_1 respectively. Define a linear map $T_1 : A_1 \to B_1$ by $T_1(\overline{a}) = \overline{Ta}$. Then Harte's theorem ([2], Theorem 3.3.8) and the fact that the spectrum of every element of A and B is totally disconnected imply that

$$\mathrm{Sp}(\overline{a}) = \bigcap_{x \in I_1} \mathrm{Sp}(a + x) = \bigcap_{y \in J_1} \mathrm{Sp}(Ta + y) = \mathrm{Sp}(T_1\overline{a}),$$

and hence T_1 is spectrum preserving. Furthermore, if $a \in I_1$, then by Lemma 3.6 we have $Ta^2 = (Ta)^2$. Continuing inductively, we define $A_n = A_{n-1}/\mathrm{kh}(\mathrm{soc}\,A_{n-1})$, $B_n = B_{n-1}/\mathrm{kh}(\mathrm{soc}\,B_{n-1})$ and let π_n and γ_n denote the canonical maps from A_{n-1} onto A_n and from B_{n-1} onto B_n, respectively. Let $I_n = \ker(\pi_n \circ \ldots \circ \pi_1)$, $J_n = \ker(\gamma_n \circ \ldots \circ \gamma_1)$ and note that $J_n = T(I_n)$. Define a linear map $T_n : A_n \to B_n$ by $T_n(\overline{a}) = \overline{T_{n-1}(a)}$. Then T_n is spectrum preserving and by using Lemma 3.6 it is easy to see that if $a \in I_n$, then $Ta^2 = (Ta)^2$. If ω is the first infinite ordinal, we define

$$I_\omega = \mathrm{kh}\Big(\bigcup_{n \geq 1} I_n\Big), \quad J_\omega = \mathrm{kh}\Big(\bigcup_{n \geq 1} J_n\Big)$$

and note that $T(\bigcup_{n\geq 1} I_n) = \bigcup_{n \geq 1} J_n$. The linear mapping T being continuous and one-to-one we have $T(\overline{\bigcup_{n\geq 1} I_n}) = \overline{\bigcup_{n\geq 1} J_n}$. Define a linear mapping

$$T'_\omega : A\Big/\overline{\bigcup_{n\geq 1} I_n} \to B\Big/\overline{\bigcup_{n\geq 1} J_n} \quad \text{by} \quad T'_\omega(\overline{a}) = \overline{Ta}.$$

Again by using Harte's theorem it follows that T'_ω is spectrum preserving and by the remark following Theorem 3.1 we have $J_\omega = T(I_\omega)$. Let $A_\omega = A/I_\omega$, $B_\omega = B/J_\omega$ and define a linear operator $T_\omega : A_\omega \to B_\omega$ by $T_\omega(\overline{a}) = \overline{Ta}$. Note that T_ω is spectrum preserving and that A_ω and B_ω are simisimple.

We claim that $Ta^2 = (Ta)^2$ for every $a \in I_\omega$. Let $a \in I_\omega$ and suppose that $u = Ta^2 - (Ta)^2$ is not zero. By applying Lemma 3.6, it is easy to see that $\gamma_n \circ \ldots \circ \gamma_1(u) \neq 0$ for $n = 1, 2, \ldots$ and by Theorem 3.2, $(\gamma_n \circ \ldots \circ \gamma_1(u))y = 0$ for every $y \in \mathrm{soc}\,B_n$. Since $u \neq 0$ and B is semisimple, there exists $b \in B$ such that $\mathrm{Sp}(ub) \neq \{0\}$ and since $ub \in J_\omega$, again by Harte's theorem we know that the intersection of all $\mathrm{Sp}(ub + y)$, for $y \in \overline{\bigcup_{n\geq 1} J_n}$, is zero.

We now prove that there exists an integer n such that $\mathrm{Sp}_{B_n}(\gamma_n \circ \ldots \circ \gamma_0(ub)) \neq \mathrm{Sp}_{B_{n+1}}(\gamma_{n+1} \circ \ldots \circ \gamma_0(ub))$, where $B_0 = B$ and γ_0 is the identity map on B. Suppose the contrary; then $\mathrm{Sp}(ub) = \mathrm{Sp}(\gamma_n \circ \ldots \circ \gamma_0(ub)) \subset \mathrm{Sp}(ub + y)$ for every $n \geq 1$ and $y \in J_n$, and consequently, by continuity of the spectrum, $\mathrm{Sp}(ub) \subset \mathrm{Sp}(ub + y)$ for every $y \in \overline{\bigcup_{n\geq 1} J_n}$. So, by a previous remark, we have $\mathrm{Sp}(ub) = \{0\}$, which is a contradiction.

Hence suppose $\mathrm{Sp}_{B_n}(\gamma_n \circ \ldots \circ \gamma_0(ub)) \neq \mathrm{Sp}_{B_{n+1}}(\gamma_{n+1} \circ \ldots \circ \gamma_0(ub))$ for some $n \geq 0$ and let $x = (\gamma_n \circ \ldots \circ \gamma_0)(ub)$. Then there exists an isolated point $\lambda \neq 0$ of $\mathrm{Sp}_{B_n}(x)$ such that $\lambda \notin \mathrm{Sp}_{B_{n+1}}(\gamma_{n+1}(x))$. If p denotes the spectral projection associated with x and λ we have $p \in \mathrm{soc}\,B_0$ and $0 \neq xp = (\gamma_n \circ \ldots \circ \gamma_0(u))(\gamma_n \circ \ldots \circ \gamma_0(b))p$, contradicting the fact that $(\gamma_n \circ \ldots \circ \gamma_0(u))y = 0$ for every $y \in \mathrm{soc}\,B_n$. So finally we have proved that $Ta^2 - (Ta)^2 = 0$ for every $a \in I_\omega$.

Continuing by transfinite induction, there exists an ordinal β in the first class of ordinals such that $A = I_\beta$ ([2], Theorem 5.7.a). By the arguments above it is easy to see that $B = J_\beta$ and $Ta^2 = (Ta)^2$ for every $a \in A$. ■

Remark. The argument used in the proof of the above theorem improves Theorem 3.2. It implies that $(Ta^2 - (Ta)^2)x = 0$ for every $x \in I$, where I is the largest closed two-sided ideal having elements with countable spectrum, that is, the union of all J_α over all ordinals α.

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On the best constant in the Khinchin–Kahane inequality

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Abstract. We prove that if r_i is the Rademacher system of functions then

$$\Big(\int \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\|^2 dt\Big)^{1/2} \le \sqrt{2}\int \Big\|\sum_{i=1}^{n} x_i r_i(t)\Big\| dt$$

for any sequence of vectors x_i in any normed linear space F.

Introduction. The classical result of Khinchin [3] states that for each $p, q > 0$ there exists a constant $c_{p,q}$ such that for any real numbers $x_1, \ldots, x_n$,

$$\Big(\int \Big|\sum_{i=1}^{n} x_i r_i(t)\Big|^p dt\Big)^{1/p} \le c_{p,q}\Big(\int \Big|\sum_{i=1}^{n} x_i r_i(t)\Big|^q dt\Big)^{1/q}. \tag{1}$$

The smallest constant $c_{p,q}$ will be denoted by $C^{\mathbb{R}}_{p,q}$. Obviously, $C^{\mathbb{R}}_{p,q} = 1$ for $p \le q$, but it took some effort to calculate the other best constants. The especially interesting case $p = 2, q = 1$ was first solved by S. J. Szarek [4], who proved $C^{\mathbb{R}}_{2,1} = \sqrt{2}$. A simpler proof was given by U. Haagerup [1] who also found $C^{\mathbb{R}}_{p,2}$ and $C^{\mathbb{R}}_{2,p}$ for each $p > 0$. A simple and elementary proof that $C^{\mathbb{R}}_{2,1} = \sqrt{2}$ was also presented by B. Tomaszewski [6].

J.-P. Kahane [2] generalized the result of Khinchin to sequences $x_1, \ldots, x_n$ in a normed linear space F, replacing in (1) the absolute value by the norm in F. Let $C_{p,q}$ denote the smallest constant in the vector-valued inequalities, over all normed linear spaces F. It is of interest to know if the constants are the same in the vector and real cases. As far as we know the best result for $p = 2$ and $q = 1$ known up to now was obtained by B. Tomaszewski [5], who proved that $C_{2,1} \le \sqrt{3}$. In this paper we show that $C_{2,1} = \sqrt{2}$; we think that our proof is simpler than the ones known for real numbers.

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