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Closed subgroups in Banach spaces

by

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Abstract. We show that zero-dimensional nondiscrete closed subgroups do exist in Banach spaces E. This happens exactly when E contains an isomorphic copy of c_0. Other results on subgroups of linear spaces are obtained.

1. Introduction. For a topological vector space E, we are interested in closed additive subgroups G of E. In case E is finite-dimensional, the structure of G is well known; namely, G is a product of a linear subspace of E and a discrete subgroup. The case when E is infinite-dimensional, in general, is far from being so simple.

Obviously, a (topological-group) isomorphism classification of groups G would provide, in particular, a classification of closed linear subspaces of E; hence, in general, it is out of our reach. Therefore, to avoid dealing with linear spaces, we shall mostly consider subgroups G which contain no nontrivial linear space. Such groups we shall call *line-free*. Note that the maximal linear subspace V contained in a group G is closed and the quotient space G/V is a line-free group.

If E is a Banach space, then the topological classification of G reduces to the line-free case as follows. Write $\kappa : E \to E/V$ for the quotient (linear) map. By a result of Bartle and Graves (see [BP2, p. 86]), there exists an (in general, nonlinear) map $\alpha : E/V \to E$ such that $\alpha \circ \kappa = \mathrm{id}_E$. It follows (see [BP2, p. 86]) that $h(x) = (\kappa(x), x - \alpha \circ \kappa(x))$, $x \in E$, establishes a

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homeomorphism of E onto $E/V \times V$, which sends G onto $G/V \times V$. One sees that G/V is a line-free closed subgroup of the Banach space E/V. By [T, Theorem 6.1], the Banach space V is homeomorphic to a Hilbert space. So the classification of G reduces to the classification of G/V. Consequently, when interested in the topological classification of closed subgroups G in Banach spaces, one can assume that G is line-free. (The above argument also works in case E is a locally convex complete metric linear space. It also applies to the isomorphism classification in case V is a complemented subspace of E, which is the case when either E is a Hilbert space or $E = \mathbb{R}^\infty$; see Sections 2 and 5.)

Some introductory facts, mainly related to the topological dimension of closed subgroups of Hilbert spaces, were provided in [DG]. The present paper is, to some extent, a continuation of [DG]. The main result of [DG] stated that nondiscrete closed subgroups of Hilbert spaces have topological dimension ≥ 1. The proof of this fact strongly used the orthogonality. A natural question arose whether the theorem could be generalized to the case of arbitrary Banach spaces. We show that this is *not* the case. The space c_0 contains a nondiscrete closed subgroup which is zero-dimensional. Moreover, we prove that a Banach space E contains a nondiscrete closed subgroup of dimension 0 if and only if E contains an isomorphic copy of c_0; this is the main result of our paper.

The paper is organized in the following way. In Section 1, we provide four equivalent conditions describing *discrete* closed subgroups G of separable complete metric linear spaces. A nontrivial condition (Theorem 1.1(e)) states that G is a free group. Section 2 contains a complete description of closed subgroups G of $E = \mathbb{R}^\infty$. It turns out that such a G is isomorphic to a product $\mathbb{R}^n \times \mathbb{Z}^k$, where $n, k = 0, 1, 2, \ldots, \infty$. What makes the case of $\mathbb{R}^\infty$ so simple is the fact that the weak topology and the original topology are the same, and that there is no strictly weaker *linear* topology on $\mathbb{R}^\infty$. The above classification result, as brought to our attention by the referee, was previously obtained by Brown, Higgins and Morris, in [BHM]. Since their approach heavily depends on Kaplan's duality theorem, we decided to include our more elementary argument here. We devote Section 3 to the study of weakly closed subgroups. We make use of basic sequences to extend to arbitrary Banach spaces some results on weakly closed subgroups in Hilbert spaces obtained in [DG]. Here, the orthogonality notion of Hilbert spaces is replaced by considering bi-orthogonal sequences; along the way, we implicitly provide an alternative proof of the main result of [DG]. Our main result is contained in Section 4. The proof depends on Proposition 4.3 which states that if a Banach space E contains a weakly unconditionally Cauchy series $\sum_{n=1}^{\infty} y_n$ for which one can find signs ε_j^n (i.e., $\varepsilon_j^n \in \{-1, 0, 1\}$) such that the sequence $\{\sum_{j=1}^{n} \varepsilon_j^n y_j\}_{n=1}^{\infty}$ is discrete, then E contains a copy of c_0. (The terminology is that of [Di].) In Section 5 we make some general remarks concerning topological isomorphism and homeomorphism classifications of closed subgroups G in Banach spaces E. We show that every infinite-dimensional Banach space E contains continuum many pairwise nonisomorphic weakly closed subgroups of dimension one. We pose some questions and formulate some conjectures concerning the topological classification of closed subgroups.

We work with the notion of the small inductive dimension ind (see [E]); however, we shall denote it by dim. Our statements related to dimension concern mostly separable metric spaces, and always metric spaces. For separable metric spaces the three basic dimension functions, ind, Ind and dim, coincide. When we deal with nonseparable spaces, we consider statements of the form "dimension $\geq k$". Since Ind = dim $\geq$ ind for all metric spaces (see [E]), our statements hold for each of these dimension functions (and the use of the symbol dim is justified).

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1. Discrete subgroups in complete metric linear spaces. Let $E = (E, |\cdot|)$ be a separable complete metric linear space ($|\cdot|$ is the so-called F-norm on E). Let G be a closed (additive) subgroup of E. Let us recall that G is *line-free* if it contains no line.

1.1. THEOREM. *The following conditions are equivalent:*

(a) *G is discrete,*
(b) *G is locally compact and line-free,*
(c) *G is countable,*
(d) *G is isomorphic to a (finite or infinite) direct sum of copies of $\mathbb{Z}$,*
(e) *G is a free abelian group.*

Proof. The implications (a)$\Rightarrow$(b) and (d)$\Rightarrow$(e) are obvious.

The implication (b)$\Rightarrow$(a) follows from the structure theorem of van Kampen for locally compact abelian groups. The group G contains an open subgroup G_0 topologically isomorphic to $\mathbb{R}^n \times H$ for some $n = 0, 1, \ldots$, and some compact group H. Since G is line-free, we have $n = 0$, and since G does not contain nontrivial compact groups, we have $H = \{0\}$. Thus $G_0 = \{0\}$ and G is discrete. Clearly, (a) implies (c).

The proof of (c)$\Rightarrow$(d) employs a standard inductive procedure for exhibiting generators of G. Let $G = \{g_n\}_{n=1}^{\infty} \cup \{0\}$, $g_n \neq 0$. The group $G_1 = G \cap \operatorname{span}\{g_1\}$ is a closed, countable subgroup of the line $\operatorname{span}\{g_1\}$. Fix x_1 to be a generator of G_1. Pick g_{i_2} to be the first element $g_2, g_3, \ldots$ so that $g_{i_2} \notin \operatorname{span}\{g_1\}$. The group $G_2 = \operatorname{span}\{g_1, g_{i_2}\} \cap G$ is a closed, countable subgroup of the plane $\operatorname{span}\{g_1, g_{i_2}\}$ (with $x_1 \in G_2$). Pick $x_2 \in G_2$ so that $\{x_1, x_2\}$ generate G_2 (consider G_2/G_1 and pick x_2 so that the coset $[x_2]$ generates G_2/G_1). Continuing this process, we inductively determine generators of G. If the set of generators is finite, then G is isomorphic to a finite product of copies of $\mathbb{Z}$. Otherwise, G is isomorphic to $\bigoplus_{i=1}^{\infty} Z_i$, $Z_i = \mathbb{Z}$.

To show (e)⇒(a) assume G is not discrete. We can inductively construct a sequence $\{x_n\}_{n=1}^{\infty} \subset G\backslash\{0\}$ such that

(i) $|x_n| \le 4^{-n}$ and $|2^{n+1}x_{n+1}| < |2^n x_n|/4$.

For $k \in \mathbb{Z}$, let

$$P(k) = \max\{n \in \mathbb{N} : 2^n \text{ divides } k\}.$$

(We agree $P(0) = \infty$.) Consider the set

$$H = \Big\{\sum_{n=1}^{\infty} m_n x_n \in E : P(m_n) \to \infty \text{ and } m_n \in \mathbb{Z}\Big\}.$$

Hence H is the set of all points of the form $\sum_{n=1}^{\infty} m_n x_n$, where $\{m_n\}_{n=1}^{\infty}$ is a sequence of integers such that $P(m_n) \to \infty$ as $n \to \infty$ and the series $\sum_{n=1}^{\infty} m_n x_n$ is convergent in E. It is easy to see that H is a group. Since G is closed in E, H is contained in G. Consequently, H is a free group. We claim that H has the cardinality of the continuum. Let $C = \{-1, 1\}^{\infty}$ be the Cantor set. For $\varepsilon = (\varepsilon_n) \in C$, consider

$$\varphi(\varepsilon) = \sum_{n=1}^{\infty} 2^n \varepsilon_n x_n.$$

By (i), $|2^n \varepsilon_n x_n| \le 2^n |x_n| \le 2^{-n}$. It follows that φ transforms C into H. We shall show that φ is injective. Assume $\sum_{n=1}^{\infty} 2^n \varepsilon_n x_n = \sum_{n=1}^{\infty} 2^n \delta_n x_n$ for some $(\varepsilon_n), (\delta_n) \in C$. Pick the first k with $\varepsilon_k \ne \delta_k$. Then

$$\pm 2^k x_k + \sum_{n=k+1}^{\infty} 2^n \eta_n x_n = 0$$

for some $(\eta_n) \in \{0, \pm 1\}$. By (i), we have

$$0 \ge |2^k x_k| - \sum_{n=k+1}^{\infty} |2^n x_n| \ge |2^k x_k| - \sum_{n=1}^{\infty} \frac{1}{4^n} |2^k x_k| = \frac{1}{2} |2^k x_k|,$$

a contradiction.

Since H has the cardinality of the continuum, so does $H/2H$. (If (x_α) is a system of generators for H, then the cosets $[x_\alpha]$ in $H/2H$ are distinct.) To get a contradiction, we shall argue that $H/2H$ is countable. Let $x = [\sum_{n=1}^{\infty} m_n x_n] \in H/2H$; hence $P(m_n) \to \infty$. Take $k_0 \in \mathbb{N}$ so large that

if $k \ge k_0$, then $P(m_k) \ge 4$. We have

$$\sum_{n=1}^{\infty} m_n x_n = \sum_{n=1}^{k_0} m_n x_n + \sum_{n=k_0+1}^{\infty} 2 \cdot \frac{m_n}{2} x_n$$

and $m_n/2 \in \mathbb{Z}$. It follows that $x = [\sum_{n=1}^{k_0} m_n x_n]$. So, every element $x \in H/2H$ is the coset of a finite sum $\sum m_n x_n$. Hence $H/2H$ is countable. ∎

2. Subgroups of $\mathbb{R}^{\infty}$. Here is a complete description of closed, additive subgroups of $\mathbb{R}^{\infty}$, the countable product of lines. This result was previously obtained by Brown, Higgins and Morris in [BHM] by analyzing the structure of the direct sum of lines and applying Kaplan's duality theorem. (Their version, formally stronger than the statement of Theorem 2.1 below, can easily be recovered from our proof.)

2.1. THEOREM. *Every closed subgroup of $\mathbb{R}^{\infty}$ is a product of lines $\mathbb{R}$ and integers $\mathbb{Z}$.*

Proof. Let V be the maximal linear subspace contained in G. Since V is closed, V is isomorphic to a finite or infinite product of lines [BPR]. Moreover, there exists a continuous linear projection of $\mathbb{R}^{\infty}$ onto V [P, Theorem 3]. This gives rise to a group-isomorphism of G onto $V \times (G/V)$ (cf. Introduction). The line-free, quotient group $\Gamma = G/V$ can then be identified as a closed subgroup of $\mathbb{R}^{\infty}$. We shall show that Γ is a finite or infinite product of copies of $\mathbb{Z}$.

In case $\operatorname{span}(\Gamma)$ is finite-dimensional, Γ is a finite product of copies of $\mathbb{Z}$. Therefore, applying the above result of [BPR], we can assume that Γ is linearly dense in $\mathbb{R}^{\infty}$.

We consider $(\mathbb{R}^{\infty})^*$, the dual of $\mathbb{R}^{\infty}$, with the weak $*$-topology. We let

$$\Gamma^* = \{x^* \in (\mathbb{R}^{\infty})^* : x^*(\Gamma) \subseteq \mathbb{Z}\}.$$

Clearly, Γ^* is a closed subgroup of $(\mathbb{R}^{\infty})^*$. Since Γ is linearly dense in $\mathbb{R}^{\infty}$, Γ^* is line-free. It follows that Γ^* is finitely discrete (i.e., the intersection of Γ^* with every finite-dimensional linear subspace of $(\mathbb{R}^{\infty})^*$ is discrete). Since $(\mathbb{R}^{\infty})^*$ is a countable union of finite-dimensional linear subspaces, Γ^* is countable. Now, the argument of the proof of Theorem 1.1(c)⇒(d) applies to yield that Γ^* is free abelian.

Let $F = \{x_n^*\}_{n=1}^{\infty}$ be a set of free generators of Γ^*. Consider $T : \mathbb{R}^{\infty} \to \mathbb{R}^{\infty}$ given by $T(x) = (x_n^*(x))$, $x \in \mathbb{R}^{\infty}$. Since F is total (cf. [DG, Corollary 2.2]), T is a continuous linear injection. For every $m \ge 1$, $T_m = (x_1^*, \ldots, x_m^*) : \mathbb{R}^{\infty} \to \mathbb{R}^m$ is a surjection, and hence T has dense image. Since there is no strictly weaker linear topology on $\mathbb{R}^{\infty}$ (see [K]), the weak topology induced by F coincides with the original one. Hence, by Banach's Open Mapping Principle, T is a surjection.

We will show that $T(\Gamma) = \mathbb{Z}^\infty$, which yields that $T : \Gamma \to \mathbb{Z}^\infty$ is an isomorphism as required. Clearly, $T(\Gamma) \subseteq \mathbb{Z}^\infty$ and it is enough to show that $T^{-1}(\mathbb{Z}^\infty) \subseteq \Gamma$. One sees that $T^{-1}(\mathbb{Z}^\infty)$ is contained in $\Gamma^{**} = \{x \in \mathbb{R}^\infty : x^*(x) \in \mathbb{Z}$ for every $x^* \in \Gamma^*\}$, which is, by [DG, Theorem 2.1], the weak closure of Γ. Since the weak topology on $\mathbb{R}^\infty$ coincides with the original topology (see, e.g., [K]), $\Gamma^{**} = \Gamma$. Hence $T^{-1}(\mathbb{Z}^\infty) \subseteq \Gamma$. ∎

3. Weakly closed subgroups in Banach spaces. Recall that a subset A of a Banach space E is called *weakly closed* if it is closed in the weak topology on E generated by E^*, the dual of E. A space is *totally disconnected* if each of its points is the intersection of a family of clopen subsets.

Below we generalize [DG, Corollary 3.3] to arbitrary separable Banach spaces.

3.1. THEOREM. *If G is a line-free, weakly closed subgroup of a separable Banach space E, then G is totally disconnected and* $\dim(G) \le 1$.

We need the following counterpart of [DG, Theorem 3.2].

3.2. PROPOSITION. *If there exists a total sequence $A \subset E^*$ such that the dimension of G relative to the weak topology generated by A is $\le k$, then* $\dim(G) \le k+1$.

Proof. First note that the weak topology generated by A coincides with the weak topology ω generated by $\mathrm{span}(A) \subset E^*$. We claim that there exists an equivalent norm $\|\!|\cdot|\!\|$ on E such that the norm topology on $\|\!|\cdot|\!\|$-spheres coincides with the ω-topology. It then easily follows that $\dim(G) \le k+1$.

Since $\mathrm{span}(A)$ is a total linear space, it is weak $*$-dense in E^* (i.e., $\mathrm{span}(A)$ is dense in the weak $*$-topology). Moreover, $\mathrm{span}(A) \cap B^*$ is weak $*$-dense in $B^* = \{x^* \in E^* : \|x^*\| \le 1\}$ (see [Da, p. 20]). Now, inspecting the proof of Theorem 3.1 in [BP2, p. 177], we see that for an arbitrary sequence $\{x_i^*\}_{i=1}^\infty \subset B^*$ which is $*$-dense in B^* one can produce an equivalent norm $\|\!|\cdot|\!\|$ on E so that if $\|\!|x_n|\!\| \to \|\!|x_0|\!\|$ and $x_i^*(x_n) \to x_i^*(x_0)$, $i = 1, 2, \ldots$, then $\|\!|x_n - x_0|\!\| \to 0$ (see also [BP2, Remark, p. 178]). If we additionally pick $\{x_i^*\}_{i=1}^\infty \subset \mathrm{span}(A)$, then the above condition assures that the original topology coincides with the ω-topology on the $\|\!|\cdot|\!\|$-spheres. ∎

Proof of 3.1. Let $G^* = \{x^* \in E^* : x^*(G) \subseteq \mathbb{Z}\}$. Since G is weakly closed, the proof of [DG, Corollary 2.2] assures that G^* is total. Since E is separable, we can select a total sequence $A \subset G^*$. (Choose a countable subcover of $\{\{x \in E : x^*(x) \ne 0\} : x^* \in G^*\}$, the cover of $E \setminus \{0\}$.) Clearly G is 0-dimensional in the weak topology induced by A. By 3.2, we get $\dim(G) \le 1$. ∎

Below we show that every infinite-dimensional Banach space contains sufficiently many nontrivial weakly closed subgroups. A sequence $\{x_n\}_{n=1}^\infty$ in a Banach space E is called a *basic sequence* if it is a Schauder basis for the closed linear subspace $\overline{\mathrm{span}}\{x_n\}_{n=1}^\infty \subset E$. (A Schauder basis will often be called a basis.) By a result of Banach, every Banach space contains a basic sequence (see [BP2, p. 215]).

3.3. THEOREM. *Let G be a nondiscrete, closed subgroup of a Banach space E. Assume that either G is line-free or G contains an infinite-dimensional linear subspace of E. Then G contains a separable, nondiscrete, weakly closed, line-free subgroup G_0.*

3.4. PROPOSITION. *There exists a basic sequence $\{x_n\}_{n=1}^\infty \subset G \subset E$ such that $\|x_n\| \to 0$.*

Proof. Assume that G contains an infinite-dimensional linear subspace E_0. We can obviously require that E_0 is a closed subspace of E. By the classical result of Banach (see [BP2, p. 215]), E_0 contains a basic sequence $\{x_n\}_{n=1}^\infty$; we can have $\|x_n\| \to 0$.

Assume that G is line-free. Pick any nonzero $v_1 \in G$. Write $G_1 = \mathrm{span}\{v_1\} \cap G$. Consider the quotient group G/G_1 endowed with the "norm"

$$\|\!|[g]|\!\| = \inf\{\|g - y\| : y \in G_1\}.$$

We will find $v_2 \in G$ so that $\|v_2\| \le 3$, $\|v_2 - v_1\| \ge 1$ and $v_2/2 \in G$. To this end, use the fact that G/G_1 is not discrete and find an integer $n_2 \in \mathbb{N}$, $n_2 > 1$, and $g \in G$ with

$$1 \le 2^{n_2} \|\!|[g]|\!\| \le 2.$$

By definition of $\|\!|[g]|\!\|$, there exists $g_2 \in [g]$ such that

$$\|g_2\| \le \|\!|[g]|\!\| + \frac{1}{2^{n_2}}.$$

Then we have

$$\|2^{n_2} g_2\| \le 2^{n_2} \|\!|[g]|\!\| + 2^{n_2} \cdot \frac{1}{2^{n_2}} \le 2 + 1 = 3;$$

and since $2^{n_2} g_2 \in [2^{n_2} g_2]$, then

$$\|2^{n_2} g_2 - v_1\| \ge 1.$$

We set $v_2 = 2^{n_2} g_2$.

Now, replace G_1 by $G_2 = \mathrm{span}\{v_1, v_2\} \cap G$ and argue in the same way to get $v_3 = 2^{n_3} g_3$, $g_3 \in G$, $n_3 > n_2$, $\|v_3\| \le 3$ and $\|v_i - v_j\| \ge 1$ for $i \ne j$, $1 \le i, j \le 3$. Inductively, we are able to find $v_1, v_2, \ldots$ so that $v_i = 2^{n_i} g_i$, $g_i \in G$ and

$$\|v_i\| \le 3 \quad \text{and} \quad \|v_i - v_j\| \ge 1 \quad \text{for } i \ne j.$$

Since $\{v_i\}_{i=1}^\infty$ is bounded, it contains a subsequence that converges to $v \in E$ in the weak topology. (We may assume that E is separable to insure that closed balls in E are metric compacta in the weak topology.) It follows that the sequence $\{v_i - v_j\}_{i \neq j}$ with the lexicographic order has a subsequence that weakly converges to $0 = v - v$. By the Bessaga–Pełczyński Selection Principle (see [Di, p. 42]) such a subsequence contains a basic sequence $\{w_k\}_{k=1}^{\infty}$. (Formally, the Bessaga–Pełczyński Selection Principle in [Di] is formulated for normalized sequences $\{x_n\}_{n=1}^{\infty}$ only; however, it is clear that the principle holds for sequences $\{x_n\}_{n=1}^{\infty}$ that are bounded away from 0.) Clearly, each w_k is of the form

$$w_k = w^{m_k} g_{n_k} - 2^{l_k} g_{r_k},$$

where $\lim m_k = \lim l_k = \infty$. We see that $w_k \in G$. Let $s_k = \min\{m_k, l_k\}$. We have

$$w_k = w^{s_k}(2^{m_k - s_k} g_{n_k} + 2^{l_k - s_k} g_{r_k}).$$

Letting $x_k = 2^{m_k - s_k} g_{n_k} + 2^{l_k - s_k} g_{r_k}$, we see that $\|x_k\| \to 0$ and so the sequence $\{x_k\}_{k=1}^{\infty}$ is as required. ■

Proof of 3.3. Let $\{x_n\}_{n=1}^{\infty}$ be a basic sequence as in 3.4. Set

$$G_0 = \Big\{ \sum_{n=1}^{\infty} m_n x_n \in E : m_n \in \mathbb{Z} \Big\}.$$

It is clear that G_0 is a subgroup of G. Since $\|x_n\| \to 0$, G_0 is nondiscrete. Using the basic functionals $x_n^*(\sum_{i=1}^{\infty} t_i x_i) = t_n$, one can readily check that G_0 is weakly closed. The fact that G_0 is line-free is shown in 5.1 below. ■

We will be interested in subgroups G with $\dim(G) \geq 1$. In searching for such groups it is convenient to isolate the following property of a sequence $\{x_n\}_{n=1}^{\infty}$ in a metric linear space $(E, |\cdot|)$:

(∗) If $\{t_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{N} \cup \{0\}$ such that $\sup\{|\sum_{n=1}^{k} t_n x_n| : k \in \mathbb{N}\} < \infty$, then $\sum_{n=1}^{\infty} t_n x_n$ converges in E.

3.5. PROPOSITION. *If a closed subgroup G of a complete metric linear space $(E, |\cdot|)$ contains a sequence $\{x_n\}_{n=1}^{\infty}$ that satisfies (∗) and $|x_n| \to 0$, then* $\dim(G) \geq 1$.

Proof. We shall show that if U is any $|\cdot|$-bounded open neighborhood of $0 \in E$, then there exists $x \in \partial U \cap G$ (∂U denotes the boundary of U).

Adopting the notation of [DG, Theorem 3.1], for $u \in U$ and $y \in E$ let $n(u, y) = \max\{n : u + ky \in U \text{ for } k = 0, 1, \ldots, n\}$. We inductively construct $n_1, n_2, \ldots$, $n_i \geq 0$, and $y_1, y_2, \ldots$ with $y_i \in G \cap U$. Set $n_1 = n(0, x_1)$ and $y_1 = n_1 x_1$. Then having defined y_k, let $n_{k+1} = n(y_k, x_{k+1})$ and $y_{k+1} = y_k + n_{k+1} x_{k+1}$. Consider the formal series $\sum_{k=1}^{\infty} n_k x_k$. Since for every p, the sum $\sum_{k=1}^{p} n_k x_k \in U$, we infer that $\sup\{|\sum_{k=1}^{p} n_k x_k| : p \in \mathbb{N}\} < \infty$. By (∗), we conclude that $\sum_{k=1}^{\infty} n_k x_k$ converges in E. Since $\mathrm{dist}_{|\cdot|}(\sum_{k=1}^{p} n_k x_k, \partial U) \leq |x_p|$, we see that $\sum_{k=1}^{\infty} n_k x_k \in \partial U$. Finally, using the fact that $\sum_{k=1}^{p} n_k x_k \in G$, we conclude that $x = \sum_{k=1}^{\infty} n_k x_k \in \partial U \cap G$. ■

Let E be a Banach space and $\{x_n\}_{n=1}^{\infty}$ be a Schauder basis for E. The basis is said to be *boundedly complete* if the condition $\sup\{\|\sum_{n=1}^{k} t_n x_n\| : k \in \mathbb{N}\} < \infty$ implies $\sum_{n=1}^{\infty} t_n x_n$ converges in E. This is always the case when E is reflexive (see [BP2, p. 241]).

3.6. COROLLARY. *Let E be a Banach space with a boundedly complete basis $\{x_n\}_{n=1}^{\infty}$ such that $\|x_n\| \to 0$. Then $G = \overline{\mathrm{gr}}\{x_n\}_{n=1}^{\infty}$, the closed group spanned on $\{x_n\}_{n=1}^{\infty}$, is weakly closed and* $\dim(G) = 1$. ■

3.7. Remark. By 3.4 and 3.6, every nondiscrete, closed subgroup which is either line-free or contains an infinite-dimensional linear subspace in a reflexive Banach space has dimension ≥ 1. This yields an alternative proof of [DG, Theorem 3.1]. ■

4. Nondiscrete, zero-dimensional subgroups in Banach spaces.

By c_0 we denote the Banach space of sequences that converge to 0 equipped with the norm $\|x\| = \sup_n |x_n|$. The ultimate goal of this section is to prove the following characterization result.

4.1. THEOREM. *A Banach space E contains a closed, nondiscrete, zero-dimensional subgroup if and only if E contains a copy of c_0.*

Let $\{x_n\}_{n=1}^{\infty}$ be a normalized Schauder basis in a Banach space E (i.e., $\|x_n\| = 1$ for $n = 1, 2, \ldots$) and let $a = (a_n) \in c_0$, $a_n \neq 0$. We define $\Gamma = \Gamma_a(\{x_n\})$ as

$$\Gamma = \Big\{ \sum_{n=1}^{\infty} t_n x_n : t_n / a_n \in \mathbb{Z} \Big\}.$$

It is clear that Γ is a weakly closed, nondiscrete subgroup of E.

4.2. LEMMA. *Let $\{e_n\}_{n=1}^{\infty}$ be the basis in c_0 consisting of the unit vectors e_n. Then for every $a = (a_n) \in c_0$, $a_n \neq 0$, the group $\Gamma = \Gamma_a(\{e_n\})$ is zero-dimensional (as well as nondiscrete and weakly closed).*

Proof. Let $A = (0, \infty) \setminus \bigcup_{n=1}^{\infty} a_n \mathbb{Z}$. We claim that for every $\varepsilon \in A$, $\partial B(\varepsilon) \cap G = \emptyset$, where $B(\varepsilon) = \{x \in c_0 : \|x\| < \varepsilon\}$. To check this let $x = (x_i) \in \partial B(\varepsilon) \cap G$. Then $\|x\| = |x_{i_0}|$ for some $i_0 \in \mathbb{N}$. Since $x_{i_0}/a_{i_0} \in \mathbb{Z}$, we infer that $\varepsilon \in a_{i_0}\mathbb{Z}$; a contradiction. ■

The sufficiency part of 4.1 requires the following fact concerning the "extraction" of c_0 from a Banach space E (cf. [BP1, Theorem 5]; see also [Di, p. 45]).

4.3. PROPOSITION. *Let* $\{y_n\}_n^\infty = 1 \subset E$ *be a sequence with* $\sum_{n=1}^\infty |x^*(y_n)| < \infty$ *for every* $x^* \in E^*$. *Assume that for every* n *there is a choice of signs* $\{\varepsilon_j^n\}_{j=1}^n \subset \{-1,0,1\}$ *such that, if we set*

$$z_n = \varepsilon_1^n y_1 + \varepsilon_2^n y_2 + \ldots + \varepsilon_n^n y_n\,,$$

for $n = 1,2,\ldots$, *then the sequence* $\{z_n\}_{n=1}^\infty$ *is discrete in* E (*i.e., no subsequence of* $\{z_n\}_{n=1}^\infty$ *converges*). *Then* E *contains a copy of* c_0.

Proof. Consider the first "column" of signs $\{\varepsilon_1^n\}_{n=1}^\infty$ and select a subsequence $\{n_k^1\}$ so that $\varepsilon_1^{n_k^1} = \text{const}$. Set $z_{n_1} = z_{n_1^1}$. Then consider the second "column" $\{\varepsilon_2^{n_k^1}\}_{k=2}^\infty$ and select a subsequence $\{n_k^2\}$ of $\{n_k^1\}_{k=2}^\infty$ so that $\varepsilon_2^{n_k^2} = \text{const}$. Set $z_{n_2} = z_{n_1^2}$. Using the diagonal procedure we can extract a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that, if $k > l$, then $z_{n_l} - z_{n_k}$ is a finite sum of $\{y_m\}_{m=l+1}^\infty$ with coefficients in $\{\pm 2, \pm 1, 0\}$. For each k we define $p(k)$ to be the "length" of z_{n_k}, i.e., if $p > p(k)$ then y_p does not appear in the description of z_{n_k} (or, more precisely, it appears with the coefficient 0). We can assume that $p(1) < p(2) < \ldots$ We consider the subsequence $\{z_{p(k)}\}_{k=1}^\infty$ of $\{z_n\}$. Since $\{z_n\}$ is discrete, $\{z_{p(k)}\}_{k=1}^\infty$ fails to satisfy the Cauchy property. Hence, there are pairs of integers (k_m, l_m) with $k_m < l_m < k_{m+1}$, $m = 1,2,\ldots$, such that

$$\|z_{p(k_m)} - z_{p(l_m)}\| \ge \delta$$

for some $\delta > 0$. By our construction, $z_{p(k_m)} - z_{p(l_m)}$, $m \ge 2$, is a sum of

$$\{y_n\}_{n=p(k_m)+1}^{p(k_{m+1})}$$

with coefficients in $\{\pm 2, \pm 1, 0\}$. Let $x_m = z_{p(k_m)} - z_{p(l_m)}$. The above property yields

$$\sum_{m=2}^\infty |x^*(x_m)| \le 2 \sum_{n=2}^\infty |x^*(y_n)| < \infty$$

for every $x^* \in E^*$. This shows that the sequence $\{x_m\}$ weakly converges to 0; moreover, we also have $\|x_m\| \ge \delta > 0$ for $m = 1,2,\ldots$ Consequently, by the Bessaga–Pełczyński Selection Principle [Di, p. 42], we can extract a subsequence of $\{x_m\}$ that is basic. Now, the result of [BP1, Lemma 3] (see also [Di, Corollary 7, p. 45]) implies that E contains a copy of c_0. ■

4.4. LEMMA. *If a Banach space* E *admits a closed, nondiscrete, zero-dimensional subgroup* G, *then one can select a sequence* $\{y_n\}_{n=1}^\infty \subset G$ *that satisfies the assumption of* 4.3.

Proof. Since G is zero-dimensional, there exists an open, bounded neighborhood of 0 in E such that $\partial U \cap G = \emptyset$. Pick a sequence $\{g_n\}_{n=1}^\infty \subset G \cap U$ with $\|g_n\| \to 0$. For $u \in U$ and $x \in E$ define

$$m(u,x) = \max\{n : u + kx \in U \text{ for } k = 0, \pm 1, \ldots, \pm n\}\,.$$

Set $y_1 = m(0,g_1)g_1$ and observe the $\pm y_1 \in G \cap U$. Moreover, there exists $\varepsilon \in \{-1,1\}$ so that

$$\operatorname{dist}_{\|\cdot\|}(\varepsilon y_1, \partial U) \le \|g_1\|\,.$$

To construct y_2, let $m = \min\{m(y_1,g_2), m(-y_1,g_2), m(0,g_2)\}$. Set $y_2 = mg_2$ and note that

$$\pm y_2, \pm y_1 \pm y_2 \in G \cap U\,.$$

Moreover, there are $\varepsilon_1, \varepsilon_2 \in \{0, \pm 1\}$ so that

$$\operatorname{dist}_{\|\cdot\|}(\varepsilon_1 y_1 + \varepsilon_2 y_2, \partial U) \le \|g_2\|\,.$$

Inductively, we can obtain vectors $\{y_n\}_{n=1}^\infty \subset G \cap U$ so that for every finite set $\Delta \subset \mathbb{N}$ we have

(i) $$\sum_{i \in \Delta} \pm y_i \in G \cap U$$

and there is $\{\varepsilon_j^n\}_{j=1}^n \subset \{0, \pm 1\}$ so that, writing $z_n = \sum_{j=1}^n \varepsilon_j^n y_j$, we obtain

(ii) $$\operatorname{dist}_{\|\cdot\|}(z_n, \partial U) \le \|g_n\|\,.$$

By (i) and [Di, Theorem 6, p. 44], we have $\sum_{n=1}^\infty |x^*(y_n)| < \infty$ for every $x^* \in E^*$. Since $\|g_n\| \to 0$ and $\partial U \cap G = \emptyset$, it follows from (ii) that no subsequence of $\{z_n\}_{n=1}^\infty$ converges in E. ■

Proof of 4.1. Combine 4.2, 4.3 and 4.4. ■

4.5. QUESTION. Let G be a (nondiscrete) zero-dimensional, closed subgroup of a separable Banach space E. Assume $G \cap E_0$ is nondiscrete for every infinite-dimensional linear subspace $E_0 \subset E$. Is then $\overline{\operatorname{span}}(G)$, the closed linear span of G, isomorphic to c_0?

5. Remarks and questions on isomorphism and topological classifications. When we say that two topological groups are isomorphic, we mean that they are topological-group isomorphic. We will show that there are continuum many nonisomorphic 0-dimensional and 1-dimensional subgroups living in Banach spaces. Let us recall that $\Gamma_a = \Gamma_a(\{x_n\}) = \{\sum_{n=1}^\infty t_n x_n : t_n/a_n \in \mathbb{Z}\}$, where $\{x_n\}_{n=1}^\infty$ is a normalized basis in a Banach space E and $a = (a_n) \in c_0$, $a_n \ne 0$.

5.1. LEMMA. *Let* E *be a Banach space with a normalized basis* $\{x_n\}_{n=1}^\infty$, *and let* $a = (a_n) \in c_0$, $a_n \ne 0$. *Then the group* Γ_a *is weakly closed and line-free.*

Proof. Clearly Γ_a is weakly closed (see the argument from the end of the proof of 3.1). To see that Γ_a is line-free, we consider the sequence of coordinate functionals $\{x_n^*\}$ associated with the basis $\{x_n\}$. For every n, $x_n^*(\Gamma_a)$ is obviously discrete. Hence, the image of any line in Γ_a under x_n^* is $\{0\}$. Since the sequence $\{x_n^*\}$ is total, it follows that Γ_a is line-free. ∎

Repeating the argument of [DG, Theorem 2.11] and using the fact that the sequence of coordinate functions $x_n^*(\sum_{i=1}^{\infty} t_i x_i) = t_n$ is equicontinuous we get the following.

5.2. PROPOSITION. *Let E be a Banach space with a normalized basis $\{x_n\}_{n=1}^{\infty}$. If $a = (a_n)$, $b = (b_n) \in c_0$ are positive monotone sequences with $(a_n/b_n) \in c_0$, then there is no continuous homomorphism of $\Gamma_a(\{x_n\})$ onto $\Gamma_b(\{x_n\})$.* ∎

We say that a basis $\{x_n\}_{n=1}^{\infty}$ is *stable* if, for any two sequences of reals $\{s_n\}$ and $\{t_n\}$ such that $0 < \inf\{s_n/t_n\} \le \sup\{s_n/t_n\} < \infty$, the series $\sum_{n=1}^{\infty} s_n x_n$ converges if and only if $\sum_{n=1}^{\infty} t_n x_n$ converges. The standard bases in l^p-spaces, $1 \le p < \infty$, and c_0 are stable.

5.3. Remark. If a basis $\{x_n\}_{n=1}^{\infty}$ is stable, then the formula

$$\sum_{i=1}^{\infty} t_i x_i \to \sum_{i=1}^{\infty} t_i \frac{b_i}{a_i} x_i$$

establishes an isomorphism of Γ_a onto Γ_b provided $\lim a_n/b_n = g$ and $0 < g < \infty$. ∎

5.4. COROLLARY. *The space c_0 contains continuum many nondiscrete, weakly closed, zero-dimensional subgroups Γ_a that are pairwise nonisomorphic. Every such Γ_a is homeomorphic to $\mathbb{Z}^{\infty}$ (which is homeomorphic to the space of irrationals).*

Proof. Let $a(t) = (t^n) \in c_0$ for $0 < t < 1$. Set $\Gamma_{a(t)} = \Gamma_{a(t)}(\{e_n\})$. If $t < s$, then $(t^n/s^n) \in c_0$; and by 5.2, $\Gamma_{a(t)}$ is not isomorphic to $\Gamma_{a(s)}$. By 4.2, each $\Gamma_{a(t)}$ is nondiscrete, weakly closed and zero-dimensional.

According to the characterization of the irrationals (due to Alexandrov and Urysohn) (see [E, p. 29]), every complete-metrizable, zero-dimensional space without a compact open set is homeomorphic to $\mathbb{Z}^{\infty}$. By 1.1(b), the groups $\Gamma_{a(t)}$ are not locally compact. ∎

5.5. COROLLARY. *Every infinite-dimensional Banach space E contains continuum many nonisomorphic, line-free, weakly closed (hence, totally disconnected, see 3.1) subgroups of dimension 1.*

Proof. We consider two cases: E is a copy of c_0, or E does not contain a copy of c_0. In both cases we define the groups using $\Gamma_{a(t)}$ with respect to some normalized basis and the sequence $a(t)$ being that of the proof of 5.4.

If E is c_0, then we let

$$z_n = (\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots).$$

Note that $\{z_n\}_{n=1}^{\infty}$ is a normalized basis that satisfies condition $(*)$ of Section 3. Indeed, if $\|\sum_{n=1}^{p} t_n z_n\| \le M$ and $t_n \ge 0(!)$, then by examining the first coordinate in c_0, we conclude that the positive series $\sum_{n=1}^{\infty} t_n$ converges. This yields the convergence of $\sum_{n=1}^{\infty} t_n z_n$ in c_0. Since the sequence $a(t) = \{a_n(t)\}$ consists of positive numbers, it is clear that condition $(*)$ also holds for $\{x_n\} = \{a_n(t) z_n\}$. It follows from 3.5 that $\dim(\Gamma_{a(t)}(\{z_n\})) \ge 1$.

If E does not contain c_0, we pick an arbitrary normalized basic sequence $\{x_n\}_{n=1}^{\infty}$ in E (see [BP2, p. 215]). Then, by 4.1, $\dim(\Gamma_{a(t)}(\{x_n\})) \ge 1$. By 3.1 and 5.1, we infer that in either case $\dim(\Gamma_{a(t)}) \le 1$. Now, apply 5.2 to conclude that the $\Gamma_{a(t)}$ are pairwise nonisomorphic. ∎

5.6. Remark. Fix the standard basis $\{e_n\}_{n=1}^{\infty}$ in l^2 and a monotone positive sequence $a = (a_n) \in c_0$ with $\lim a_n/a_{2n} < \infty$. The group $\Gamma = \Gamma_a(\{e_n\})$ has the following properties:

(a) Γ is a complete-metrizable group,
(b) Γ is totally disconnected,
(c) $\dim(\Gamma) = 1$,
(d) Γ is isomorphic to its finite product,
(e) $\dim(\Gamma^{\infty}) = 1$.

To show (d) it is enough to check that Γ is isomorphic to its square. Consider the standard isomorphism $\varphi : l^2 \to l^2 \times l^2$ given by $\varphi((t_n)) = ((t_{2n-1}), (t_{2n}))$. If we represent $\Gamma \times \Gamma$ by $\Gamma^1 \times \Gamma^2$, where $\Gamma^1 = \{(x_i) \in l^2 : x_i/a_{2i-1} \in \mathbb{Z}\}$ and $\Gamma^2 = \{(y_i) \in l^2 : y_i/a_{2i} \in \mathbb{Z}\}$, then we see that φ is an isomorphism of Γ onto $\Gamma \times \Gamma$. By 5.3 and the property of (a_n), both Γ^1 and Γ^2 are isomorphic to Γ. To verify (e), represent Γ^{∞} as the limit of the inverse sequence of finite products of Γ. Since the limit of an inverse sequence of separable metric spaces with dimension $\le n$ is itself of dimension $\le n$, Γ^{∞} is of dimension 1. ∎

Finally, let us ask some questions related to the topological classification of closed subgroups G in Banach spaces.

5.7. QUESTION. Does there exist a line-free G with $1 < \dim(G) < \infty$?

5.8. QUESTION. Let G_0 be the connected component of 0 in G. Is G homeomorphic to $G_0 \times (G/G_0)$?

5.9. QUESTION. Assume G is locally connected. Is then G a Hilbert space manifold?

The separable case of 5.9 reduces via a result of [DT] to the verification that G is an absolute neighborhood retract; it is, however, unclear whether such a G must be even locally connected in dimension 1.

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Almost everywhere convergence of Laguerre series

by

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Abstract. Let $a \in \mathbb{Z}^+$ and $f \in L^p(\mathbb{R}^+), 1 \le p \le \infty$. Denote by c_j the inner product of f and the Laguerre function $\mathcal{L}_j^a$. We prove that if $\{c_j\}$ satisfies

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{n<j\le\lambda n} |\Delta^k c_j| j^{k/2-1/4} = 0 \quad \text{and} \quad |c_j| j^{k/2-1/4} = o(1) \quad \text{as } j \to \infty$$

for some $k \in \mathbb{N}$, then the Laguerre series $\sum c_j \mathcal{L}_j^a$ converges to f almost everywhere.

1. Introduction. Let $L_n^a(t)$ denote the nth Laguerre polynomial of order a on $\mathbb{R}$,

$$L_n^a(t) = \frac{1}{n!} t^{-a} e^t \frac{d^n}{dt^n}(t^{n+a} e^{-t}), \quad a > -1,\ n = 0, 1, 2, \ldots,$$

or, equivalently,

$$L_n^a(t) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \binom{n+a}{n-k} t^k, \quad a > -1,\ n = 0, 1, 2, \ldots$$

The Laguerre polynomials form a complete orthogonal system in $L^2(\mathbb{R}^+, t^a e^{-t} dt)$ and satisfy the summation formula [13, p. 102]

$$\sum_{k=0}^{n} L_k^a(t) = L_n^{a+1}(t)\,. \tag{1}$$

It is well known (cf. [9, p. 348]) that

$$|L_n^a(t)| = O(e^{t/2} t^{-a/2-1/4} n^{a/2-1/4})\,.$$

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