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Complemented ideals of group algebras

by

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Abstract. The existence of a projection onto an ideal $\mathcal{I}$ of a commutative group algebra $L^1(G)$ depends on its hull $Z(\mathcal{I}) \subseteq \widehat{G}$. Existing methods for constructing a projection onto $\mathcal{I}$ rely on a decomposition of $Z(\mathcal{I})$ into simpler hulls, which are then reassembled one at a time, resulting in a chain of projections which can be composed to give a projection onto $\mathcal{I}$. These methods are refined and examples are constructed to show that this approach does not work in general. Some answers are also given to previously asked questions concerning such hulls and some conjectures are presented concerning the classification of these complemented ideals.

0. Introduction. In this paper, we will examine the problem of determining when a closed ideal $\mathcal{I}$ of a commutative group algebra $L^1(G)$ has a Banach space complement. This examination will be based on the techniques and results developed in the paper [4] of D. Alspach, A. Matheson and J. Rosenblatt. It will be shown that the techniques from this paper are not always able to construct a projection onto an ideal when one does exist. The ideals illustrating this are constructed in $L^1(\mathbb{R}^3)$ and $L^1(\mathbb{R}^4)$, so that if we are to continue the successful sequence of papers [3], [1], ..., then we will require some techniques beyond those described in [4]. We will also consider the questions asked in the concluding section of [4], and consider questions that seem appropriate to the search for a complete characterization of the complemented ideals of group algebras.

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1. Definitions and basic concepts. For a commutative semisimple Banach algebra $\mathfrak{A}$ with carrier space (or maximal ideal space) $\Phi_{\mathfrak{A}} \subseteq \mathfrak{A}^*$, the *hull* of a set $\mathfrak{X} \subseteq \mathfrak{A}$ is $Z_{\mathfrak{A}}(\mathfrak{X}) = \mathfrak{X}^{\perp} \cap \Phi_{\mathfrak{A}}$, and the *kernel* of a set $X \subseteq \Phi_{\mathfrak{A}}$

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is $\mathcal{I}_{\mathfrak{A}}(X) = {}^{\perp}X$. The subscript $\mathfrak{A}$ will usually be omitted. We say $X \subseteq \Phi_{\mathfrak{A}}$ is *a hull* if it is the hull of some $\mathfrak{X} \subseteq \mathfrak{A}$; in particular, $X = Z(\mathcal{I}(X))$. Basic properties of hull and kernel can be found in [6, Section 23]. We say $\mathfrak{A}$ is *regular* if every closed $X \subseteq \Phi_{\mathfrak{A}}$ is a hull and we say a hull $X \subseteq \Phi_{\mathfrak{A}}$ is a set of *synthesis* if $\mathcal{I}(X)$ is the unique closed ideal with hull X.

The commutative harmonic analysis we will use is largely contained in the book [14] of W. Rudin, and we will mostly follow the notational conventions therein. In particular, G is always a locally compact abelian group with dual Γ, and each has group operation $+$. We will denote the set of compact symmetric neighbourhoods of the identity in G by $\mathcal{U}_G$. We identify Γ with the carrier space of the convolution algebra $L^1(G)$ and also with a subset of the carrier space of the convolution algebra $M(G)$. We will also use the identification of $L^1(G)$ with its Fourier transform $A(\Gamma)$, and likewise identify $M(G)$ with $B(\Gamma)$. A *coset* in Γ is a translate E of a subgroup Λ. Note that $\Lambda = E - E$. It is a simple matter to transfer many objects and results associated with Λ that are of a translation-invariant nature to corresponding objects and results associated with E. For example, an open coset is closed. The *coset ring* of E, written $\mathcal{R}(E)$, is the Boolean ring generated by the (relatively) open subcosets of E. This consists of translates of sets in $\mathcal{R}(\Lambda)$. We also define $A(E)$ and $B(E)$ to be the algebras of functions on E that are translates of functions in $A(\Lambda)$ and $B(\Lambda)$ respectively, with norms such that this translation is isometric. Then $A(E) = A(\Gamma)|_E = \{f|_E : f \in A(\Gamma)\}$ is a regular Banach algebra of continuous functions on E such that $\Phi_{A(E)} = E$. Similarly $B(E) = B(\Gamma)|_E$ is a Banach algebra of continuous functions on E, and $A(E)$ is clearly a closed ideal of $B(E)$. Cohen's idempotent measure theorem ([14, Theorem 3.1.3]) gives us that $\{\chi_X : X \in \mathcal{R}(E)\}$ is the set of idempotents in $B(E)$. With E_{d} the coset E with its discrete topology (a coset in Γ_{d}) we denote by $\mathcal{R}_{\mathrm{c}}(E)$ the set of closed subsets X of E with $X \in \mathcal{R}(E_{\mathrm{d}})$. In specifying subsets of locally compact abelian groups, we will follow the notational conventions of [4] that algebraic operations take precedence over cartesian products, which in turn take precedence over set operations. We may also use redundant parentheses as an aid to clarity. In this context, if $X \subseteq G$, then nX is the set $X + \ldots + X = \{x_1 + \ldots + x_n : x_1, \ldots, x_n \in X\}$, rather than its subset $\{nx : x \in X\}$, *except* in the case $n\mathbb{Z} = \{nm : m \in \mathbb{Z}\}$.

If $\mathfrak{X}$ is a Banach space with a closed subspace $\mathfrak{Y}$ we will use two equivalent criteria for the existence of a Banach space complement to $\mathfrak{Y}$—firstly the existence of a continuous projection (henceforth a *projection*) $\mathfrak{X} \to \mathfrak{Y}$ and secondly the existence of a linear transformation $T \in \mathcal{B}(\mathfrak{X}/\mathfrak{Y}, \mathfrak{X})$ with $Q_{\mathfrak{Y}} \circ T = I_{\mathfrak{X}/\mathfrak{Y}}$. (Here $Q_{\mathfrak{Y}}$ is the quotient mapping $\mathfrak{X} \to \mathfrak{X}/\mathfrak{Y}$ and $I_{\mathfrak{X}/\mathfrak{Y}}$ is the identity mapping on $\mathfrak{X}/\mathfrak{Y}$.) Such a continuous linear right inverse to the quotient mapping is commonly called a *splitting map*, since it gives a splitting of the short exact sequence $0 \to \mathfrak{Y} \hookrightarrow \mathfrak{X} \to \mathfrak{X}/\mathfrak{Y} \to 0$. Such a criterion appeared in a different guise in [4], where a closed subspace $\mathfrak{Z}$ of $\mathfrak{X}$ was sought such that $Q_{\mathfrak{Y}}|_{\mathfrak{Z}} : \mathfrak{Z} \to \mathfrak{X}/\mathfrak{Y}$ is an isomorphism.

As in [4], our approach to the problem of characterizing the complemented ideals in group algebras starts with the result [12, Theorem 1.4] of H. P. Rosenthal, which states that if $\mathcal{I}$ is complemented, then $X = Z(\mathcal{I}) \in \mathcal{R}_{\mathrm{c}}(\Gamma)$, and since such X is of synthesis (see [7, Theorem 3.9]), we have $\mathcal{I} = \mathcal{I}(X)$. Thus the problem is that of characterizing those $X \in \mathcal{R}_{\mathrm{c}}(\Gamma)$ for which $\mathcal{I}(X)$ has a Banach space complement, that is, those X with *complemented kernel*.

It is clear that if $E_1, \ldots, E_n$ are closed cosets in Γ and $X_k \in \mathcal{R}(E_k)$ $(1 \le k \le n)$, then $\bigcup_{k=1}^n X_k$ lies in $\mathcal{R}_{\mathrm{c}}(\Gamma)$. Also, by [7, Theorem 3.1], the converse holds, in that for any $X \in \mathcal{R}_{\mathrm{c}}(\Gamma)$, there exist closed cosets $E_1, \ldots, E_n$ and sets $X_k \in \mathcal{R}(E_k)$ with $X = \bigcup_{k=1}^n X_k$. Moreover, we can assume that each X_k is of the form

$$X_k = E_k \Big\backslash \Big(\bigcup_{1 \le j \le m_k} E_{kj} \Big),$$

where $m_k \ge 0$ and each E_{kj} is a relatively open subcoset of E_k. Such X_k we will call *elementary sets*, and we will denote the set of all such sets by $\mathcal{R}_{\mathrm{e}}(\Gamma)$. In particular, closed cosets and the empty set are elementary sets, and a finite intersection of elementary sets is elementary. We could, if we liked, also assume that a decomposition into elementary sets has the property that each E_{kj} is of infinite index in E_k. Little seems to be gained by this, and the resultant class of elementary sets is not closed under finite intersections. The example $2\mathbb{Z}\backslash\{0\} = (4\mathbb{Z}\backslash\{0\}) \cup (4\mathbb{Z}+2) = \mathbb{Z}\backslash((2\mathbb{Z}+1)\cup\{0\})$ shows that the decomposition of a set $X \in \mathcal{R}_{\mathrm{c}}(\Gamma)$ into elementary sets is not always uniquely determined and the coset E_k is not always uniquely determined by the elementary set X_k. Elementary sets were shown to have complemented kernel in [11, Lemma 11].

A decomposition of $X \in \mathcal{R}_{\mathrm{c}}(\Gamma)$ into elementary sets provides a possibility for constructing a projection $L^1(G) \to \mathcal{I}(X)$, since we have

$$L^1(G) \supseteq \mathcal{I}(X_1) \supseteq \mathcal{I}(X_1 \cup X_2) \supseteq \ldots \supseteq \mathcal{I}(X),$$

so that if there exists projections

$$L^1(G) \xrightarrow{P_1} \mathcal{I}(X_1) \xrightarrow{P_2} \mathcal{I}(X_1 \cup X_2) \xrightarrow{P_3} \ldots \xrightarrow{P_n} \mathcal{I}(X), \tag{1.1}$$

then $P_n \circ \ldots \circ P_1$ is the required projection onto $\mathcal{I}(X)$. We will call (1.1) an *elementary chain of projections* onto $\mathcal{I}(X)$. One aspect of this type of construction relevant to the current paper is that there are configurations of elementary sets $X_1, \ldots, X_n$ for which there is no elementary chain of projections as in (1.1), but if $X_1, \ldots, X_n$ are permuted, then an elementary chain of projections does exist. Examples illustrating this phenomenon will be discussed below, and we will consider how these can be used to construct hulls $X \in \mathcal{R}_c(\Gamma)$ which have complemented kernel, but for which we cannot construct an elementary chain of projections.

The first such example constructed in this paper is one where the hull X is a finite union of subgroups $\Lambda_1, \ldots, \Lambda_n$ such that $\mathcal{I}(X)$ is complemented, but for which there is no permutation π of $\{1, \ldots, n\}$ for which there exists a chain of projections

$$(1.2)\quad L^1(G) \to \mathcal{I}(\Lambda_{\pi(1)}) \to \mathcal{I}(\Lambda_{\pi(1)} \cup \Lambda_{\pi(2)}) \to \ldots \\ \ldots \to \mathcal{I}(\Lambda_{\pi(1)} \cup \ldots \cup \Lambda_{\pi(n)}).$$

Thus, there is no elementary chain of projections relying on the given decomposition of X into elementary sets. This answers Question 4.1 of [2]. However, we will show in Proposition 3.4 that in this case there is an elementary chain of projections based on an alternative decomposition of X into elementary sets. This example does, nonetheless, cast doubt on the existence of an elementary chain of projections onto an arbitrary complemented ideal in a group algebra. This doubt is borne out by a second example, which is similar, but for which no representation of the hull as a union of elementary sets can give a chain of projections.

Associated with these two examples are many ideals which will need to be proven either complemented or non-complemented. Such demonstrations require some new methods, due to the nature of the examples. The next section develops these, as well as surveying some known results.

2. Construction of splitting maps. We start with a description of the quotient $A(\Gamma)/\mathcal{I}(X)$ for sets $X \in \mathcal{R}_c(\Gamma)$. Standard Banach algebra theory (for example, [6, Section 23]) tells us that since X is a hull, we can identify this quotient with the algebra of functions $A(\Gamma)|_X$, which has carrier space X. We denote this algebra by $A(X)$. With this identification, the quotient mapping $A(\Gamma) \to A(\Gamma)/\mathcal{I}(X)$ corresponds to the restriction mapping $\varrho_X : f \mapsto f|_X$ and the quotient norm is given by $\|f\|_{A(X)} = \inf\{\|g\| : g \in A(\Gamma)$ and $g|_X = f\}$. Also, the splitting map we need for complementation corresponds to a continuous linear right inverse to ϱ_X—that is, a continuous linear map $T : A(X) \to A(\Gamma)$ such that for each $f \in A(X)$, $T(f)$ is an extension of f. The existence of such a map T, which we will also call a splitting map, is a convenient criterion for complementation that will be widely used below.

To facilitate use of this criterion, we now consider the structure of the quotient algebras $A(X)$. The above definition of $A(X)$ agrees with the standard one in the case where X is a closed subgroup of Γ, by [14, Theorem 2.7.4], and consequently in the case where E is a closed coset in Γ. Given this, the description of $A(X)$ for elementary X is particularly simple.

Throughout this section, if we have functions $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ satisfying $f|_{X\cap Y} = g|_{X\cap Y}$, then $f \cup g$ will denote their common extension to a function on $X \cup Y$. This corresponds to a union of graphs.

2.1. Lemma. *If E is a closed coset in Γ and $S \in \mathcal{R}(E)$, then*

$$A(S) = \{f \in C(S) : f \cup 0_{E\setminus S} \in A(E)\},$$

which is isomorphic, as a Banach algebra, to $\mathcal{I}_{A(E)}(E \setminus S)$.

Proof. If $f \in A(S)$, say $f = g|_S$ for some $g \in A(\Gamma)$, then $f \cup 0_{E\setminus S} = g|_E \cdot \chi_S \in A(E)$, as $\chi_S \in B(E)$. The remaining statements follow easily. ∎

2.2. Lemma. *Suppose $X, Y \in \mathcal{R}_c(\Gamma)$ and $f \in A(X)$, $g \in A(Y)$ are such that $f|_{X\cap Y} = g|_{X\cap Y}$. Then $f \cup g \in A(X \cup Y)$.*

Proof. Let $f_1, g_1 \in A(\Gamma)$ have $f_1|_X = f$ and $g_1|_Y = g$. Then $f_1 - g_1 \in \mathcal{I}(X \cap Y)$. However, as noted in [4, Section 1], $\mathcal{I}(X \cap Y) = \mathcal{I}(X) + \mathcal{I}(Y)$. Hence there exist $f_2 \in \mathcal{I}(X)$ and $g_2 \in \mathcal{I}(Y)$ with $f_1 - g_1 = f_2 - g_2$. Then $f_1 - f_2 = g_1 - g_2 \in A(\Gamma)$ has $(f_1 - f_2)|_X = f$ and $(g_1 - g_2)|_Y = g$, so that $f \cup g = (f_1 - f_2)|_{X\cup Y} \in A(X \cup Y)$. ∎

In the following theorem, the notation ϱ_X is again used to denote a restriction mapping, although in this case it occurs as a mapping $A(Y) \to A(X)$, where $X, Y \in \mathcal{R}_c(\Gamma)$ and $X \subseteq Y$. This is clearly a continuous algebra homomorphism. Also, if $\mathfrak{X}, \mathfrak{Y}_1, \ldots, \mathfrak{Y}_n$ are Banach spaces and for each k, $T_k : \mathfrak{X} \to \mathfrak{Y}_k$ is continuous and linear, then the continuous linear map $\mathfrak{X} \to \bigoplus_{k=1}^n \mathfrak{Y}_k$ given by $x \mapsto (T_1(x), \ldots, T_n(x))$ will be denoted by $(T_1, \ldots, T_n)$.

2.3. Theorem. *Supposing $X_1, \ldots, X_n \in \mathcal{R}_c(\Gamma)$ and $X = \bigcup_{k=1}^n X_k$, put*

$$\mathfrak{A} = \Big\{(f_1, \ldots, f_n) \in \bigoplus_{k=1}^n A(X_k) : f_j|_{X_j\cap X_k} = f_k|_{X_j\cap X_k}\ (1 \le j < k \le n)\Big\}.$$

Then $(\varrho_{X_1}, \ldots, \varrho_{X_n})$ is an algebra isomorphism from $A(X)$ onto $\mathfrak{A}$ whose continuous inverse is given by $(f_1, \ldots, f_n) \mapsto f_1 \cup \ldots \cup f_n$.

Proof. Clearly $(\varrho_{X_1}, \ldots, \varrho_{X_n})$ is a monomorphism into $\mathfrak{A}$. If $(f_1, \ldots, f_n) \in \mathfrak{A}$, then by Lemma 2.2, $f = f_1 \cup \ldots \cup f_n \in A(X)$, so that $(f_1, \ldots, f_n) = (\varrho_{X_1}, \ldots, \varrho_{X_n})(f)$. Hence $(\varrho_{X_1}, \ldots, \varrho_{X_n})$ is a continuous isomorphism onto $\mathfrak{A}$, a closed subalgebra of $\bigoplus_{k=1}^n A(X_k)$. By Banach's inversion theorem, $(\varrho_{X_1}, \ldots, \varrho_{X_n})^{-1}$ is continuous. ∎

2.3.1. Corollary. *For $X, Y \in \mathcal{R}_c(\Gamma)$, $\varrho_X : \mathcal{I}_{A(X\cup Y)}(Y) \to \mathcal{I}_{A(X)}(X \cap Y)$ is a continuous algebra isomorphism with inverse $f \mapsto f \cup 0_Y$.*

Theorem 2.3 can immediately be combined with the characterization of $\mathcal{R}_c(\Gamma)$ and Lemma 2.1 to give the next corollary, a description of the quotient algebra with which we are working. It is interesting to compare this with

[10, Theorem A], which uses essentially the same method to characterize the range of homomorphisms between commutative group algebras.

2.3.2. Corollary. *Suppose $X \in \mathcal{R}_c(\Gamma)$, say $X = \bigcup_{k=1}^n X_k$ where for each $1 \le k \le n$ there is a closed coset E_k in Γ such that $X_k \in \mathcal{R}(E_k)$. Then*

$$A(X) = \{f \in C(X) : f|_{X_k} \cup 0_{E_k \setminus X_k} \in A(E_k)\ (1 \le k \le n)\}.$$

The above results do not differ much in content from any of the results used in [4, Section 2]—they are just stated in a more convenient form.

We now consider some results whereby we can combine splitting maps for ideals $\mathcal{I}(X_1), \ldots, \mathcal{I}(X_n)$, to give a splitting map for $\mathcal{I}(X_1 \cup \ldots \cup X_n)$. The situation for disjoint $X_1, \ldots, X_n$ is quite straightforward. Recall from [5] that two sets $X, Y \subseteq \Gamma$ are called *uniformly separated* if there exists $U \in \mathcal{U}_\Gamma$ with $(X + U) \cap Y = \emptyset$. We say that $X \subseteq \Gamma$ is *uniformly discrete* if there exists $U \in \mathcal{U}_\Gamma$ with $(\gamma + U) \cap X = \{\gamma\}$ for each $\gamma \in X$. By [5, Theorem 0.2], uniform separation of $X, Y \in \mathcal{R}_c(\Gamma)$ is equivalent to the existence of a measure $\mu \in M(G)$ with $\widehat{\mu}(X) = \{0\}$ and $\widehat{\mu}(Y) = \{1\}$. A significant proportion of the results in this paper rely on this equivalence. The following proposition is a special case of [1, Theorem 2.2], and the corollaries are similar to results in [4, Section 2].

2.4. Proposition. *If $X, Y \in \mathcal{R}_c(\Gamma)$ are disjoint then $\mathcal{I}(X \cup Y)$ is complemented if and only if $\mathcal{I}(X)$, $\mathcal{I}(Y)$ are complemented and X and Y are uniformly separated.*

2.4.1. Corollary [4, Theorem 2.3]. *If $X \in \mathcal{R}_c(\Gamma)$ is discrete, then $\mathcal{I}(X)$ is complemented if and only if X is uniformly discrete.*

2.4.2. Corollary. *If $X, Y \in \mathcal{R}_c(\Gamma)$ and X is discrete, then $\mathcal{I}(X \cup Y)$ is complemented if and only if $\mathcal{I}(Y)$ is complemented and $X \setminus Y$ is uniformly discrete and uniformly separated from Y.*

Proposition 2.4 and its corollaries lead to a characterization of the complemented ideals in $L^1(G)$ in the case where the only closed subgroups of Γ are either discrete or open. Here $X \in \mathcal{R}_c(\Gamma)$ has complemented kernel if and only if ∂X, the boundary of X in Γ, is uniformly discrete. The only such groups Γ are those with an open subgroup isomorphic to one of $\{0\}$, $\mathbb{T}$ or $\mathbb{R}$. The case $\Gamma = \mathbb{R}$ is dealt with in this way in [3], and the case $\Gamma = \mathbb{R} \times \mathbb{Z}$ can also be seen as a consequence of the characterization of the complemented ideals of $L^1(\mathbb{R}^2)$ in [1].

The situation for non-disjoint sets $X_1, \ldots, X_n \in \mathcal{R}_c(\Gamma)$ is less straightforward. Considerable progress on this situation was made in Sections 2 and 3 of [4]. We start by considering some of the results therein. The following result is useful for showing that certain ideals are not complemented.

2.5. Proposition [4, Proposition 1.9]. *If $X, Y \in \mathcal{R}_c(\Gamma)$ and $\mathcal{I}(X)$, $\mathcal{I}(Y)$ and $\mathcal{I}(X \cup Y)$ are complemented, then $\mathcal{I}(X \cap Y)$ is complemented.*

The positive results of [4] concerning a situation involving intersecting elementary sets relies on a property of pairs of subgroups. Recall that a pair (Λ, Ξ) of closed subgroups of Γ satisfies (D) if

$$(\Lambda + \Xi)/(\Lambda \cap \Xi) = \Lambda/(\Lambda \cap \Xi) \oplus \Xi/(\Lambda \cap \Xi),$$

so that we have a *topological* direct sum, and in particular, $\Lambda + \Xi$ is closed. Note that by [8, Theorem 6.12], if Γ is σ-compact with closed subgroups Λ and Ξ, then (Λ, Ξ) satisfies (D) if and only if $\Lambda + \Xi$ is closed. We will mostly be considering hulls in Euclidean groups (those of the form $\mathbb{R}^n$), which are σ-compact so that we can use this criterion for (D).

If we denote rational dependence in $\mathbb{R}$ by $\equiv$, then for $\xi, \eta \in \mathbb{R}$, the pair $(\xi\mathbb{Z}, \eta\mathbb{Z})$ in $\mathbb{R}$ satisfies (D) if and only if $\xi \equiv \eta$. Similarly, in $\mathbb{R}^3$, $(\mathbb{R} \times \xi\mathbb{Z} \times \{0\}, \{0\} \times \eta\mathbb{Z} \times \mathbb{R})$ satisfies (D) if and only if $\xi \equiv \eta$. Also, if Λ, Ξ are closed subgroups of a Euclidean group with $\Lambda \cong \mathbb{R}^p$ and $\Xi \cong \mathbb{R}^q \times \mathbb{Z}$ for some $p, q \ge 0$, then (Λ, Ξ) satisfies (D). A different type of example arises if we consider any non-discrete locally compact abelian group Γ with a dense subgroup Λ. Give Λ its discrete topology and consider $\Delta_\Lambda = \{(\lambda, \lambda) : \lambda \in \Lambda\}$ as a subgroup of $\Gamma \times \Lambda$. Clearly $(\{0\} \times \Lambda) \cap \Delta_\Lambda = \{0\}$ and each of $\{0\} \times \Lambda$ and Δ_Λ is discrete, but $(\{0\} \times \Lambda) + \Delta_\Lambda = \Lambda \times \Lambda$ is not discrete, since it is dense in $\Gamma \times \Lambda$. Hence $(\{0\} \times \Lambda, \Delta_\Lambda)$ fails (D). Note that in the case $\Lambda = \Gamma_{\mathrm{d}}$, $(\{0\} \times \Lambda) + \Delta_\Lambda = \Gamma \times \Gamma_{\mathrm{d}}$, which is closed. Hence (D) in a pair (Λ, Ξ) is not in general equivalent to $\Lambda + \Xi$ being closed. The utility of property (D) is evident in the following result, which follows from Theorems 3.11 and 4.4 of [4].

2.6. Proposition. *If G is σ-compact and Ξ, Λ are closed subgroups of Γ, then $\mathcal{I}(\Xi \cup \Lambda)$ is complemented if and only if (Ξ, Λ) satisfies* (D).

As noted in [4, Section 3], it seems likely that Proposition 2.6 will also hold for non-σ-compact groups G. This will not concern us here, as all the specific hulls considered below occur in metrizable Γ, so that G is σ-compact. Some generally applicable criteria for (D) are contained in the following lemma, whose proof is straightforward.

2.7. Lemma. *If Λ, Ξ are closed subgroups of Γ, then the following are equivalent:*

(i) *The pair (Λ, Ξ) satisfies* (D).

(ii) *For each $U \in \mathcal{U}_\Gamma$, there exists $V \in \mathcal{U}_\Gamma$ with $(\Lambda + \Xi) \cap V \subseteq (\Lambda \cap U) + (\Xi \cap U)$.*

(iii) *For each $U \in \mathcal{U}_\Gamma$, there exists $V \in \mathcal{U}_\Gamma$ with $(\Lambda + V) \cap (\Xi + V) \subseteq (\Lambda \cap \Xi) + U$.*

It is interesting to compare (iii) with uniform separation—two disjoint sets $X, Y \subseteq \Gamma$ are uniformly separated if and only if there exists $V \in \mathcal{U}_\Gamma$ such that $(X+V)\cap(Y+V)=\emptyset$. This suggests some generalizations of (D) applicable to pairs of elementary sets. If S_1 and S_2 are elementary sets, we say that (S_1,S_2) satisfies

(D′) if for all $U \in \mathcal{U}_\Gamma$ there exists $V \in \mathcal{U}_\Gamma$ with $(S_1+V)\cap(S_2+V)\subseteq (S_1\cap S_2)+U$;

(D_0) if either S_1 and S_2 are uniformly separated or there exists $\gamma \in S_1 \cap S_2$ and a pair of closed subgroups (Λ_1,Λ_2) which satisfy (D) such that $S_1 \in \mathcal{R}(\Lambda_1+\gamma)$ and $S_2 \in \mathcal{R}(\Lambda_2+\gamma)$.

It is straightforward to show that (D_0) implies (D′) and that (D′) implies (D_0) for pairs of closed cosets. This second implication can also be shown to hold for any pair of elementary sets, although the technicalities in the proof render it unsuitable for the present discussion.

The inductive procedure developed in [4, Section 3] gives an elementary chain of projections for a large class of hulls. The strongest result explicitly stated therein was Theorem 3.11, and the comment following this theorem states that a small improvement can be made. With this improvement, we obtain the following theorem. Note that the apparent reduction in the number of pairs of subgroups requiring a (D)-like condition is merely a result of a re-indexing—the index s in [4] is replaced by $k+1$ below, and the pairs (Γ_i,Γ_n) in [4] appear below in the form $(X_j\cap\bigcap_{i=n+1}^n X_i, \bigcap_{i=n}^n X_i)=(X_j,X_n)$. The corollaries follow directly from the theorem, and are included to facilitate the application of the theorem in this paper.

2.8. THEOREM. *Suppose G is σ-compact and $X=\bigcup_{k=1}^n X_k$, where $X_1,\ldots,X_n \in \mathcal{R}_e(\Gamma)$. If all pairs $(X_j\cap\bigcap_{i=k+1}^n X_i, \bigcap_{i=k}^n X_i)$ $(1\le j<k\le n)$ satisfy (D_0), then $\mathcal{I}(X)$ is complemented.*

2.8.1. COROLLARY. *If $E_1,\ldots,E_m$ are Euclidean cosets in $\mathbb{R}^n$, E_0 is a closed coset in $\mathbb{R}^n$ affinely homeomorphic to some $\mathbb{R}^p\times\mathbb{Z}$, and $S\in\mathcal{R}(E_0)$, then $\mathcal{I}(E_1\cup\ldots\cup E_m\cup S)$ is complemented.*

2.8.2. COROLLARY. *Suppose G is σ-compact, and $E_0,E_1,\ldots,E_n$ are closed cosets in Γ such that for each $1\le i<j\le n$, $E_i\cap E_j=E_0$ and the pair (E_i,E_n) satisfies (D′). Then $\mathcal{I}(E_1\cup\ldots\cup E_n)$ is complemented.*

The next theorem develops a technique which was used in [4, Section 4] and [1, Section 4] to show that certain hulls therein have complemented kernel. These examples will be discussed below.

2.9. THEOREM. *Suppose $X_1,\ldots,X_n,X_1',\ldots,X_n' \in \mathcal{R}_c(\Gamma)$ are such that*

(i) $X_k \subseteq X_k'$ $(1\le k\le n)$,

(ii) $X_j\cap X_k = X_j'\cap X_k'$ $(1\le j<k\le n)$,

(iii) *$\mathcal{I}(X_k)$ is complemented* $(1\le k\le n)$,

(iv) *$\mathcal{I}(\bigcup_{k=1}^n X_k')$ is complemented.*

Then $\mathcal{I}(\bigcup_{k=1}^n X_k)$ is complemented.

Proof. Put $Z=\bigcup_{k=1}^n X_k$, and let $T_0: A(\bigcup_{k=1}^n X_k')\to A(\Gamma)$ and $T_k: A(X_k)\to A(\Gamma)$ $(1\le k\le n)$ be splitting maps. If $1\le j<k\le n$, then for each $f\in A(Z)$ and each $\gamma\in X_j'\cap X_k'$ we have $\gamma\in X_j\cap X_k$, so that $T_j(f|_{X_j})(\gamma)=f(\gamma)=T_k(f|_{X_k})(\gamma)$. Hence, by Theorem 2.3,

$$T'(f)=(\varrho_{X_1'}\circ T_1\circ\varrho_{X_1}(f))\cup\ldots\cup(\varrho_{X_n'}\circ T_n\circ\varrho_{X_n}(f))$$

defines a continuous linear mapping $T': A(Z)\to A(\bigcup_{k=1}^n X_k')$. Moreover, if $\gamma\in Z$, then $T'(f)(\gamma)=f(\gamma)$, so that $T=T_0\circ T': A(Z)\to A(\Gamma)$ is a splitting map, as required. ∎

We now apply the above theorem to some examples. These occur in Euclidean groups, so that some geometric insight is possible. To aid this, we will adopt the terms *line*, *plane* and *grille* for cosets in a Euclidean group affinely homeomorphic to the groups $\mathbb{R}$, $\mathbb{R}^2$ and $\mathbb{R}\times\mathbb{Z}$ respectively. The terms *collinear*, *coplanar* and *parallel* will be used in this sense. We will refer to the union of two coplanar non-parallel grilles as a *grid*.

2.10. EXAMPLE. Let $x_0,x_1,x_2\in\mathbb{R}^n$ be linearly independent and let $\xi_1,\xi_2,\eta_1,\eta_2\in\mathbb{R}$ be such that $\xi_1\not\equiv\xi_2$. Put $\Xi=\mathbb{R}x_0$, $E_1=(\xi_1\mathbb{Z}+\eta_1)x_0+\mathbb{R}x_1$ and $E_2=(\xi_2\mathbb{Z}+\eta_2)x_0+\mathbb{R}x_2$. Each of these is a closed coset in $\mathbb{R}^n$. (We have a line Ξ and two planes through Ξ in which there are grilles E_1 and E_2 intersecting Ξ.) Then $\mathcal{I}(E_1\cup E_2)$ is not complemented, since (E_1,E_2) fails (D). However, with $\Pi_r=\mathbb{R}x_0+\mathbb{R}x_r$ $(r=1,2)$, put $X_1=E_1\cup\Xi$, $X_1'=\Pi_1$, $X_2=E_2\cup\Xi$ and $X_2'=\Pi_2$. By Proposition 2.6, each of X_1, X_2 and $X_1'\cup X_2'$ has complemented kernel, and $X_1'\cap X_2'=\Xi=X_1\cap X_2$. Hence, by Theorem 2.9, $\mathcal{I}(E_1\cup E_2\cup\Xi)$ is complemented. Note that there exists a chain of projections $L^1(\mathbb{R}^3)\to\mathcal{I}(E_1)\to\mathcal{I}(E_1\cup\Xi)\to\mathcal{I}(E_1\cup E_2\cup\Xi)$ but none $L^1(\mathbb{R}^3)\to\mathcal{I}(E_1)\to\mathcal{I}(E_1\cup E_2)\to\mathcal{I}(E_1\cup E_2\cup\Xi)$.

The previous example occurred as [4, Example 0.1(v)] where it took the form $\mathcal{I}((\mathbb{R}\times\mathbb{Z}\times\{0\})\cup(\{0\}\times\sqrt{2}\mathbb{Z}\times\mathbb{R})\cup(\{0\}\times\mathbb{R}\times\{0\}))\subseteq L^1(\mathbb{R}^3)$. An instance of the next example also occurs in the literature, as [1, Example 4.1]. There will be further discussion of these examples and the issues surrounding them in Section 5.

2.11. EXAMPLE. Continuing with the notation of Example 2.10, we let $x_3,x_4\in\mathbb{R}^n$ be such that x_3 does not lie in either of the planes Π_1, Π_2, and x_4 lies in the plane $\Pi_3=\Xi+\mathbb{R}x_3$ but not on either of the lines Ξ, $\mathbb{R}x_3$. Let $E_3=(\xi_1\mathbb{Z}+\eta_1)x_0+\mathbb{R}x_3=(E_1\cap\Xi)+\mathbb{R}x_3$ and $E_4=(\xi_2\mathbb{Z}+\eta_2)x_0+\mathbb{R}x_4=(E_2\cap\Xi)+\mathbb{R}x_4$. (We have introduced a third plane Π_3 through Ξ

and a grid $E_3 \cup E_4$ on Π_3 such that $E_3 \cap \Xi = E_1 \cap \Xi$ and $E_4 \cap \Xi = E_2 \cap \Xi$.) By Theorem 2.9 with $X_1 = E_3 \cup E_4$, $X_1' = \Pi_3$ and $X_2 = X_2' = E_1$, $\mathcal{I}(E_1 \cup E_3 \cup E_4)$ is complemented. Now note that by Corollary 2.8.1, $\mathcal{I}(\Pi_1 \cup \Pi_3 \cup E_2)$ is complemented, so that we can again apply Theorem 2.9, this time with $X_1 = E_1 \cup E_3 \cup E_4$, $X_1' = \Pi_1 \cup \Pi_3$ and $X_2 = X_2' = E_2$. Hence $\mathcal{I}(E_1 \cup E_2 \cup E_3 \cup E_4)$ is complemented. It can be shown that neither $\mathcal{I}(E_1 \cup E_2 \cup E_3)$ nor $\mathcal{I}(E_1 \cup E_2 \cup E_4)$ is complemented. This will be left for Lemma 4.4, where a stronger result is proven. Given this, if there is a chain of projections $L^1(G) \to \mathcal{I}(E_{\pi(1)}) \to \ldots \to \mathcal{I}(E_{\pi(1)} \cup \ldots \cup E_{\pi(4)})$ for some permutation π of $\{1,2,3,4\}$, then $\pi(4) \notin \{3,4\}$.

The result [4, Lemma 1.4] (the "if" part of Proposition 2.4) uses certain Fourier–Stieltjes transforms to combine sets which have complemented kernel to yield another set with complemented kernel. The following result is similar, but more general. One useful feature is that it can be used in two ways. The first of these is to show that under certain conditions, we can combine hulls, each with complemented kernel, to give a larger hull with complemented kernel. The second is to show that if a hull has complemented kernel, certain parts of it can also be shown to have complemented kernel. An example of a construction of the first type will be seen in Proposition 4.3, and an example of the second type is given in Corollary 2.12.1, which will prove quite useful in subsequent sections.

2.12. THEOREM. *Suppose $X, X_1, \ldots, X_n \in \mathcal{R}_c(\Gamma)$ and $F_1, \ldots, F_n \in B(\Gamma)$ are such that*

(i) $X \subseteq \bigcup_{k=1}^n X_k$,
(ii) *for each k, either $X \subseteq X_k$ or $X_k \subseteq X$,*
(iii) *for each k, $F_k = 0$ on $X \triangle X_k = (X_k \setminus X) \cup (X \setminus X_k)$,*
(iv) $\sum_{k=1}^n F_k = 1$ *on X,*
(v) *each $\mathcal{I}(X_k)$ is complemented in $L^1(G)$.*

Then $\mathcal{I}(X)$ is complemented in $L^1(G)$.

Proof. Let $T_k : A(X_k) \to A(\Gamma)$ $(1 \le k \le n)$ be splitting maps and let $f \in A(X)$. For each k such that $X_k \subseteq X$, we have $f|_{X_k} \in A(X_k)$, so that we can define $T_k'(f) = F_k \cdot (T_k \circ \varrho_{X_k}(f)) \in A(\Gamma)$. Clearly $T_k'(f) = F_k \cdot f$ on X_k, and since $F_k = 0$ on $X \setminus X_k$, we have $T_k'(f) = F_k \cdot f$ on X. Moreover, $T_k' : A(X) \to A(\Gamma)$ is continuous and linear.

For each k such that $X_k \supseteq X$, we have $Y_k = \overline{X_k \setminus X} \in \mathcal{R}_c(\Gamma)$, by [7, Theorem 3.2]. Then $F_k(Y_k) = \{0\}$, so $F_k|_X \cdot f \in \mathcal{I}_{A(X)}(X \cap Y_k)$. By Corollary 2.3.1, the mapping $f \mapsto (F_k|_X \cdot f) \cup 0_{Y_k}$ is a continuous linear mapping $A(X) \to \mathcal{I}_{A(X_k)}(Y_k)$. Hence $T_k' : f \mapsto T_k((F_k|_X \cdot f) \cup 0_{Y_k})$ defines a mapping $A(X) \to \mathcal{I}_{A(\Gamma)}(Y_k)$ which is continuous and linear with $T_k'(f) = F_k \cdot f$ on X.

Thus $T = \sum_{k=1}^n T_k'$ is continuous and linear and satisfies $T(f)(\gamma) = f(\gamma) \sum_{k=1}^n F_k(\gamma) = f(\gamma)$ for all $\gamma \in X$. Hence T is a splitting map and $\mathcal{I}(X)$ is complemented. ∎

2.12.1. COROLLARY. *Suppose that $Y, Y_1, \ldots, Y_n \in \mathcal{R}_c(\Gamma)$ are such that $\bigcup_{k=1}^n Y_k \subseteq Y$ and each ideal $\mathcal{I}(Y), \mathcal{I}(Y_1), \ldots, \mathcal{I}(Y_n)$ is complemented. If there exists $U \in \mathcal{U}_\Gamma$ such that*

$$Y \cap \bigcup_{\substack{i,j\\ i \ne j}} ((Y_i + U) \cap (Y_j + U)) \subseteq \bigcup_k Y_k,$$

then $\mathcal{I}(Y_1 \cup \ldots \cup Y_n)$ is complemented.

Proof. Let $X = Y_1 \cup \ldots \cup Y_n$. By [5, Proposition 0.2], there exist measures $\mu_1, \ldots, \mu_n \in M(G)$ such that for each k, $\widehat{\mu}_k(Y_k) = \{1\}$ and $\widehat{\mu}_k(\Gamma \setminus (Y_k + U)) = \{0\}$. Put

$$F_k = \widehat{\mu}_k \prod_{\substack{1 \le j \le n\\ j \ne k}} (1 - \widehat{\mu}_j) \quad (1 \le k \le n)$$

and

$$F = 1 - \prod_{1 \le j \le n} (1 - \widehat{\mu}_j) - \sum_{1 \le j \le n} F_j.$$

For each k, $Y_k \subseteq X$, $\mathcal{I}(Y_k)$ is complemented and $F_k = 0$ on $\bigcup_{j \ne k} Y_j \supseteq X \setminus Y_k$. Also $X \subseteq Y$, $\mathcal{I}(Y)$ is complemented and $Y \setminus X \subseteq Y \setminus \bigcup_{i \ne j}((Y_i + U) \cap (Y_j + U))$, so that for each $\gamma \in Y \setminus X$, there is at most one k with $\widehat{\mu}_k(\gamma) \ne 0$. However, if $\widehat{\mu}_j(\gamma) = 0$ for all $j \ne k$, then $1 - \prod(1 - \widehat{\mu}_j(\gamma)) = \widehat{\mu}_k(\gamma)$ and $\sum F_j(\gamma) = F_k(\gamma) = \widehat{\mu}_k(\gamma)$, so that $F(\gamma) = 0$. Thus $F(Y \setminus X) = \{0\}$. Finally, $F + \sum F_j = 1 - \prod(1 - \widehat{\mu}_j)$, which takes the value 1 on X. Hence, by Theorem 2.12, $\mathcal{I}(X)$ is complemented. ∎

3. A hull in $\mathbb{R}^3$. We are now ready to construct our first hull. This will be a union of closed subgroups $\Lambda_1, \ldots, \Lambda_9$ in $\mathbb{R}^3$ such that $\mathcal{I}(\bigcup_{j=1}^9 \Lambda_j)$ is complemented without there being a permutation π of $\{1, \ldots, 9\}$ and an elementary chain of projections as in equation (1.2). The idea behind this construction is quite simple. We saw from Examples 2.10 and 2.11 that there were hulls which had to be assembled in a specific order so as to yield an elementary chain of projections. In particular, these hulls contained "crucial" elementary sets which could not be added last. The example we consider below contains nine instances $X_1, \ldots, X_9$ of Example 2.10, arranged so that each subgroup Λ_k is crucial to X_k, and the regions in which there is a failure of uniform separation can be isolated from each other by an appropriate partition of the identity, allowing the application of Theorem 2.12.

In order for this construction to work, we need to choose the subgroups Λ_k carefully. The situation we want is for each Λ_k to be a grille whose component of the identity Ξ_k plays the rôle of Ξ in the kth instance of Example 2.10. Thus, the relations that we want satisfied are as follows:

(i) For each $1 \le k \le 9$, there are exactly two other indices, i and j say, for which Ξ_k intersects each of Λ_i, Λ_j nontrivially. (That is, $\Lambda_i \cap \Xi_k \ne \{0\}$ and $\Lambda_j \cap \Xi_k \ne \{0\}$.)

(ii) With k, i and j as in (i), $\Lambda_k \cap \Lambda_i \subseteq \Xi_k$ and $\Lambda_k \cap \Lambda_j \subseteq \Xi_k$.

(iii) With k, i and j as in (i), $\Xi_k \cup \Lambda_i \cup \Lambda_j$ is an instance of Example 2.10, so that $\mathcal{I}(\Xi_k \cup \Lambda_i \cup \Lambda_j)$ is complemented, but $\mathcal{I}(\Lambda_i \cup \Lambda_j)$ is not.

(iv) If r and s are such that (Λ_r, Λ_s) fails (D), then for some $k \in \{1, \ldots, 9\} \setminus \{r, s\}$, the line Ξ_k intersects each of Λ_r and Λ_s nontrivially.

Each of these properties will be verified in the course of the construction, and it will then be shown that these properties imply that $\mathcal{I}(X)$ is complemented and that no $\mathcal{I}(\bigcup_{j \ne k} \Lambda_j)$ is complemented. This latter property excludes the possibility of a chain of projections as in equation (1.2).

To help our arithmetic, we have the following simple lemma, in which the ratio x/y of two vectors $x, y \in \mathbb{R}^n$ is defined when $x \in \mathbb{R}y \ne \{0\}$ to be the unique $\xi \in \mathbb{R}$ with $x = \xi y$.

3.1. Lemma. *Suppose $E \subseteq \mathbb{R}^n$ is a line with $0 \notin E$, $x, y, z \in E$. Then for $\eta, \zeta \in \mathbb{R}$,*

$$(\eta(\mathbb{Z} + \zeta)x + \mathbb{R}y) \cap \mathbb{R}z = \frac{x - y}{z - y}\,\eta(\mathbb{Z} + \zeta)z.$$

3.2. Example. Define points $p_1, \ldots, p_9 \in \mathbb{R}^2 \times \{1\} \subseteq \mathbb{R}^3$ as follows:

$$\begin{aligned} &p_1 = (0, 42, 1), &&p_4 = (0, 21, 1), &&p_7 = (14, 14, 1),\\ &p_2 = (42, 0, 1), &&p_5 = (21, 0, 1), &&p_8 = (0, -14, 1),\\ &p_3 = (-42, -42, 1), &&p_6 = (-21, -21, 1), &&p_9 = (-14, 0, 1), \end{aligned} \tag{3.1}$$

then the following nine triplets of indices $\{i, j, k\}$ have $\{p_i, p_j, p_k\}$ collinear:

$$\begin{aligned} &\mathbb{I}_1 = \{1, 5, 7\}, &&\mathbb{I}_4 = \{2, 4, 7\}, &&\mathbb{I}_7 = \{3, 6, 7\},\\ &\mathbb{I}_2 = \{2, 6, 8\}, &&\mathbb{I}_5 = \{3, 5, 8\}, &&\mathbb{I}_8 = \{1, 4, 8\},\\ &\mathbb{I}_3 = \{3, 4, 9\}, &&\mathbb{I}_6 = \{1, 6, 9\}, &&\mathbb{I}_9 = \{2, 5, 9\}. \end{aligned} \tag{3.2}$$

For each k, let E_k be the line through $\{p_i : i \in \mathbb{I}_k\}$. The relationships between these points and lines can be verified algebraically, or by reference to Figure 3.2.1. (There are many such planar configurations of 9 points with 9 collinear triplets such that each point is in three triplets. See, for instance, [9, Section 17].) There are seven extra points $q_1, \ldots, q_7 \in \mathbb{R}^2 \times \{1\}$ which occur as the intersection of two or more of the lines $E_1, \ldots, E_9$. These are:

$$\begin{aligned} &q_1 = (6, 30, 1) = E_1 \cap E_3, &&q_5 = (-24, -30, 1) = E_5 \cap E_6,\\ &q_2 = (24, -6, 1) = E_2 \cap E_1, &&q_6 = (-6, 24, 1) = E_6 \cap E_4,\\ &q_3 = (-30, -24, 1) = E_3 \cap E_2, &&q_7 = (0, 0, 1) = E_7 \cap E_8 \cap E_9.\\ &q_4 = (30, 6, 1) = E_4 \cap E_5, \end{aligned} \tag{3.3}$$

Put $q_8 = q_9 = q_7$, so that each $q_k \in E_k$.

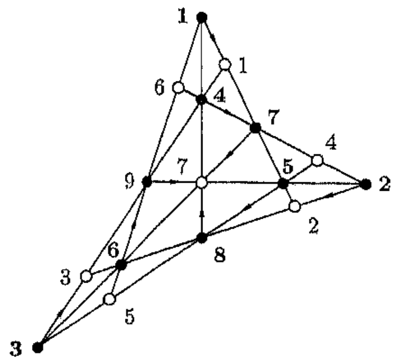

Fig. 3.2.1. The points $p_1, \ldots, p_9$ (filled circles) and $q_1, \ldots, q_7$ (open circles) and the lines $E_1, \ldots, E_9$ of Example 3.2. An arrow from p_k indicates E_k.

For $1 \le k \le 9$, let $\Xi_k = \mathbb{R}p_k$ and let Π_k be the plane through E_k and $(0, 0, 0)$. Clearly

$$\Xi_i \subseteq \Pi_k \Leftrightarrow p_i \in E_k \Leftrightarrow i \in \mathbb{I}_k. \tag{3.4}$$

Now take $\xi_1, \ldots, \xi_9 \in \mathbb{R}$ such that $\xi_1 = \xi_2 = \xi_3$, $\xi_4 = \xi_5 = \xi_6$, $\xi_7 = \xi_8 = \xi_9$, and ξ_3, ξ_6 and ξ_9 are pairwise rationally independent. For each $1 \le k \le 9$, put $\Lambda_k = \xi_k \mathbb{Z} q_k + \mathbb{R}p_k$. Clearly Λ_k is a grille contained in the plane Π_k, and Ξ_k is the component of the identity of Λ_k. Put $X = \bigcup_{k=1}^{9} \Lambda_k$.

We now verify the above properties (i)–(iv). Note that $\Xi_k \cap \Lambda_i \ne \{0\}$ if and only if $\Xi_k \subseteq \Pi_i$, which by (3.4) is equivalent to the condition $k \in \mathbb{I}_i$. However, since each k occurs in exactly three of the $\mathbb{I}_i$, one of these being $\mathbb{I}_k$, we have (i). For (ii), note that $E_i \ne E_j$ and $p_k \in E_i \cap E_j$, so that $E_i \cap E_j = \{p_k\}$. It follows that $\Pi_i \cap \Pi_j = \mathbb{R}p_k = \Xi_k$, so that $\Lambda_i \cap \Lambda_j \subseteq \Xi_k$.

It is clear from (3.2) that if $k \in \mathbb{I}_i \cap \mathbb{I}_j \cap \mathbb{I}_k$, then each of the sets $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$ contain exactly one of i, j, k. Hence ξ_i, ξ_j and ξ_k are distinct. Now, $k \in \mathbb{I}_i$, so that $p_k \in E_i$. Also $p_i, q_i \in E_i$, so that p_i, p_k and q_i are collinear. Similarly $p_j, p_k, q_j \in E_j$ are collinear. Hence, by Lemma 3.1, we have

$$\begin{aligned} \Lambda_i \cap \Xi_k &= (\xi_i \mathbb{Z} q_i + \mathbb{R}p_i) \cap \mathbb{R}p_k = \frac{q_i - p_i}{p_k - p_i}\,\xi_i \mathbb{Z} p_k,\\ \Lambda_j \cap \Xi_k &= (\xi_j \mathbb{Z} q_j + \mathbb{R}p_j) \cap \mathbb{R}p_k = \frac{q_j - p_j}{p_k - p_j}\,\xi_j \mathbb{Z} p_k. \end{aligned} \tag{3.5}$$

However, $\xi_i \neq \xi_j$, so that $\xi_i \not\equiv \xi_j$, and since each of q_i, p_i, q_j, p_j, p_k has rational co-ordinates, we see that

$$\eta = \frac{q_i - p_i}{p_k - p_i}\,\xi_i \quad \text{and} \quad \zeta = \frac{q_j - p_j}{p_k - p_j}\,\xi_j$$

are rationally independent. Moreover, $\Lambda_i = (\Lambda_i \cap \Xi_k) + \mathbb{R}p_i$ and $\Lambda_j = (\Lambda_j \cap \Xi_k) + \mathbb{R}p_j$, so that

$$\Xi_k \cup \Lambda_i \cup \Lambda_j = \mathbb{R}p_k \cup (\eta\mathbb{Z}p_k + \mathbb{R}p_i) \cup (\zeta\mathbb{Z}p_k + \mathbb{R}p_j). \tag{3.6}$$

Since p_i, p_j and p_k are not collinear, they are linearly independent so that $\Xi_k \cup \Lambda_i \cup \Lambda_j$ is an instance of Example 2.10, satisfying (iii).

Finally, we verify (iv). For r, s such that $\Pi_r \cap \Pi_s = \mathbb{R}q_k$ $(1 \le k \le 7)$, we have $\xi_r = \xi_s$. A comparison of the spacings of $\Lambda_r \cap \mathbb{R}q_k$ and $\Lambda_s \cap \mathbb{R}q_k$ as in (3.5) now gives us that (Λ_r, Λ_s) satisfies (D). The only other pairs of subgroups (Λ_r, Λ_s) are those for which $\Pi_r \cap \Pi_s = \Xi_k$, for some $1 \le k \le 9$. Here there are three cases: $r = k$, $s = k$ and $\mathbb{I}_k = \{k, r, s\}$. In the first two cases, (Λ_r, Λ_s) satisfies (D), and in the last, (Λ_r, Λ_s) fails (D), but each intersects Ξ_k nontrivially.

3.3. PROPOSITION. *Let $\Lambda_1, \ldots, \Lambda_9$ be as specified above. Then for any $1 \le k \le 9$, the ideal $\mathcal{I}(\bigcup_{j\neq k} \Lambda_j)$ is not complemented.*

Proof. Suppose $\mathcal{I}(\bigcup_{j\neq k} \Lambda_j)$ is complemented for some $1 \le k \le 9$. Let i, j be as in (i) above and put

$$Y = \bigcup_{j\neq k} \Lambda_j, \quad Y_1 = \Lambda_i \setminus \Xi_i, \quad \text{and} \quad Y_2 = \Lambda_j \setminus \Xi_j.$$

Clearly $Y_1 \cup Y_2 \subseteq Y$. We claim that there exists $U \in \mathcal{U}_{\mathbb{R}^3}$ such that $(Y_1+U) \cap (Y_2+U) \cap Y \subseteq Y_1 \cap Y_2$. If this is the case, then we could apply Corollary 2.12.1 to deduce that $\mathcal{I}(Y_1 \cup Y_2)$ is complemented. However, $\Xi_i \cup \Xi_j$ is easily verified to be uniformly separated from $Y_1 \cup Y_2$, so that by Proposition 2.4, $\mathcal{I}(Y_1 \cup Y_2 \cup \Xi_i \cup \Xi_j) = \mathcal{I}(\Lambda_i \cup \Lambda_j)$ is complemented. This contradicts property (iii) above, completing the proof.

We now prove the claim. Let $U_0 \in \mathcal{U}_{\mathbb{R}^3}$ have $3U_0$ disjoint from $Y_1 \cup Y_2$. For each $r \in \{i, j\}$, the pair (Ξ_k, Ξ_r) satisfies (D), and for $r \in \{1, \ldots, 9\} \setminus \{i, j, k\}$, the pair (Ξ_k, Λ_r) satisfies (D). Each of the above pairs has intersection $\{0\}$. Hence, by Lemma 2.7, there exists $U_1 \in \mathcal{U}_{\mathbb{R}^3}$ with $U_1 \subseteq U_0$ and

$$(\Xi_k + U_1) \cap (\Xi_r + U_1) \subseteq U_0 \quad (r \in \{i, j\}),$$
$$(\Xi_k + U_1) \cap (\Lambda_r + U_1) \subseteq U_0 \quad (r \in \{1, \ldots, 9\} \setminus \{i, j, k\}).$$

Hence, for $r \in \{i, j\}$,

$$\Xi_r \cap ((\Xi_k \setminus U_0) + U_1) \subseteq ((\Xi_r + U_1) \cap \Xi_k \setminus U_0) + U_1 \subseteq (U_0 \setminus U_0) + U_1 = \emptyset,$$

and similarly for each $r \in \{1, \ldots, 9\} \setminus \{i, j, k\}$, we have $\Lambda_r \cap ((\Xi_k \setminus U_0) + U_1) = \emptyset$. Hence $Y \cap ((\Xi_k \setminus U_0) + U_1) \subseteq Y_1 \cup Y_2$.

Since (Π_i, Π_j) satisfies (D) with $\Pi_i \cap \Pi_j = \Xi_k$, there exists $U_2 \in \mathcal{U}_{\mathbb{R}^3}$ with $U_2 \subseteq U_1$ and $(\Pi_i + U_2) \cap (\Pi_j + U_2) \subseteq \Xi_k + U_1$. But $\Xi_k + U_1 \subseteq ((\Xi_k \setminus U_0) + U_1) \cup (U_0 + U_1)$ and $(Y_1 + U_2) \cap (U_0 + U_1) \subseteq (Y_1 \cap 3Y_0) + U_0 = \emptyset$, so that

$$\begin{aligned} Y \cap (Y_1 + U_2) \cap (Y_2 + U_2) &\subseteq Y \cap (\Pi_i + U_2) \cap (\Pi_j + U_2) \cap (Y_1 + U_2) \\ &\subseteq Y \cap ((\Xi_k \setminus U_0) + U_1) \subseteq Y_1 \cup Y_2, \end{aligned}$$

as claimed. ∎

To complete this example, we need to show that $\mathcal{I}(\bigcup_{k=1}^{9} \Lambda_k)$ is complemented. One way to approach this is to apply Theorem 2.12 with $X = \bigcup_{k=1}^{9} \Lambda_k$ and

$$\{X_1, \ldots, X_{26}\} = \Big\{\Xi_1 \cup \Lambda_6 \cup \Lambda_8,\ \Xi_2 \cup \Lambda_4 \cup \Lambda_9, \ldots, \Xi_9 \cup \Lambda_3 \cup \Lambda_6; \\ \Lambda_1 \cup \Lambda_3, \Lambda_1 \cup \Lambda_2, \ldots, \Lambda_7 \cup \Lambda_8 \cup \Lambda_9; \Lambda_1, \ldots, \Lambda_9; \bigcup_{1\le j\le n} \Xi_j\Big\}.$$

(Note that here the first nine sets are the instances of Example 2.10 occurring in the equations (3.6), and each of the next seven sets consists of those subgroups Λ_i which intersect one of the lines $\mathbb{R}q_k$ nontrivially. These arise from the relations in (3.3).) The specification of the functions $F_1, \ldots, F_{26} \in B(\mathbb{R}^3)$ will not be presented here, as the proposition below shows the ideal $\mathcal{I}(X)$ to be complemented without such a construction. A demonstration of the construction of such functions will be given for the hull constructed in the next section, where it seems that there is no alternative.

3.4. PROPOSITION. *With notation as in Example 3.2, there is an elementary chain of projections*

$$\begin{aligned} &L^1(\mathbb{R}^3) \to \mathcal{I}(\Xi_1) \to \ldots \\ &\qquad \ldots \to \mathcal{I}(\Xi_1 \cup \ldots \cup \Xi_9) \to \mathcal{I}(\Lambda_1 \cup \Xi_2 \cup \ldots \cup \Xi_9) \to \ldots \\ &\qquad\qquad \ldots \to \mathcal{I}(\Lambda_1 \cup \ldots \cup \Lambda_8 \cup \Xi_9) \to \mathcal{I}(X). \end{aligned}$$

Proof. The existence of a chain of projections

$$L^1(\mathbb{R}^3) \to \mathcal{I}(\Xi_1) \to \ldots \to \mathcal{I}(\Xi_1 \cup \ldots \cup \Xi_9)$$

follows from [4, Corollary 3.12]. We proceed from here by induction on $k \in \{1, \ldots, 9\}$, using Theorem 2.9 to show that at each stage, there is a projection

$$\mathcal{I}(\Lambda_1 \cup \ldots \cup \Lambda_{k-1} \cup \Xi_k \cup \ldots \cup \Xi_9) \to \mathcal{I}(\Lambda_1 \cup \ldots \cup \Lambda_k \cup \Xi_{k+1} \cup \ldots \cup \Xi_9).$$

Let $X_0 = \Lambda_1 \cup \ldots \cup \Lambda_{k-1} \cup \Xi_k \cup \ldots \cup \Xi_9$, and assume that $\mathcal{I}(X_0)$ is complemented. We show that $\mathcal{I}(X_0 \cup \Lambda_k)$ is complemented, so that it is complemented in $\mathcal{I}(X_0)$.

Put $Y = \Lambda_k \cap \bigcup_{m=1}^{7} \mathbb{R}q_m$ and $X_1 = X_0 \cup Y$. Since each $\Lambda_k \cap \mathbb{R}q_m$ is a proper subgroup of $\mathbb{R}q_m$, Y is discrete. Moreover, if $n \neq m$, then $\Lambda_k \cap \mathbb{R}q_m \setminus \{0\}$ and $\Lambda_k \cap \mathbb{R}q_n \setminus \{0\}$ are uniformly separated. Hence Y is uniformly discrete. Also, due to rational dependence of $\Lambda_k \cap \mathbb{R}q_m$ and $\Lambda_j \cap \mathbb{R}q_m$ ($1 \leq j < k$), $Y \setminus X_0$ is uniformly separated from X_0. Hence, by Corollary 2.4.2, $\mathcal{I}(X_1)$ is complemented. Now put

$$X_1' = \Pi_1 \cup \ldots \cup \Pi_{k-1} \cup \Xi_k \cup \ldots \cup \Xi_9 \cup \bigcup_{1 \leq m \leq 7} (\Lambda_k \cap \mathbb{R}q_m),$$
$$X_2 = X_2' = \Lambda_k \cup \bigcup_{j \in \mathbb{I}_k} \Xi_j.$$

By Corollary 2.8.1, X_2 and $X_1' \cup X_2' = \Pi_1 \cup \ldots \cup \Pi_{k-1} \cup \Lambda_k \cup \Xi_{k+1} \cup \ldots \cup \Xi_9$ each has complemented kernel. Moreover,

$$\begin{aligned} X_1 \cap X_2 \subseteq X_1' \cap X_2' &= (X_1 \cap X_2) \cup \Big((\Pi_1 \cup \ldots \cup \Pi_{k-1}) \cap \Big(\Lambda_k \cup \bigcup_{j \in \mathbb{I}_k} \Xi_j\Big)\Big) \\ &\subseteq (X_1 \cap X_2) \cup \bigcup_{j \in \mathbb{I}_k} \Xi_j \cup \bigcup_{1 \leq j < k} (\Pi_j \cap \Lambda_k). \end{aligned}$$

Clearly $\bigcup_{j \in \mathbb{I}_k} \Xi_j \subseteq X_1 \cap X_2$. For $1 \leq j < k$, either $\Pi_j \cap \Pi_k = \Xi_i$ for some i or $\Pi_j \cap \Pi_k = \mathbb{R}q_m$ for some m. In the first case, $\Pi_j \cap \Lambda_k \subseteq \Xi_i \subseteq X_1 \cap X_2$, and in the second, $\Pi_j \cap \Lambda_k = \mathbb{R}q_m \cap \Lambda_k \subseteq X_1 \cap X_2$. Hence $X_1' \cap X_2' \subseteq X_1 \cap X_2$. Thus, by Theorem 2.9, $X_1 \cup X_2 = X_0 \cup \Lambda_k$ has complemented kernel. ∎

Remark. This alternative decomposition of X into elementary sets relies on the subgroups Λ_k being disconnected. A simple modification of this example alters this situation. Let $\widetilde{\Gamma} = \mathbb{R}^3 \times \mathbb{T}^9$, and let $\theta_0 : \mathbb{R}^3 \to \widetilde{\Gamma}$ and $\theta_k : \mathbb{T} \to \widetilde{\Gamma}$ ($1 \leq k \leq 9$) be the natural injections. Put $\widetilde{\Lambda}_k = \{\theta_0(\xi_k \eta q_k + \zeta p_k) + \theta_k(e^{2\pi i \eta}) : \eta, \zeta \in \mathbb{R}\}$ ($1 \leq k \leq 9$) and $\widetilde{X} = \bigcup_{k=1}^{9} \widetilde{\Lambda}_k \in \mathcal{R}_c(\widetilde{\Gamma})$. Note that for each k, we have $\widetilde{\Lambda}_k \cong \mathbb{R}^2$ and $\widetilde{\Lambda}_k \cap \theta_0(\mathbb{R}^3) = \theta_0(\xi_k \mathbb{Z} q_k + \mathbb{R}p_k) = \theta_0(\Lambda_k)$. Two applications of Theorem 2.9 give $\mathcal{I}(\widetilde{X})$ complemented—firstly with $n = 9$, $X_k = \theta_0(\mathbb{R}^3) \cup \widetilde{\Lambda}_k$ and $X_k' = \theta_0(\mathbb{R}^3) + \theta_k(\mathbb{T})$ ($1 \leq k \leq 9$), to give $\mathcal{I}(\widetilde{X} \cup \theta_0(\mathbb{R}^3))$ complemented, and secondly with $n = 10$, $X_k = X_k' = \widetilde{\Lambda}_k$ ($1 \leq k \leq 9$), $X_{10} = \theta_0(X)$ and $X_{10}' = \theta_0(\mathbb{R}^3)$. Moreover, each $\widetilde{\Lambda}_k$ is connected, so it seems that the only elementary chain of projections we could consider is that based on a permutation π of $\{1, \ldots, 9\}$, as in equation (1.2). However, it can be shown (cf. Proposition 3.3) that for any such π, $\mathcal{I}(\bigcup_{k=1}^{8} \widetilde{\Lambda}_{\pi(k)})$ is not complemented. Despite this, there is an elementary chain of eighteen projections onto $\mathcal{I}(\widetilde{X})$:

$$\begin{aligned} &L^1(\mathbb{R}^3 \times \mathbb{Z}^9) \to \mathcal{I}(\theta_0(\Xi_1)) \to \ldots \\ &\quad \ldots \to \mathcal{I}(\theta_0(\Xi_1 \cup \ldots \cup \Xi_9)) \to \mathcal{I}(\widetilde{\Lambda}_1 \cup \theta_0(\Xi_2 \cup \ldots \cup \Xi_9)) \to \ldots \\ &\qquad \ldots \to \mathcal{I}(\widetilde{\Lambda}_1 \cup \ldots \cup \widetilde{\Lambda}_8 \cup \theta_0(\Xi_9)) \to \mathcal{I}(\widetilde{X}), \end{aligned}$$

similar to that in Proposition 3.4. An unexpected property of this example is that the $\theta_0(\Xi_k)$ have empty interior in $\widetilde{X}$, and so they are superfluous to any elementary decomposition of $\widetilde{X}$.

4. A hull in $\mathbb{R}^4$. In this section we will construct a hull X with complemented kernel that is similar to Example 3.2, but for which there is no decomposition of X into elementary sets yielding a chain of projections. This set X will again be a union of grilles contained within a union of planes $\Pi_1 \cup \ldots \cup \Pi_N$ intersecting three-at-a-time in lines $\Xi_1, \ldots, \Xi_N$. However, instead of having each threesome of intersecting planes containing an instance of Example 2.10, we will have an instance of Example 2.11. We use this "building block" because even when it is decomposed into elementary sets smaller than its constituent cosets, the same restrictions on the order in which it is assembled apply. One difficulty with such a construction is that it requires a grid on each plane Π_k. If there are lines Ξ other than $\Xi_1, \ldots, \Xi_N$ which occur as the intersection of some of the planes $\Pi_1, \ldots, \Pi_N$ (such as the lines $\mathbb{R}q_k$ in Example 3.2), then the behaviour of the grids intersecting Ξ needs consideration, to ensure that $\mathcal{I}(X)$ is complemented. This difficulty is avoided by using an arrangement of planes for which such intersections do not occur. We construct this in $\mathbb{R}^4$, instead of $\mathbb{R}^3$.

4.1. Example. Define points $p_1, \ldots, p_{10} \subseteq \mathbb{R}^3 \times \{1\} \subseteq \mathbb{R}^4$ as follows:

$$\begin{array}{lll} p_1 = (-2,2,2,1), & p_5 = (2,-2,2,1), & p_8 = (1,1,-2,1), \\ p_2 = (2,2,-2,1), & p_6 = (4,-2,4,1), & p_9 = (2,0,0,1), \\ p_3 = (4,2,-4,1), & p_7 = (-2,0,0,1), & p_{10} = (1,-1,2,1). \\ p_4 = (-2,-2,-2,1), & & \end{array} \tag{4.1}$$

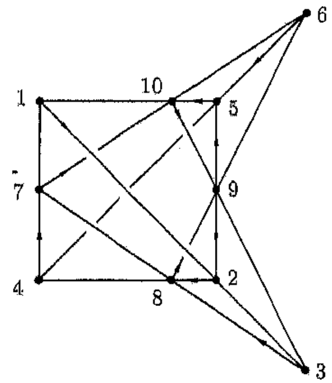

Fig. 4.1.1. The points $p_1, \ldots, p_{10}$ and the lines $E_1, \ldots, E_{10}$ of Example 4.1, projected onto the first & third coordinates.

(Here p_1, p_2, p_4, p_5 are the vertices of a regular tetrahedron, and $p_3, p_6, \ldots, p_{10}$ are coplanar points, each occurring on an edge of the tetrahedron. Such a configuration is discussed in [9, Section 19].) We have ten triplets of indices $\{i,j,k\}$ with $\{p_i,p_j,p_k\}$ collinear:

$$\begin{array}{llll} \mathbb{I}_1=\{1,2,3\}, & \mathbb{I}_4=\{1,4,7\}, & \mathbb{I}_7=\{6,7,10\}, & \mathbb{I}_9=\{2,5,9\}, \\ \mathbb{I}_2=\{2,4,8\}, & \mathbb{I}_5=\{1,5,10\}, & \mathbb{I}_8=\{6,8,9\}, & \mathbb{I}_{10}=\{3,9,10\}, \\ \mathbb{I}_3=\{3,7,8\}, & \mathbb{I}_6=\{4,5,6\}. & & \end{array} \tag{4.2}$$

As in Example 3.2 for each index $1\le k\le 10$, let E_k be the line containing $\{p_i : i\in\mathbb{I}_k\}$, put $\Xi_k=\mathbb{R}p_k$, and let Π_k be the plane containing E_k and $(0,0,0,0)$. Again,

$$\Xi_i\subseteq \Pi_k \Leftrightarrow p_i\in E_k \Leftrightarrow i\in\mathbb{I}_k. \tag{4.3}$$

Furthermore, if $i\ne j$, then

$$\begin{aligned} \Pi_i\cap\Pi_j\ne\{0\} \Leftrightarrow E_i\cap E_j\ne\emptyset \Leftrightarrow \mathbb{I}_i\cap\mathbb{I}_j=\{k\}, &\text{ for some } k \\ \Leftrightarrow \Pi_i\cap\Pi_j=\Xi_k, &\text{ for some } k. \end{aligned} \tag{4.4}$$

Each plane Π_k contains the three lines Ξ_k, Ξ_r, and Ξ_s, where $\mathbb{I}_k=\{k,r,s\}$. We require two grilles $S_{k,r}$ and $S_{k,s}$ in Π_k such that $S_{k,r}$ intersects Ξ_k and Ξ_r, but not Ξ_s, and similarly for $S_{k,s}$. Putting $\widetilde{\mathbb{Z}}=\mathbb{Z}+1/2$, these will be of the form $S_{k,r}=\xi_{k,r}\widetilde{\mathbb{Z}}p_k+\mathbb{R}p_s$ and $S_{k,s}=\xi_{k,s}\widetilde{\mathbb{Z}}p_k+\mathbb{R}p_r$ for some $\xi_{k,r},\xi_{k,s}\in\mathbb{R}$. Then

$$\begin{array}{ll} S_{k,r}\cap\Xi_k=\xi_{k,r}\widetilde{\mathbb{Z}}p_k, & S_{k,s}\cap\Xi_k=\xi_{k,s}\widetilde{\mathbb{Z}}p_k, \\ S_{k,r}\cap\Xi_r=\dfrac{p_k-p_s}{p_r-p_s}\,\xi_{k,r}\widetilde{\mathbb{Z}}p_r, & S_{k,s}\cap\Xi_r=\emptyset, \\ S_{k,r}\cap\Xi_s=\emptyset, & S_{k,s}\cap\Xi_s=\dfrac{p_k-p_r}{p_s-p_r}\,\xi_{k,s}\widetilde{\mathbb{Z}}p_s. \end{array} \tag{4.5}$$

Note that we are only defining $S_{k,r}$ when $(k,r)\in\mathbb{J}=\{(k,r): r\in\mathbb{I}_k\setminus\{k\}\}$. Note also that the fractions in (4.5), given by Lemma 3.1, are rational.

For each line Ξ_k, there are three planes containing Ξ_k, say Π_k, Π_i and Π_j, so that if $\mathbb{I}_k=\{k,r,s\}$, the four grilles intersecting Ξ_k are $S_{i,k}$, $S_{j,k}$, $S_{k,r}$ and $S_{k,s}$. Supposing these comprise an instance of Example 2.11, we have

$$\begin{aligned} &S_{i,k}\cap\Xi_k=S_{k,r}\cap\Xi_k\ne S_{j,k}\cap\Xi_k=S_{k,s}\cap\Xi_k \\ \text{or}\quad &S_{i,k}\cap\Xi_k=S_{k,s}\cap\Xi_k\ne S_{j,k}\cap\Xi_k=S_{k,r}\cap\Xi_k. \end{aligned} \tag{4.6}$$

Then either $\xi_{i,k}\equiv\xi_{k,r}\not\equiv\xi_{j,k}\equiv\xi_{k,s}$ or $\xi_{i,k}\equiv\xi_{k,s}\not\equiv\xi_{j,k}\equiv\xi_{k,r}$. If we partition $\mathbb{J}$ into sets $\mathbb{J}_M=\{(k,r)\in\mathbb{J}:\xi_M\equiv\eta_M\}$ (for some $\eta_M\in\mathbb{R}$), then for each $(i,k)\in\mathbb{J}_M$ there exists r with $(k,r)\in\mathbb{J}_M$, and if $(k,s)\in\mathbb{J}$ with $s\ne r$, then $(k,s)\notin\mathbb{J}_M$.

Consider the following partition of $\mathbb{J}$:

$$\begin{aligned} \mathbb{J}_1&=\{(1,2),(2,4),(4,1)\},\\ \mathbb{J}_2&=\{(2,8),(8,9),(9,2)\},\\ \mathbb{J}_3&=\{(3,7),(7,10),(10,3)\},\\ \mathbb{J}_4&=\{(4,7),(7,6),(6,4)\},\\ \mathbb{J}_5&=\{(5,10),(10,9),(9,5)\},\\ \mathbb{J}_6&=\{(1,3),(3,8),(8,6),(6,5),(5,1)\}. \end{aligned} \tag{4.7}$$

These satisfy the requirements above. Suppose $(i,k),(k,r)\in\mathbb{J}_M$ with, say, $\mathbb{I}_i=\{i,j,k\}$ and $\mathbb{I}_k=\{k,r,s\}$. Then to have $S_{k,r}\cap\Xi_k=S_{i,k}\cap\Xi_k$, as in (4.6), equation (4.5) gives

$$|\xi_{k,r}|=\left|\frac{p_i-p_j}{p_k-p_j}\right||\xi_{i,k}| \tag{4.8}$$

As an example, consider the indices in $\mathbb{J}_1$. These require

$$\begin{aligned} |\xi_{2,4}|&=\left|\frac{p_1-p_3}{p_2-p_3}\right||\xi_{1,2}|=3|\xi_{1,2}|,\\ |\xi_{4,1}|&=\left|\frac{p_2-p_8}{p_4-p_8}\right||\xi_{2,4}|=\frac{1}{3}|\xi_{2,4}|,\\ |\xi_{1,2}|&=\left|\frac{p_4-p_7}{p_1-p_7}\right||\xi_{4,1}|=1|\xi_{4,1}|. \end{aligned}$$

Which is possible with nonzero values $\xi_{k,r}$, since

$$\left|\frac{p_1-p_3}{p_2-p_3}\right|\cdot\left|\frac{p_2-p_8}{p_4-p_8}\right|\cdot\left|\frac{p_4-p_7}{p_1-p_7}\right|=1.$$

Similarly, we have

$$\begin{aligned} \left|\frac{p_2-p_4}{p_8-p_4}\right|\cdot\left|\frac{p_8-p_6}{p_9-p_6}\right|\cdot\left|\frac{p_9-p_5}{p_2-p_5}\right|&=\frac{4}{3}\cdot\frac{3}{2}\cdot\frac{1}{2}=1,\\ \left|\frac{p_3-p_8}{p_7-p_8}\right|\cdot\left|\frac{p_7-p_6}{p_{10}-p_6}\right|\cdot\left|\frac{p_{10}-p_9}{p_3-p_9}\right|&=1\cdot 2\cdot\frac{1}{2}=1,\\ \left|\frac{p_4-p_1}{p_7-p_1}\right|\cdot\left|\frac{p_7-p_{10}}{p_6-p_{10}}\right|\cdot\left|\frac{p_6-p_5}{p_4-p_5}\right|&=2\cdot 1\cdot\frac{1}{2}=1,\\ \left|\frac{p_5-p_1}{p_{10}-p_1}\right|\cdot\left|\frac{p_{10}-p_3}{p_9-p_3}\right|\cdot\left|\frac{p_9-p_2}{p_5-p_2}\right|&=\frac{4}{3}\cdot\frac{3}{2}\cdot\frac{1}{2}=1, \end{aligned}$$

$$\left|\frac{p_1-p_2}{p_3-p_2}\right|\cdot\left|\frac{p_3-p_7}{p_8-p_7}\right|\cdot\left|\frac{p_8-p_9}{p_6-p_9}\right|\cdot\left|\frac{p_6-p_4}{p_5-p_4}\right|\cdot\left|\frac{p_5-p_{10}}{p_1-p_{10}}\right| = 2\cdot 2\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{3}=1,$$

so that if we specify that $\eta_1, \ldots, \eta_6 \in \mathbb{R}$ are pairwise rationally independent, then we can define all the $\{\xi_{k,s} : (k,s) \in \mathbb{J}\}$ via the formulae

$$\begin{aligned} 3\eta_1 &= 3\xi_{1,2} = \xi_{2,4} = 3\xi_{4,1}, \\ 4\eta_2 &= 4\xi_{2,8} = 3\xi_{8,9} = 2\xi_{9,2}, \\ 2\eta_3 &= 2\xi_{3,7} = 2\xi_{7,10} = \xi_{10,3}, \\ 2\eta_4 &= 2\xi_{4,7} = \xi_{7,6} = \xi_{6,4}, \\ 4\eta_5 &= 4\xi_{5,10} = 3\xi_{10,9} = 2\xi_{9,5}, \\ 12\eta_6 &= 12\xi_{1,3} = 6\xi_{3,8} = 3\xi_{8,6} = 6\xi_{6,5} = 4\xi_{5,1}. \end{aligned}$$

It is easily verified that these satisfy all instances of equation (4.8), and that if $\mathbb{I}_k = \{k, r, s\}$, then $\xi_{k,r} \not\equiv \xi_{k,s}$. To complete the construction, we use these values $\xi_{k,r}$ to give the cosets $S_{k,r}$ as above. That is, if $1 \le k \le 10$ and $\mathbb{I}_k = \{k, r, s\}$, then $S_{k,r} = \xi_{k,r}\widetilde{\mathbb{Z}}p_k + \mathbb{R}p_s$ and $S_{k,s} = \xi_{k,s}\widetilde{\mathbb{Z}}p_k + \mathbb{R}p_r$. (Remember that $\widetilde{\mathbb{Z}} = \mathbb{Z} + 1/2$.) Finally, put $X = \bigcup_{(k,r)\in\mathbb{J}} S_{k,r}$.

We now start proving that this example has the properties claimed.

4.2. Lemma. *With notation as in Example* 4.1, *there exist $U_1, U_2 \in \mathcal{U}_{\mathbb{R}^4}$ such that if $(i,j) \in \mathbb{J}$ and $k, r \in \{1, \ldots, 10\}$ are such that $k \notin \{i,j\}$ and $r \ne i$, then*

(i) $(\Xi_k + 2U_1) \cap S_{i,j} = \emptyset$,

(ii) $(\Pi_r + U_2) \cap S_{i,j} \subseteq (\Pi_r + U_2) \cap (\Pi_i + U_2) \subseteq (\Pi_r \cap \Pi_i) + U_1 \subseteq \bigcup_{s=1}^{10} (\Xi_s + U_1)$,

(iii) $X \cap (\Xi_i + 2U_1) \cap (\Xi_r + 2U_1) = \emptyset$.

Proof. Let $U_0 \in \mathcal{U}_{\mathbb{R}^4}$ have $3U_0 \cap X = \emptyset$. For any j, k, the Euclidean subgroups Ξ_j and Π_k satisfy (D), and so there exists $U_{j,k} \in \mathcal{U}_{\mathbb{R}^4}$ with $U_{j,k} \subseteq U_0$ and

$$(\Xi_j + 2U_{j,k}) \cap (\Pi_k + 2U_{j,k}) \subseteq (\Xi_j \cap \Pi_k) + U_0. \tag{4.9}$$

By replacing each $U_{j,k}$ with $U_1 = \bigcap_{j,k} U_{j,k} \in \mathcal{U}_{\mathbb{R}^4}$, we see that (4.9) holds with a fixed U_1 in place of $U_{j,k}$. Similarly, there exists $U_2 \in \mathcal{U}_{\mathbb{R}^4}$ such that $U_2 \subseteq U_1$ and

$$(\Pi_i + U_2) \cap (\Pi_r + U_2) \subseteq (\Pi_i \cap \Pi_r) + U_1 \qquad (i \ne r). \tag{4.10}$$

Equations (4.4) and (4.10) give (ii) immediately. Part (iii) follows by noting that we have either $r \notin \mathbb{I}_i$ or $i \notin \mathbb{I}_r$. In the first case, (4.9) gives $(\Xi_r + 2U_1) \cap (\Xi_i + 2U_1) \subseteq (\Xi_r \cap \Pi_i) + U_0 = U_0$, so that $X \cap (\Xi_r + 2U_1) \cap (\Xi_i + 2U_1) = \emptyset$. The second case follows symmetrically.

For (i), there are two cases. If $\mathbb{I}_i = \{i,j,k\}$, then $S_{i,j} = \xi_{i,j}\widetilde{\mathbb{Z}}p_i + \Xi_k$, and by (4.9), $(S_{i,j} + \Xi_k) \cap 2U_1 = S_{i,j} \cap 2U_1 \subseteq X \cap 2U_0 = \emptyset$. Hence $S_{i,j} \cap (\Xi_k + 2U_1) = \emptyset$. Otherwise $\mathbb{I}_i \ne \{i,j,k\}$, giving $k \notin \mathbb{I}_i$, so that by equation (4.3), $\Pi_i \cap \Xi_k = \{0\}$. Then by (4.9), $\Xi_k \cap (S_{i,j} + 2U_1) \subseteq U_0$. Hence $(\Xi_k + 2U_1) \cap S_{i,j} \subseteq S_{i,j} \cap (U_0 + 2U_1) = \emptyset$. ∎

4.3. Proposition. *The ideal $\mathcal{I}(X)$ of Example* 4.1 *is complemented.*

Proof. For $1 \le k \le 10$, let X_k be the instance of Example 2.11 occurring in the vicinity of the line Ξ_k and let X_{k+10} be the grid on the plane Π_k. That is,

$$X_k = \bigcup\{S_{i,j} : (i,j) \in \mathbb{J},\ i = k \text{ or } k = j\}$$

and

$$X_{k+10} = \bigcup\{S_{i,j} : (i,j) \in \mathbb{J},\ i = k\}.$$

We have (i), (ii) and (v) of Theorem 2.12. Let $U_1, U_2 \in \mathcal{U}_{\mathbb{R}^4}$ be as in Lemma 4.2. For $1 \le k \le n$, let $\mu_k, \nu_k \in M(\mathbb{R}^4)$ be such that

$$\begin{aligned} \widehat{\mu}_k &= 1 \quad \text{on } \Xi_k + U_1, & \widehat{\nu}_k &= 1 \quad \text{on } \Pi_k, \\ \widehat{\mu}_k &= 0 \quad \text{off } \Xi_k + 2U_1, & \widehat{\nu}_k &= 0 \quad \text{off } \Pi_k + U_2. \end{aligned}$$

(These measures are easily obtained from [14, Theorems 2.6.2 and 2.7.1]—for instance, as in the proof of [13, Theorem 2.3].) Define $F_k = \widehat{\mu}_k$ and $F_{k+10} = \widehat{\nu}_k \prod_{r=1}^{10} (1 - \widehat{\mu}_r) \in B(\mathbb{R}^4)$.

For each $1 \le k \le 10$ and for each $(i,j) \in \mathbb{J}$ such that $k \notin \{i,j\}$, by Lemma 4.2(i) we have $\widehat{\mu}_k = 0$ on $S_{i,j}$. Hence $F_k(X \setminus X_k) = \{0\}$. Similarly, if $(i,j) \in \mathbb{J}$ is such that $k \ne i$, then by Lemma 4.2(ii), $\widehat{\nu}_k = 0$ on $S_{i,j} \setminus \bigcup_{r=1}^{10} (\Xi_r + U_1)$, so that $F_{k+10} = 0$ on $S_{i,j}$. Hence $F_{k+10}(X \setminus X_{k+10}) = \{0\}$.

Finally, if $\gamma \in X$, then by Lemma 4.2(iii), $\prod_{k=1}^{10} (1 - \widehat{\mu}_k(\gamma)) = 1 - \sum_{r=1}^{10} \widehat{\mu}_r(\gamma)$, so that

$$1 - \sum_{k=1}^{20} F_k(\gamma) = \Big(1 - \sum_{k=1}^{10} \widehat{\nu}_k(\gamma)\Big) \prod_{r=1}^{10} (1 - \widehat{\mu}_r(\gamma)).$$

Clearly, if $\gamma \in \bigcup_{r=1}^{10} (\Xi_r + U_1)$ then $\sum_{k=1}^{20} F_k(\gamma) = 1$. Otherwise, there is some $(i,j) \in \mathbb{J}$ with $\gamma \in S_{i,j}$. Then $\widehat{\nu}_i(\gamma) = 1$ and if $k \ne i$, then $\widehat{\nu}_k(\gamma) = 0$, by Lemma 4.2(ii). Hence $\sum_{k=1}^{10} \widehat{\nu}_k(\gamma) = \widehat{\nu}_i(\gamma) = 1$, again giving $\sum_{k=1}^{20} F_k(\gamma) = 1$. Thus, by Theorem 2.12, the ideal $\mathcal{I}(X)$ is complemented. ∎

Remark. The number of sets X_k in the above proof can be halved by taking $X_1, \ldots, X_{10}$ as defined but with $F_k = \widehat{\mu}_k + \widehat{\nu}_k \prod_1^{10} (1 - \widehat{\mu}_r)$. The proof is given in the above form as a demonstration of a general method of applying Theorem 2.12. For a hull $X = \bigcup_{k=1}^{n} S_k$, this involves considering regions of the form $W_{\mathbb{I}} = (\bigcap_{i\in\mathbb{I}} S_i) + U_{\mathbb{I}}$, for suitably chosen $U_{\mathbb{I}} \in \mathcal{U}_{\Gamma}$. Then it is a matter of constructing the Fourier–Stieltjes transforms $F_{\mathbb{I}}$, each supported within $W_{\mathbb{I}} \setminus \bigcup_{\mathbb{J}\supsetneq\mathbb{I}} W_{\mathbb{J}}$. The difficulty here is assessing which $\bigcup_{i\in\mathbb{I}} S_i$ are complemented.

4.4. LEMMA. *Let $E_1, \ldots, E_4$ be as in Example 2.11, and suppose $Y_k \in \mathcal{R}(E_k)$ $(1 \le k \le 4)$ are such that $\Xi \cap E_1 \setminus Y_1$ and $\Xi \cap E_2 \setminus Y_2$ are finite. Then $\mathcal{I}(Y_1 \cup Y_2 \cup Y_3 \cup Y_4)$ is complemented if and only if $\Xi \cap E_3 \setminus Y_3$ and $\Xi \cap E_4 \setminus Y_4$ are finite.*

Proof. Suppose each $\Xi \cap (E_k \setminus Y_k)$ is finite. By Theorem 2.8 and Corollary 2.4.2, $Z = Y_3 \cup Y_4 \cup (\Xi \cap E_3 \setminus Y_3) \cup (\Xi \cap E_4 \setminus Y_4)$ has complemented kernel, and since $Y_1 \cap \Pi_3 \subseteq Z$ and $Y_2 \cap \Pi_3 \subseteq Z$, we can use an argument similar to that of Example 2.11 to show that $\mathcal{I}(Y_1 \cup Y_2 \cup Y_3 \cup Y_4)$ is complemented.

Conversely, suppose $\mathcal{I}(Y_1 \cup Y_2 \cup Y_3 \cup Y_4)$ is complemented. By Corollary 2.8.1, the hulls $\Pi_1 \cup \Pi_3$ and $\Pi_1 \cup \Pi_3 \cup Y_1 \cup Y_2 \cup Y_3 \cup Y_4 = \Pi_1 \cup \Pi_3 \cup Y_2$ have complemented kernel, so that by Proposition 2.5, $\mathcal{I}((\Pi_1 \cup \Pi_3) \cap (Y_1 \cup Y_2 \cup Y_3 \cup Y_4))$ is complemented. However,

$$\begin{aligned}(\Pi_1 \cup \Pi_3) &\cap (Y_1 \cup Y_2 \cup Y_3 \cup Y_4)\\ &= Y_1 \cup Y_3 \cup Y_4 \cup ((\Xi \cap Y_2) \setminus (Y_1 \cup Y_3 \cup Y_4)).\end{aligned}$$

Put $Y_0 = (\Xi \cap Y_2) \setminus (Y_1 \cup Y_3 \cup Y_4)$ and $E_0 = \Xi \cap E_2 = (\xi_2\mathbb{Z} + \eta_2)x_0$. Then E_0 is discrete and $Y_0 \in \mathcal{R}(E_0)$, so that $Y_0 \in \mathcal{R}_c(\mathbb{R}^4)$ is discrete and disjoint from $Y_1 \cup Y_3 \cup Y_4$. Hence, by Proposition 2.5, Y_0 is uniformly separated from $Y_1 \cup Y_3 \cup Y_4$.

Suppose Y_0 is infinite. Since E_0 is affinely homeomorphic to $\mathbb{Z}$, we see by [12, A0] that E_0 has an infinite subcoset E_0' such that $F_1 = E_0' \setminus Y_0$ is finite. Then $E_0' \cap Y_0 = E_0' \setminus F_1$ is uniformly separated from $Y_1 \cap \Xi = \Xi \cap E_1 \setminus F_2$, where $F_2 = \Xi \cap E_1 \setminus Y_1$ is finite. Since the addition of finite sets does not alter uniform separation, E_0' and $\Xi \cap E_1 \setminus E_0'$ are uniformly separated. However, $E_0' = (\xi_2(m\mathbb{Z} + n) + \eta_2)x_0$, for some $m, n \in \mathbb{Z}$ and $\Xi \cap E_1 = (\xi_1\mathbb{Z} + \eta_1)x_0$. Then $\xi_1 \not\equiv m\xi_2$ contradicts the uniform separation of E_0' and $\Xi \cap E_1 \setminus E_0'$. Hence Y_0 is finite.

Finally, $E_0 \cap (Y_1 \cup Y_3) \subseteq (\xi_2\mathbb{Z} + \eta_2) \cap (\xi_1\mathbb{Z} + \eta_1)$ consists of at most one point, so that

$$\begin{aligned}\Xi \cap E_4 \setminus Y_4 &= E_0 \setminus Y_4\\ &= E_0 \setminus (Y_2 \cup Y_4) \cup Y_0 \cup ((E_0 \cap Y_2 \setminus Y_4) \cap (Y_1 \cup Y_3))\end{aligned}$$

is also finite. Similarly, $\Xi \cap E_3 \setminus Y_3$ is finite. ∎

4.5. PROPOSITION. *With the notation of Example 4.1, suppose $Y_1, \ldots, Y_M \in \mathcal{R}_e(\mathbb{R}^4)$ are such that $Y_1 \cup \ldots \cup Y_M = X$. Then for some $1 \le m \le M$, $\mathcal{I}(Y_1 \cup \ldots \cup Y_m)$ is not complemented.*

Proof. For each $1 \le m \le M$, the set $Z_m = \overline{X \setminus \bigcup_{k=1}^m Y_k} \in \mathcal{R}_c(\mathbb{R}^4)$ has no isolated points, and lies within X, so that Z_m consists of a union of lines. Let $m < M$ be the maximum index for which Z_{m+1} consists of only finitely many lines. Then $Z_m \setminus Z_{m+1} \subseteq Y_{m+1}$, so that Y_{m+1} consists of infinitely many lines. Since Y_{m+1} is an elementary set, these lines are parallel, and it follows that $Y_{m+1} \subseteq S_{j,i}$, for some $(j, i) \in \mathbb{J}$. Thus, there are an infinite number of lines from Z_m in the coset $S_{j,i}$ and if $(r, s) \in \mathbb{J} \setminus \{(j, i)\}$, then there are only a finite number of lines from Z_m contained in $S_{r,s}$.

For each $(r, s) \in \mathbb{J}$, put $W_{r,s} = \overline{S_{r,s} \setminus Z_m}$ and $W = \bigcup_{(r,s)\in\mathbb{J}} W_{r,s}$. Then $W_{r,s} \in \mathcal{R}_c(S_{r,s})$ has no isolated points, and since $S_{r,s}$ is affinely homeomorphic to $\mathbb{R} \times \mathbb{Z}$, we have $W_{r,s} \in \mathcal{R}(S_{r,s})$. Moreover,

$$W = \overline{X \setminus Z_m} = \bigcup \{Y_k : 1 \le k \le m \text{ and } Y_k \text{ is not discrete}\}.$$

Let k, r, s be such that $(j, k), (r, j), (s, j) \in \mathbb{J}$, $k \ne i$, and $r \ne s$. Then $S_{j,i} \setminus W_{j,i} \subseteq Z_m$ consists of an infinite number of lines, so that $\Xi_j \cap S_{j,i} \setminus W_{j,i}$ is infinite. Similarly $\Xi_j \cap S_{j,k} \setminus W_{j,k}$, $\Xi_j \cap S_{r,j} \setminus W_{r,j}$ and $\Xi_j \cap S_{s,j} \setminus W_{s,j}$ are finite. Hence, by Lemma 4.4, $\mathcal{I}(W_{j,i} \cup W_{j,k} \cup W_{r,j} \cup W_{s,j})$ is not complemented.

Clearly each of $\widetilde{W}_r = W_{r,j}$, $\widetilde{W}_s = W_{s,j}$ and $\widetilde{W}_j = W_{j,i} \cup W_{j,k}$ has complemented kernel. Let $U_1, U_2 \in \mathcal{U}_{\mathbb{R}^4}$ be as in Lemma 4.2, then since $\widetilde{W}_r \subseteq \Pi_r$ and $\widetilde{W}_s \subseteq \Pi_s$, we see by Lemma 4.2(ii) that $(\widetilde{W}_r + U_2) \cap (\widetilde{W}_s + U_2) \subseteq \Xi_j + U_1$. Then by Lemma 4.2(i), $W \cap (\Xi_j + U_1) \subseteq W_{r,j} \cup W_{s,j} \cup W_{j,i} \cup W_{j,k} = \widetilde{W}_r \cup \widetilde{W}_s \cup \widetilde{W}_j$. Similarly $W \cap (\widetilde{W}_r + U_2) \cap (\widetilde{W}_j + U_2) \subseteq \widetilde{W}_r \cup \widetilde{W}_s \cup \widetilde{W}_j$ and $W \cap (\widetilde{W}_s + U_2) \cap (\widetilde{W}_j + U_2) \subseteq \widetilde{W}_r \cup \widetilde{W}_s \cup \widetilde{W}_j$. Hence, by Corollary 2.12.1, $\mathcal{I}(W)$ is not complemented. However, $W \subseteq \bigcup_{k=1}^m Y_k$ and $(\bigcup_{k=1}^m Y_k) \setminus W$ is discrete, so that by Corollary 2.4.2, $\mathcal{I}(\bigcup_{k=1}^m Y_k)$ is not complemented. ∎

5. Hulls containing sets failing (D′).

To have a characterization of complementation and a general algorithm for constructing projections or splitting maps, we need to address the cases which Theorem 2.8 cannot deal with—those where there are pairs of elementary sets in the hull failing (D_0). Our main technique here is Theorem 2.9, for which we are required to decompose X, the hull in question, into subsets $X_k \in \mathcal{R}_c(\Gamma)$, and then construct appropriate supersets $X_k' \in \mathcal{R}_c(\Gamma)$. We would obviously like a systematic method of specifying these sets X_k, X_k'. The examples in this section demonstrate some of the difficulties that have to be overcome.

We begin by considering Questions 4.1–4.3 of [4], which were aimed at addressing this issue. These questions were an attempt to generalize the phenomenon occurring in [4, Example 0.1(v)] (and consequently in Example 2.10), where the addition of the subgroup $\{0\} \times \mathbb{R} \times \{0\}$ to the set $(\mathbb{R} \times \mathbb{Z} \times \{0\}) \cup (\{0\} \times \sqrt{2}\mathbb{Z} \times \mathbb{R})$ converted a hull with noncomplemented kernel into one with complemented kernel, and the subgroup $\mathbb{R} \times \mathbb{R} \times \{0\}$ was instrumental in demonstrating this fact. The obvious question, generalized in [4, Question 4.3], is whether every complemented ideal $\mathcal{I}(X)$ for which $(\mathbb{R} \times \mathbb{Z} \times \{0\}) \cup (\{0\} \times \sqrt{2}\mathbb{Z} \times \mathbb{R}) \subseteq X$ has $\{0\} \times \mathbb{R} \times \{0\} \subseteq X$. This is answered in the negative by [1, Example 4.1], on which Example 2.11 of the current paper is based.

However, the other two questions are still of interest. These relate to the existence of a groups analogous to $\{0\}\times\mathbb{R}\times\{0\}$ and $\mathbb{R}\times\mathbb{R}\times\{0\}$ in a more general situation. For these, suppose (Γ_1,Γ_2) is a pair of closed subgroups failing (D). Question 4.1 asked for the existence of a (unique) minimal closed subgroup amongst those Γ_2' for which $\Gamma_2\subseteq\Gamma_2'$ and (Γ_1,Γ_2') satisfies (D). Question 4.2 asked for the existence of a (unique) minimal closed subgroup amongst those H for which each of the pairs (H,Γ_1), (H,Γ_2), $(H+\Gamma_1,\Gamma_2)$ and $(H+\Gamma_2,\Gamma_1)$ satisfies (D).

The example providing a negative answer to Question 4.3 does not render such subgroups Γ_2', H redundant—they were used in the application of Theorem 2.9 in Example 2.11. These subgroups Γ_2' and H are also useful for the construction of the Fourier–Stieltjes transforms to be used in Theorem 2.12. Examples of this occurred in Proposition 4.3, where each line Ξ_k was used in the definition of $\widehat{\mu}_k$ and each plane Π_k was used in the definition of $\widehat{\nu}_k$. In this case, none of the subgroups Ξ_k,Π_k were contained in the hull in question.

Unfortunately, we will see that each of Questions 4.1 and 4.2 of [4] has a negative answer. For the "unique" aspect of each question, we need look no further than the example given as an illustration. Put $\Gamma_1=\mathbb{R}\times\mathbb{Z}\times\{0\}$ and $\Gamma_2=\{0\}\times\sqrt{2}\mathbb{Z}\times\mathbb{R}$, and let H be *any* linear subgroup of $\mathbb{R}^3$ not lying in the plane $\mathbb{R}\times\{0\}\times\mathbb{R}$. Then H is minimal with respect to the conditions in [4, Question 4.2], but H can differ from $\{0\}\times\mathbb{R}\times\{0\}$, the subgroup stated as being the unique such subgroup. This also provides a negative answer to [4, Question 4.1], in that for any such H, the subgroup Γ_2+H is minimal amongst those Γ_2' for which (Γ_2',Γ_1) satisfies (D), and so there is no unique minimal such subgroup.

This also provides a different type of hull providing a negative answer to [4, Question 4.3]. If we take any H as above and put $\Lambda=(\Gamma_1+H)\cap(\Gamma_2+H)$, then $\mathcal{I}(\Gamma_1\cup\Gamma_2\cup\Lambda)$ is complemented. There are three cases here—the first where $\Lambda=H=\{0\}\times\mathbb{R}\times\{0\}$, the second where H is contained in either $\mathbb{R}\times\mathbb{R}\times\{0\}$ or $\{0\}\times\mathbb{R}\times\mathbb{R}$, in which case $\Lambda\cong\mathbb{R}\times\mathbb{Z}$, and the third where H does not lie in $(\mathbb{R}\times\mathbb{R}\times\{0\})\cup(\{0\}\times\mathbb{R}\times\mathbb{R})$, in which case $\Lambda\cong\mathbb{R}\times\mathbb{Z}^2$.

The following example shows that the "minimal" aspects of Questions 4.1 and 4.2 of [4] have negative answers. As in Section 2, Δ_X is the set $\{(x,x): x\in X\}$.

5.1. Example. Let Λ be an infinite totally disconnected compact abelian group (for instance, $\prod_{\mathbb{N}}\mathbb{Z}_2$), and put $\Gamma=\Lambda\times\Lambda_{\mathrm{d}}$, $\Gamma_1=\Delta_\Lambda$, and $\Gamma_2=\{0\}\times\Lambda_{\mathrm{d}}$. Suppose Γ_2' is a closed subgroup of Γ containing Γ_2. Then $\Gamma_2'=\Xi\times\Lambda_{\mathrm{d}}$ for some closed subgroup Ξ of Λ, so that $\Gamma_1\cap\Gamma_2'=\Delta_\Xi$. Now, Γ_1 is discrete, so that if (Γ_1,Γ_2') satisfies (D), then Γ_2'/Δ_Ξ is open in $(\Gamma_1+\Gamma_2')/\Delta_\Xi=\Gamma/\Delta_\Xi$. Hence Γ_2' is open in Γ and Ξ is open in Λ. Since Λ is compact, Ξ is of finite index in Λ. Conversely, if Ξ is of finite index in Λ then Γ_1/Δ_Ξ is finite, so that (Γ_1,Γ_2') satisfies (D). Hence (Γ_1,Γ_2') satisfies (D) if and only if Ξ is of finite index in Λ. However, an infinite totally disconnected compact abelian group has no minimal finite-index subgroup. Hence there is no minimal closed subgroup amongst those Γ_2' for which $\Gamma_2\subseteq\Gamma_2'$ and (Γ_1,Γ_2') satisfies (D).

It can be similarly shown that for a closed subgroup H of Γ, all four pairs (H,Γ_1), (H,Γ_2), $(H+\Gamma_1,\Gamma_2)$ and $(H+\Gamma_2,\Gamma_1)$ satisfy (D) if and only if H is of finite index in Γ, so that there is no minimal such H. Note that in this case, G is σ-compact precisely when $\widehat{\Lambda}$ is countable, whereas Γ is never σ-compact.

Thus, if we are to apply Theorems 2.9 and 2.12 to construct projections onto a specific ideal $\mathcal{I}(Z)$, we will need some alternative way of specifying the sets X_k, X_k' required in Theorem 2.9 and the Fourier–Stieltjes transforms required in Theorem 2.12. Each of these will have to rely on the elementary sets which make up the hull in question.

A possibility for finding the regions and Fourier–Stieltjes transforms was suggested by the remark following Proposition 4.3. Here we try to find regions of the form $W_{\mathbb{I}}=(\bigcap_{i\in\mathbb{I}}S_i)+U_{\mathbb{I}}$, where $U_{\mathbb{I}}\in\mathcal{U}_\Gamma$. and the S_i are some elementary sets comprising X such that $\mathcal{I}(\bigcup_{i\in\mathbb{I}}S_i)$ is complemented. A method based on that of Proposition 4.3 that may work in general is to specify the sets $U_{\mathbb{I}}$ for "large" $\mathbb{I}$ first. Then for $\mathbb{J}$, $\mathbb{K}$ with $\mathbb{J}\cup\mathbb{K}=\mathbb{I}$, choose $U_{\mathbb{J}}, U_{\mathbb{K}}$ such that $((\bigcap_{j\in\mathbb{J}}S_j)+U_{\mathbb{J}})\cap((\bigcap_{k\in\mathbb{K}}S_k)+U_{\mathbb{K}})\subseteq(\bigcap_{i\in\mathbb{I}}S_i)+U_{\mathbb{I}}$. These last choices are intended to rely on (D′) in the pair $(\bigcap_{j\in\mathbb{J}}S_j,\bigcap_{k\in\mathbb{K}}S_k)$.

The situation for the general application of Theorem 2.9 seems less straightforward. For this we require X to be represented as a union of sets $X_1,\ldots,X_n\in\mathcal{R}_c(\Gamma)$ such that there exist sets $\{X_k'\}_{k=1}^n$, each larger than the corresponding X_k. In previous applications of Theorem 2.9 (for instance, in 2.10, 2.11, 3.4 and 4.4), these supersets were constructed from sets of the form $\Lambda_1+\Lambda_2$, where (Λ_1,Λ_2) was a pair of closed subgroups satisfying (D) which were in some way extracted from the hull X. Obviously, a similar method can be applied for pairs of closed cosets (E_1,E_2) satisfying (D′), using $E_1+E_2-(E_1\cap E_2)$ in place of $\Lambda_1+\Lambda_2$. The following example demonstrates that we may have to build the supersets from other pieces. It consists of a union of 5 subgroups such that if we want to prove complementation, we are forced to build the supersets using pairs of subgroups failing (D).

5.2. Example. Let Γ_0 be a countable dense subgroup of $\mathbb{R}$, given its discrete topology, and put $\Gamma=\mathbb{R}^3\times\Gamma_0$, a σ-compact and metrizable group. Define closed subgroups $\Lambda_1,\ldots,\Lambda_5$ of Γ as follows:

$$\Lambda_1=\{(x,-\xi,4(\xi-x),\xi): x\in\mathbb{R},\ \xi\in\Gamma_0\},$$
$$\Lambda_2=\{(0,y,3\xi-y,\xi): y\in\mathbb{R},\ \xi\in\Gamma_0\},$$

$$\Lambda_3 = \{(0,y,0,\xi) : y \in \mathbb{R},\ \xi \in \Gamma_0\},$$
$$\Lambda_4 = \{(3\xi + y, y, 0, \xi) : y \in \mathbb{R},\ \xi \in \Gamma_0\},$$
$$\Lambda_5 = \{(4(\xi - z), \xi, z, \xi) : z \in \mathbb{R},\ \xi \in \Gamma_0\}.$$

Put $X = \bigcup_{k=1}^5 \Lambda_k$. A straightforward application of some arithmetic and the results in this paper verify the following:

(i) If $1 \le i < j \le 5$, then (Λ_i, Λ_j) satisfies (D) if and only if $\Lambda_i \cap \Lambda_j \neq \{0\}$, which occurs if and only if $j = i + 1$.

(ii) If $1 \le i < j \le 4$, then the pairs $(\Lambda_i \cap \Lambda_{i+1}, \Lambda_{j+1})$, $(\Lambda_i, \Lambda_j \cap \Lambda_{j+1})$ and $(\Lambda_i \cap \Lambda_{i+1}, \Lambda_j \cap \Lambda_{j+1})$ all fail (D).

(iii) If $2 \le n \le 5$, then $\mathcal{I}(\bigcup_{i=1}^{n-1} \overline{\Lambda_i + (\Lambda_{n-1} \cap \Lambda_n)})$ is complemented. (This uses Corollary 2.8.2.)

(iv) If $2 \le n \le 5$, then $(\bigcup_{i=1}^{n-1} \overline{\Lambda_i + (\Lambda_{n-1} \cap \Lambda_n)}) \cap \Lambda_n = \Lambda_{n-1} \cap \Lambda_n$ and since $\Lambda_{n-1} \cap \Lambda_n = (\bigcup_{i=1}^{n-1} \Lambda_i) \cap \Lambda_n$, we can apply a simple induction argument, involving Theorem 2.9, to show that each ideal $\mathcal{I}(\bigcup_{i=1}^{n} \Lambda_i)$ is complemented.

(v) For any nonempty $\mathbb{I} \subseteq \{1, \ldots, 4\}$, the ideal $\mathcal{I}(X \cup \bigcup_{i\in\mathbb{I}} (\Lambda_i + \Lambda_{i+1}))$ is not complemented. This requires the assessment of a number of cases using Proposition 2.5, but is essentially routine. The problem here is actually caused by intersections such as $(\Lambda_2 + \Lambda_3) \cap \Lambda_5$.

Thus, although $\mathcal{I}(X)$ is complemented, we see from (v) that it is not possible to apply Theorem 2.9 in the way we might have expected—that is, with the supersets constructed by extracting from the hull pairs of subgroups (Ξ_1, Ξ_2) satisfying (D), and using sums $\Xi_1 + \Xi_2$ as the constituents of these supersets. Moreover, it can be shown that we cannot avoid using sets such as $\overline{\Lambda_i + (\Lambda_{n-1} \cap \Lambda_n)}$ by using either of Theorems 2.8 or 2.12.

Hence, we have to accept that in a general algorithm for constructing projections, we may have to use sums of pairs of subgroups failing (D). This, unfortunately, causes other problems. For instance, suppose each subgroup Λ_k in the previous example was replaced by an elementary set $X_k \in \mathcal{R}(\Lambda_k)$. Then we could again consider subgroups such as $\overline{\Lambda_i + (\Lambda_{n-1} \cap \Lambda_n)}$, but we could not guarantee the existence of a suitable set $Z \in \mathcal{R}(\overline{\Lambda_i + (\Lambda_{n-1} \cap \Lambda_n)})$ that adequately reflects the structure of the X_k. This could occur, since the topology on $\overline{\Lambda_i + (\Lambda_{n-1} \cap \Lambda_n)}$ is "more connected" than the product topology on $\Lambda_i \oplus (\Lambda_{n-1} \cap \Lambda_n)$, resulting in a smaller coset ring.

6. Uniformities on hulls. This final section deals with a topological property of certain sets $X \in \mathcal{R}_c(\Gamma)$ that seems promising as a component in a characterization of when $\mathcal{I}(X)$ is complemented. Since this discussion is mostly speculative, much of the technical detail will be omitted.

By a *uniformity* (or *uniform structure*) on a set X, we will mean a filter $\mathcal{W}$ of subsets of $X \times X$ such that each $W \in \mathcal{W}$ contains Δ_X and for each $W \in \mathcal{W}$, there exists $W_0 \in \mathcal{W}$ with $W_0 \circ W_0 \subseteq W$, where $W_0 \circ W_0 = \{(x,z) \in X \times X : \text{there exists } y \in X \text{ with } (x,y),(y,z) \in W_0\}$ is the usual composition of relations. The *discrete uniformity* on X is $\{W \subseteq X \times X : \Delta_X \subseteq W\}$. Unless stated to the contrary, a subset Y of a uniform space X will be assumed to be endowed with the *relative uniformity* $\mathcal{W}|_Y = \{W \cap (Y \times Y) : W \in \mathcal{W}\}$. If $W \in \mathcal{W}$ and $x \in X$, put $W(x) = \{y \in X : (x,y) \in W\}$, then $\mathcal{W}(x) = \{W(x) : W \in \mathcal{W}\}$ is a filter of subsets of X. The unique topology on X such that the filter of neighbourhoods at each $x \in X$ is $\mathcal{W}(x)$ we call the topology *induced by* $\mathcal{W}$.

For $U \in \mathcal{U}_\Gamma$, define $W_U = \{(x,y) \in \Gamma \times \Gamma : x - y \in U\}$. Then $\{W_U : U \in \mathcal{U}_\Gamma\}$ forms a base for the standard uniformity on Γ, which we denote by $\mathcal{W}_\Gamma$. Suppose $X \in \mathcal{R}_c(\Gamma)$, and let $X = \bigcup_{k=1}^n X_k$ be a fixed decomposition of X into elementary sets. Then define

$$\begin{aligned} W_{X,U} &= \{(x_0, x_n) \in X \times X : \text{there exist } x_1, \ldots, x_{n-1} \in X, j_1, \ldots, j_n \le n, \\ &\qquad \text{with } (x_{k-1}, x_k) \in W_U \cap (X_{j_k} \times X_{j_k}) (1 \le k \le n)\} \\ &= \Big(W_U \cap \bigcup_{1\le j\le n} (X_j \times X_j)\Big)^{\circ n}, \end{aligned}$$

where $(\ldots)^{\circ n}$ denotes n-fold composition. It is easily verified that:

(i) $\{W_{X,U} : U \in \mathcal{U}_\Gamma\}$ forms the base for a uniformity $\mathcal{W}_X$ on X.

(ii) The topology induced by this uniformity coincides with the relative topology on X as a subspace of Γ.

(iii) For any elementary $Z \subseteq X$, $\mathcal{W}_X|_Z = \mathcal{W}_\Gamma|_Z$.

(iv) $\mathcal{W}_X$ is the strongest (finest, largest) uniformity such that each injection $X_k \hookrightarrow X$ is uniformly continuous.

(v) $\mathcal{W}_X$ is independent of the chosen decomposition of X into elementary sets.

If $X \in \mathcal{R}_c(\Gamma)$, we say X *satisfies* (U) if $\mathcal{W}_X = \mathcal{W}_\Gamma|_X$. This means that for each $U \in \mathcal{U}_\Gamma$, there exist $V \in \mathcal{U}_\Gamma$ such that for any $x, y \in X$ with $x - y \in U$, there exist $x_0, \ldots, x_n \in X$ such that $x_0 = x$, $x_n = y$, and for each k, $x_k - x_{k-1} \in V$ and x_{k-1}, x_k both lie in some X_j.

It follows by [4, Lemma 2.2] that if $X \in \mathcal{R}_c(\Gamma)$ is discrete, then $\Delta_X \in \mathcal{W}_X$, so that $\mathcal{W}_X$ is the discrete uniformity on X, whereas $\mathcal{W}_\Gamma|_X$ is the discrete uniformity if and only if X is uniformly discrete. Hence a discrete set $X \in \mathcal{R}_c(\Gamma)$ satisfies (U) if and only if X is uniformly discrete, which occurs if and only if $\mathcal{I}(X)$ is complemented.

Another instructive example is $X = \Lambda_1 \cup \Lambda_2$, where Λ_1, Λ_2 are closed subgroups of Γ. In this case, X satisfies (U) if and only if for each $U \in \mathcal{U}_\Gamma$, there exists $V \in \mathcal{U}_\Gamma$ such that for any $x \in \Lambda_1$ and $z \in \Lambda_2$ with $x - z \in V$, there exists $y \in \Lambda_1 \cap \Lambda_2$ with $x - y, y - z \in U$. Equivalently, $(\Lambda_1 + \Lambda_2) \cap V \subseteq (\Lambda_1 \cap U) + (\Lambda_2 \cap U)$. Hence, by Lemma 2.7, $\Lambda_1 \cup \Lambda_2$ satisfies (U) if and only if (Λ_1, Λ_2) satisfies (D). Similarly, if $S_1, S_2 \in \mathcal{R}_e(\Gamma)$ then $S_1 \cup S_2$ satisfies (U) if and only if (S_1, S_2) satisfies (D′). Thus, in the case of σ-compact G (and possibly for all G), if X is the union of a pair of elementary sets, then $\mathcal{I}(X)$ is complemented if and only if X satisfies (U).

This equivalence does not always hold. The hull $\mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z} \cup {}_\theta\mathbb{R}$ of [4, 0.1(iii)] satisfies (U), but has noncomplemented kernel, due to Proposition 2.5. (A more intricately constructed criterion that takes Proposition 2.5 into consideration is considered below.) It seems likely that we will have a one-way implication.

6.1. Conjecture. If $X \in \mathcal{R}_c(\Gamma)$ has $\mathcal{I}(X)$ complemented, then X satisfies (U).

It seems likely that this will yield to an argument involving weak compactness as in [4, Proposition 2.1 and Theorem 4.4]. In each of these two cases, the argument relied on the fact that failure of either uniform separation or (D) results in nets of points $\{x_\alpha\}$, $\{y_\alpha\}$ in X such that $y_\alpha - x_\alpha \to 0$ but for which there were functions $f_\alpha \in A(X)$ with $f_\alpha(x) = 0$ and $f_\alpha(y) = 1$ such that $\{f_\alpha\}$ is relatively weakly compact. Then it was shown that if $T : A(X) \to A(\Gamma)$ is a splitting map, then $T(\{f_\alpha\})$ cannot be relatively weakly compact, since weak compactness in a group algebra relies on uniform integrability, which conflicts with the separation properties required of the functions $T(f_\alpha)$. The difficulty in the more general case here is to construct the required functions $f_\alpha \in A(X)$.

6.2. Conjecture. If $X, Y \in \mathcal{R}_c(\Gamma)$ are such that $\mathcal{I}(X)$, $\mathcal{I}(Y)$ and $\mathcal{I}(X \cap Y)$ are complemented and $X \cup Y$ satisfies (U), then $\mathcal{I}(X \cup Y)$ is complemented.

Finally, we consider a criterion that, in the author's judgement, could provide a characterization of the complemented ideals of a group algebra. The examples we have seen so far seem to indicate that to determine whether $\mathcal{I}(\bigcup_{k=1}^n X_k)$ is complemented, we need to examine not only the relationships between the elementary sets $X_1, \ldots, X_n$, but also between elementary sets of the form $\bigcap_{j\in\mathbb{J}} X_j$. A convenient way of viewing this might be to consider the lattice of sets generated by $X_1, \ldots, X_n$.

Supposing $X_1, \ldots, X_n \in \mathcal{R}_e(\Gamma)$ and $X = \bigcup_{k=1}^n X_k$, let $\mathcal{L}$ be the lattice of sets generated by $X_1, \ldots, X_n$. For each $Z \in \mathcal{L}$, let $\mathcal{L}_Z = \{Y \in \mathcal{L} : Y \subset Z\}$. We define the property (U′) on sets $Z \in \mathcal{L}$ to be satisfied when

(i) Z satisfies (U), and

(ii) for any maximal $Y_0 \in \{Y \in \mathcal{L}_Z : Y \text{ fails (U′)}\}$, there is a unique minimal $Y_1 \in \{Z\} \cup \{Y \in \mathcal{L}_Z : Y \text{ satisfies (U′) and } Y_0 \subseteq Y\}$.

Since $\mathcal{L}$ is finite and the definition of (U′) in $Z \in \mathcal{L}$ depends only on (U) in Z and (U′) in the sets $Y \in \mathcal{L}$ for which $Y \subset Z$, this property is well defined throughout $\mathcal{L}$. Moreover, it can be shown that (U′) is independent of the decomposition into elementary sets. In particular, if $Z \in \mathcal{L}$ as above, and $\mathcal{L}'$ is the lattice generated by a set of elementary sets comprising Z, then we can suppose $\mathcal{L}' \subseteq \mathcal{L}$, so that Z having (U′) as a member of $\mathcal{L}$ coincides with property of having (U′) in its own right. In the case $G = \mathbb{R}^2$, (U′) can be shown to be equivalent to the criterion involving the *dependent intersection property* of [1, Theorem 3.1], so that (U′) characterizes complementation in this case. It is hoped that this will hold for general G.

6.3. Conjecture. If $X \in \mathcal{R}_c(\Gamma)$ then $\mathcal{I}(X)$ is complemented if and only if X satisfies (U′).

These last two conjectures are based partially on the author's experience with examples and partially on the available techniques for constructing projections and proving that projections do not exist. To see why the latter conjecture is plausible, suppose X is such that the conjecture holds for all $Z \in \mathcal{L} \setminus \{X\}$. Suppose also that Conjectures 6.1 and 6.2 hold throughout $\mathcal{L}$.

If X fails (U′), then either X fails (U) or there exists a maximal $Y_0 \in \{Y \in \mathcal{L}_X : Y \text{ fails (U′)}\}$ and distinct minimal $Y_1, Y_2 \in \{X\} \cup \{Y \in \mathcal{L}_X : Y \text{ satisfies (U′) and } Y_0 \subseteq Y\}$. In the first case, $\mathcal{I}(X)$ is not complemented, by Conjecture 6.1. In the second case, $Y_1 \cap Y_2$ is smaller than each of Y_1, Y_2, so $Y_1 \cap Y_2$ fails (U′). However, $Y_0 \subseteq Y_1 \cap Y_2$, so that $Y_0 = Y_1 \cap Y_2$. Now, each of Y_0, Y_1, Y_2 is properly contained in X, so that $\mathcal{I}(Y_1)$ and $\mathcal{I}(Y_2)$ are complemented and $\mathcal{I}(Y_0)$ is not complemented. Thus, by Proposition 2.5, $\mathcal{I}(Y_1 \cup Y_2)$ is not complemented. If $Y_1 \cup Y_2 \subset X$, then $Y_1 \cup Y_2$ fails (U′). However, $Y_0 \subset Y_1 \cup Y_2$, which contradicts the maximality of Y_0. Hence $X = Y_1 \cup Y_2$ has noncomplemented kernel.

On the other hand, if X satisfies (U′), then properties (i) and (ii) ensure that there is no way of using results such as Propositions 2.4 and 2.5 to prove that $\mathcal{I}(X)$ is not complemented. A more positive result seems possible, but more difficult. The presence of (U) in many of the subsets of X results in relations such as $(Y_1 + V) \cap (Y_2 + V) \subseteq (Y_1 \cap Y_2) + U$ for $U, V \in \mathcal{U}_\Gamma$ which in turn can aid the construction of the Fourier–Stieltjes transforms required in Theorem 2.12. Many of the problems outlined in the previous section come into play here. One of the main obstacles to the development of a general procedure for constructing projections is the structural complexity that can occur in a set $X \in \mathcal{R}_c(\Gamma)$.

Note. The author has recently proven Conjecture 6.1. This result, along with complete arguments concerning the material in this section, will appear in a forthcoming paper *Uniformities and complemented group algebra ideals*.

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Topologies and bornologies determined by operator ideals, II

by

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Abstract. Let $\mathfrak{A}$ be an operator ideal on LCS's. A continuous seminorm p of a LCS X is said to be $\mathfrak{A}$-*continuous* if $\widetilde{Q}_p \in \mathfrak{A}^{\mathrm{inj}}(X, \widetilde{X}_p)$, where $\widetilde{X}_p$ is the completion of the normed space $X_p = X/p^{-1}(0)$ and $\widetilde{Q}_p$ is the canonical map. p is said to be a Groth($\mathfrak{A}$)-*seminorm* if there is a continuous seminorm q of X such that $p \le q$ and the canonical map $\widetilde{Q}_{pq} : \widetilde{X}_q \to \widetilde{X}_p$ belongs to $\mathfrak{A}(\widetilde{X}_q, \widetilde{X}_p)$. It is well known that when $\mathfrak{A}$ is the ideal of absolutely summing (resp. precompact, weakly compact) operators, a LCS X is a nuclear (resp. Schwartz, infra-Schwartz) space if and only if every continuous seminorm p of X is $\mathfrak{A}$-continuous if and only if every continuous seminorm p of X is a Groth($\mathfrak{A}$)-seminorm. In this paper, we extend this equivalence to arbitrary operator ideals $\mathfrak{A}$ and discuss several aspects of these constructions which were initiated by A. Grothendieck and D. Randtke, respectively. A bornological version of the theory is also obtained.

1. Introduction. Let X be a LCS (locally convex space) and p a continuous seminorm of X. Denote by X_p the quotient space $X/p^{-1}(0)$ equipped with the quotient seminorm (in fact, norm) $\|\cdot\|_p$. Q_p denotes the canonical map from X onto X_p and $\widetilde{Q}_p$ denotes the unique map induced by Q_p from X into the completion $\widetilde{X}_p$ of X_p. If q is a continuous seminorm of X such that $p \le q$ (i.e. $p(x) \le q(x)$, $\forall x \in X$), the canonical maps $Q_{pq} : X_q \to X_p$ and $\widetilde{Q}_{pq} : \widetilde{X}_q \to \widetilde{X}_p$ are continuous.

Let $\mathfrak{A}$ be an operator ideal on Banach spaces. Following A. Pietsch [10], we call a LCS X a Groth($\mathfrak{A}$)-*space* if for each continuous seminorm p of X there is a continuous seminorm q of X such that $p \le q$ and $\widetilde{Q}_{pq} \in \mathfrak{A}(\widetilde{X}_q, \widetilde{X}_p)$. This amounts to saying that the completion $\widetilde{X}$ of X is a topological projective limit $\varprojlim \widetilde{Q}_{pq}\widetilde{X}_q$ of Banach spaces of type $\mathfrak{A}$ (cf. [7]). A. Grothendieck's construction of nuclear spaces is a model of Groth($\mathfrak{A}$)-spaces. In fact, a LCS X is a nuclear (resp. Schwartz, infra-Schwartz) space if it is a Groth($\mathfrak{N}$)-

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