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The Iwasawa λ-invariants of $\mathbb{Z}_p$-extensions of real quadratic fields

by

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1. Introduction. Let k be a totally real number field. Let p be a fixed prime number and $\mathbb{Z}_p$ the ring of all p-adic integers. We denote by $\lambda = \lambda_p(k)$, $\mu = \mu_p(k)$ and $\nu = \nu_p(k)$ the Iwasawa invariants of the cyclotomic $\mathbb{Z}_p$-extension k_∞ of k for p (cf. [10]).

Then Greenberg's conjecture states that both $\lambda_p(k)$ and $\mu_p(k)$ always vanish (cf. [8]). In other words, the order of the p-primary part of the ideal class group of k_n remains bounded as n tends to infinity, where k_n is the nth layer of k_∞/k. We know by the Ferrero–Washington theorem (cf. [2], [15]) that $\mu_p(k)$ always vanishes when k is an abelian (not necessarily totally real) number field. However, the conjecture remains unsolved up to now except for some special cases (cf. [1], [3], [5]–[8], [13]).

This paper is a continuation of our previous papers [3], [5]–[7] and [12], that is to say, we investigate Greenberg's conjecture when k is a real quadratic field and p is an odd prime number which splits in k. The purpose of this paper is to extend our previous results, and to give basic numerical data of $k = \mathbb{Q}(\sqrt{m})$ for $0 \le m \le 10000$ and $p = 3$. On the basis of these data, we can verify Greenberg's conjecture for most of these k's.

2. Notation and statement of the results. Let k be a real quadratic field with class number h and ε the fundamental unit of k. Let p be an odd prime number which splits in k, namely, $(p) = \mathfrak{p}\mathfrak{p}'$ in k where $\mathfrak{p} \ne \mathfrak{p}'$. Then we can choose $\alpha \in k$ such that $\mathfrak{p}'^h = (\alpha)$. In [6], we defined two invariants $n_1, n_2 \in \mathbb{N}$ for k and p by

$$\mathfrak{p}^{n_1} \parallel (\alpha^{p-1} - 1), \qquad \mathfrak{p}^{n_2} \parallel (\varepsilon^{p-1} - 1).$$

Here $\mathfrak{p}^n \parallel \mathfrak{a}$ means that $\mathfrak{p}^n \mid \mathfrak{a}$ and $\mathfrak{p}^{n+1} \nmid \mathfrak{a}$ for an ideal $\mathfrak{a}$ of k. In spite of ambiguity of α, n_1 is uniquely determined under the condition $n_1 \le n_2$.

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For the cyclotomic $\mathbb{Z}_p$-extension

$$k = k_0 \subset k_1 \subset \ldots \subset k_n \subset \ldots \subset k_\infty = \bigcup_{n=1}^{\infty} k_n$$

with Galois group $\Gamma = \mathrm{Gal}(k_\infty/k)$, let A_n be the p-primary part of the ideal class group of k_n, and $\mathfrak{p}_n$ (resp. $\mathfrak{p}'_n$) the unique prime ideal of k_n lying above $\mathfrak{p}$ (resp. $\mathfrak{p}'$). We put

$$A_n^\Gamma = \{a \in A_n \mid a^\sigma = a \text{ for all } \sigma \in \Gamma\} \quad \text{and} \quad D_n = \langle Cl(\mathfrak{p}_n)\rangle \cap A_n,$$

where $Cl(\mathfrak{p}_n)$ denotes the ideal class represented by $\mathfrak{p}_n$. Then we have $A_n^\Gamma \supset D_n$. These groups are closely related to Greenberg's conjecture (cf. Theorem 2 in [8]).

Moreover, we introduce two other invariants $n_0^{(r)}$ and $n_2^{(r)}$ following [13]. Let E_n be the group of units in k_n and d_n the order of $Cl(\mathfrak{p}_n)$ (so the order of $Cl(\mathfrak{p}'_n)$) in the ideal class group of k_n. For each $m \ge n \ge 0$, we denote by $N_{m,n}$ the norm map from k_m to k_n. Fix an integer $r \ge 0$. Then we can choose $\beta_r \in k_r$ such that $\mathfrak{p}'^{d_r}_r = (\beta_r)$. We define the invariants $n_0^{(r)}, n_2^{(r)} \in \mathbb{N}$ for k and p by

$$\mathfrak{p}^{n_0^{(r)}} \parallel (N_{r,0}(\beta_r)^{p-1} - 1), \qquad p^{n_2^{(r)}} = p^{n_2}(E_0 : N_{r,0}(E_r)).$$

As in the case of n_1, $n_0^{(r)}$ is uniquely determined under the condition $n_0^{(r)} \le n_2^{(r)}$, though the choice of β_r is not unique. Here we note that $r+1 \le n_0^{(r)}$ because k_∞/k is totally ramified at p. Furthermore, it is easy to see that

$$n_0^{(r)} \le n_0^{(r+1)} \le n_0^{(r)} + 1 \quad \text{and} \quad n_2^{(r)} \le n_2^{(r+1)} \le n_2^{(r)} + 1$$

for each $r \ge 0$. Put $n_0 = n_0^{(0)}$ in particular. We then see that $n_0 \le n_1 \le n_2$.

Remark 1. By the definitions of $n_0^{(r)}$ and $n_2^{(r)}$, we see that $n_0^{(r)}$ is the maximal integer n such that $\mathfrak{p}^n \mid (N_{r,0}(\beta_r)^{p-1} - 1)$ for all elements β_r of k_r satisfying $\mathfrak{p}'^{d_r}_r = (\beta_r)$ and that $n_2^{(r)}$ is the maximal integer n such that $\mathfrak{p}^n \mid (N_{r,0}(\varepsilon_r)^{p-1} - 1)$ for all elements ε_r of E_r. Indeed, it follows from the definition of $n_2^{(r)}$ that $\mathfrak{p}^{n_2^{(r)}} \mid (N_{r,0}(\varepsilon_r)^{p-1} - 1)$ for all $\varepsilon_r \in E_r$. Moreover, there exists $\eta_r \in E_r$ such that $\varepsilon^{u_r} = N_{r,0}(\eta_r)$, so that $\mathfrak{p}^{n_2^{(r)}} \parallel (N_{r,0}(\eta_r)^{p-1} - 1)$, where u_r denotes the integer such that $p^{u_r} = (E_0 : N_{r,0}(E_r))$. Hence the second assertion follows. The first one immediately follows from the inequality $n_0^{(r)} \le n_2^{(r)}$.

Remark 2. When we put $r = 0$, we have

$$n_1 = \min\{n_0 + v_p(h) - v_p(d_0)\,, n_2\}$$

where $v_p(a)$ denotes the exact power of p dividing a. Hence, if $A_0 = D_0$, then $n_0 = n_1$.

Let ζ_p be a primitive pth root of unity and $k^* = k(\zeta_p)$. For the *CM*-field k^*, let $(k^*)^+$ be the maximal real subfield of k^* and put $\lambda_p^-(k^*) = \lambda_p(k^*) - \lambda_p((k^*)^+)$. Our main theorems are as follows.

THEOREM 1 (Generalization of Proposition in [3] and Theorem 2 in [12]). *Let k be a real quadratic field and p an odd prime number which splits in k. Assume that*

(i) $\lambda_p^-(k^*) = 1$ *and*

(ii) $n_0^{(r)} \neq n_2^{(r)}$ *for some* $r \geq 0$.

Then $\lambda_p(k) = \mu_p(k) = 0$.

Remark 3. Let χ be the non-trivial Dirichlet character associated with k and ω the Teichmüller character of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$. We denote by $\lambda_p(k^*)_{\omega\chi^{-1}}$ the $\omega\chi^{-1}$-component of $\lambda_p(k^*)$. Then we may replace assumption (i) of Theorem 1 by a weaker assumption that $\lambda_p(k^*)_{\omega\chi^{-1}} = 1$ (cf. Proposition 1 in [9]).

Putting $r = 0$ in Theorem 1, we obtain the following

COROLLARY 1 (cf. Theorem 2 in [6]). *Let k and p be as in Theorem* 1. *If* $\lambda_p^-(k^*) = 1$ *and* $n_0 \neq n_2$, *then* $\lambda_p(k) = \mu_p(k) = 0$.

THEOREM 2. *Let k be a real quadratic field and p an odd prime number which splits in k. Assume that $A_0 = D_0$. Then the following conditions are equivalent.*

(i) $n_0^{(r)} = r + 1$ *for some* $r \geq 0$.

(ii) $n_0^{(r)} = r + 1$ *for all sufficiently large* r.

(iii) $n_2^{(r)} = r + 1$ *for some* $r \geq 0$.

(iv) $n_2^{(r)} = r + 1$ *for all sufficiently large* r.

(v) $A_n^{\Gamma} = D_n$ *for all sufficiently large* n.

In particular, one of these conditions holds if and only if $\lambda_p(k) = \mu_p(k) = 0$.

Putting $r = 0$ in the condition (i) of Theorem 2, we obtain the following.

COROLLARY 2 (cf. Theorem 1 in [6]). *Let k and p be as in Theorem* 2. *If* $A_0 = D_0$ *and* $n_0 = 1$ (*i.e.*, $n_1 = 1$), *then* $\lambda_p(k) = \mu_p(k) = 0$.

Moreover, putting $r = n_2 - 1$ in condition (iii) of Theorem 2, we obtain the following by Lemma 8 (cf. Section 5).

COROLLARY 3 (cf. Theorem in [5] and Lemma in [7]). *Let k and p be as in Theorem* 2. *If* $A_0 = D_0$ *and* $N_{n_2-1,0}(E_{n_2-1}) = E_0$, *then* $\lambda_p(k) = \mu_p(k) = 0$.

The notation defined in this section will be used throughout this paper. We also denote by $\beta_r \in k_r$ a generator of $\mathfrak{p}_r'^{d_r}$ satisfying

$$\mathfrak{p}^{n_0^{(r)}} \parallel (N_{r,0}(\beta_r)^{p-1} - 1) \quad \text{and} \quad n_0^{(r)} \leq n_2^{(r)}.$$

Namely, $\beta_r \in k_r$ is a generator of $\mathfrak{p}_r'^{d_r}$ which determines $n_0^{(r)}$. Since p splits in k, we have $k_{\mathfrak{p}} \simeq \mathbb{Q}_p$, where $k_{\mathfrak{p}}$ is the completion of k at $\mathfrak{p}$. So, by identifying $\mathfrak{p} \in k_{\mathfrak{p}}$ with $p \in \mathbb{Q}_p$, we may write $N_{r,0}(\beta_r)^{p-1} \in k$ as in the following form of a p-adic integer:

$$N_{r,0}(\beta_r)^{p-1} = 1 + p^{n_0^{(r)}} x_r, \qquad x_r \in \mathbb{Z}_p^{\times}.$$

3. Some fundamental lemmas. We first refer to the following three lemmas.

Lemma 1 (cf. Theorem 2 in [8]). *Let k and p be as in Section 2. Then $A_n^{\Gamma} = D_n$ for all sufficiently large n if and only if $\lambda_p(k) = \mu_p(k) = 0$.*

Lemma 2 (cf. Proposition 1 in [6]). *Let k and p be as in Section 2. Then*

$$|A_n^{\Gamma}| = \begin{cases} |A_0|p^n & \text{if } n < n_2 - 1, \\ |A_0|p^{n_2-1} & \text{if } n \geq n_2 - 1. \end{cases}$$

Lemma 3 (cf. Lemma 3 in [12]). *Let k and p be as in Section 2. If A_n is cyclic for all $n \geq 0$ and if D_r is non-trivial for some $r \geq 0$, then $\lambda_p(k) = \mu_p(k) = 0$.*

Next we prove two more lemmas. Since $\mathfrak{p}_r = \mathfrak{p}_{r+1}^p$, we have $d_{r+1} = d_r$ or pd_r; in particular, $|D_{r+1}| = |D_r|$ or $p|D_r|$. If we write $d_r = cp^j$ with an integer c prime to p, then c is independent of r.

Lemma 4. *Let r be a fixed non-negative integer. Assume that $|D_{r+1}| = p|D_r|$. Then*

$$n_0^{(r+1)} = \begin{cases} n_0^{(r)} & \text{if } n_0^{(r)} = n_2^{(r)} = n_2^{(r+1)}, \\ n_0^{(r)} + 1 & \text{otherwise.} \end{cases}$$

Proof. Since $d_{r+1} = pd_r$, we have

$$\mathfrak{p}_{r+1}'^{d_{r+1}} = \mathfrak{p}_{r+1}'^{pd_r} = \mathfrak{p}_r'^{d_r} = (\beta_r) \quad \text{in } k_{r+1}.$$

Thus we may take β_r as a generator of $\mathfrak{p}_{r+1}'^{d_{r+1}}$. Then we obtain

$$\begin{aligned} N_{r+1,0}(\beta_r)^{p-1} &= N_{r,0}(\beta_r)^{p(p-1)} \\ &= (1 + p^{n_0^{(r)}} x_r)^p, \qquad x_r \in \mathbb{Z}_p^{\times}, \\ &= 1 + p^{n_0^{(r)}+1} x_r', \qquad x_r' \in \mathbb{Z}_p^{\times}, \end{aligned}$$

therefore

$$\mathfrak{p}^{n_0^{(r)}+1} \,\|\, (N_{r+1,0}(\beta_r)^{p-1} - 1).$$

Hence it follows from the definition of $n_0^{(r+1)}$ that

$$n_0^{(r+1)} = \min\{n_0^{(r)} + 1, n_2^{(r+1)}\},$$

which yields the desired result. ∎

Lemma 5. *Let r be a fixed non-negative integer. Assume that $|D_r| = |D_{r+1}|$. Then*

(i) *If $n_0^{(r)} < n_2^{(r)}$, then $n_0^{(r+1)} = n_0^{(r)}$.*
(ii) *If $n_2^{(r)} = n_2^{(r+1)}$, then $n_0^{(r+1)} = n_0^{(r)}$.*

Proof. (i) Since $d_{r+1} = d_r$, we have

$$(\beta_r) = \mathfrak{p}_r'^{d_r} = N_{r+1,r}(\mathfrak{p}_{r+1}'^{d_r}) = N_{r+1,r}(\mathfrak{p}_{r+1}'^{d_{r+1}}) = (N_{r+1,r}(\beta_{r+1})) \quad \text{in } k_r.$$

Hence $N_{r+1,r}(\beta_{r+1}) = \beta_r\varepsilon_r$ for some $\varepsilon_r \in E_r$. Taking the norm from k_r to k, we have $N_{r+1,0}(\beta_{r+1}) = N_{r,0}(\beta_r)N_{r,0}(\varepsilon_r)$. Therefore we obtain the following p-adic expansion:

$$(1)\quad 1+p^{n_0^{(r+1)}}x_{r+1} = 1+p^{n_0^{(r)}}x_r+p^{n_2^{(r)}}y_r+\ldots, \qquad x_r, x_{r+1} \in \mathbb{Z}_p^\times,\ y_r \in \mathbb{Z}_p.$$

This implies the desired result.

(ii) Suppose that $n_0^{(r+1)} \ne n_0^{(r)}$. Then it follows from (1) that $n_0^{(r)} = n_2^{(r)}$. Therefore $n_0^{(r+1)} > n_0^{(r)} = n_2^{(r)} = n_2^{(r+1)}$, which contradicts the definition of $n_0^{(r+1)}$. This completes the proof. ∎

Remark 4. Lemmas 4 and 5 can be used for determining $n_0^{(r+1)}$ from $n_0^{(r)}$, $n_2^{(r)}$ and $n_2^{(r+1)}$. However, Lemma 5 does not work in the case where $n_0^{(r)} = n_2^{(r)} < n_2^{(r+1)}$. Actually, when $p = 3$, we see that $n_0 = n_2 = 2 < n_2^{(1)} = 3$ and $n_0^{(1)} = 2$ for $k = \mathbb{Q}(\sqrt{106})$, and that $n_0 = n_2 = 2 < n_2^{(1)} = 3$ and $n_0^{(1)} = 3$ for $k = \mathbb{Q}(\sqrt{295})$ (cf. Table 1). Hence, in this situation the practical calculation of β_{r+1} is necessary to the determination of $n_0^{(r+1)}$.

4. The proof of Theorem 1 and some examples. In order to prove Theorem 1, we need the following lemma.

Lemma 6. *Let r be a fixed non-negative integer. If $n_0^{(r)} \ne n_2^{(r)}$, then $|D_n| > |D_r|$ for all $n \ge n_0^{(r)}$.*

Proof. Suppose that $|D_n| = |D_r|$ for some $n \ge n_0^{(r)}$. Then since $d_n = d_r$, we have $N_{n,r}(\beta_n) = \beta_r\varepsilon_r$ for some $\varepsilon_r \in E_r$, as in the proof of Lemma 5. Taking the norm and expanding it in the p-adic form, we obtain

$$(2)\quad 1+p^{n_0^{(n)}}x_n = 1+p^{n_0^{(r)}}x_r+p^{n_2^{(r)}}y_r+\ldots, \qquad x_r, x_n \in \mathbb{Z}_p^\times,\ y_r \in \mathbb{Z}_p.$$

Since $n_0^{(n)} \ge n+1 \ge n_0^{(r)}+1 > n_0^{(r)}$ for all $n \ge n_0^{(r)}$, it follows from (2) that $n_0^{(r)} = n_2^{(r)}$. This completes the proof. ∎

By Lemma 2, $|A_n^\Gamma|$ remains bounded as n tends to infinity, hence so does $|D_n|$. Therefore we obtain the following as a corollary to Lemma 6.

COROLLARY 4. *Let k and p be as in Section 2. Then $n_0^{(r)} = n_2^{(r)}$ for all sufficiently large r.*

Proof of Theorem 1. Let k_n^* be the nth layer of the cyclotomic $\mathbb{Z}_p$-extension k_∞^*/k^* and A_n^* the p-primary part of the ideal class group of k_n^*. Since k_n^* is a CM-field, we can define $(A_n^*)^+$ by the p-primary part of the ideal class group of its maximal real subfield and $(A_n^*)^-$ by the kernel of the norm map from A_n^* to $(A_n^*)^+$. The Ferrero–Washington theorem guarantees the vanishing of $\mu_p(k^*)$, hence, by assumption (i), $(A_n^*)^-$ is cyclic for all $n \geq 0$. It follows from the reflection theorem that $(A_n^*)^+$ is cyclic, hence so is A_n for all $n \geq 0$. By Lemma 6, we also have the inequality $|D_n| > |D_r|$ under assumption (ii), hence $D_n \neq 1$, for all $n \geq n_0^{(r)}$. Therefore Theorem 1 immediately follows from Lemma 3. ∎

EXAMPLE 1. Let $k = \mathbb{Q}(\sqrt{26893})$ and $p = 3$, for which we could not verify that $\lambda_3(k) = \mu_3(k) = 0$ in [13]. Then $n_0 = n_1 = n_2 = 4$, and moreover, $\lambda_3^-(k^*) = 1$ and $n_0^{(1)} = 4 \neq n_2^{(1)} = 5$ (see Table 2 of [13]). Therefore it follows from Theorem 1 that $\lambda_3(k) = \mu_3(k) = 0$.

EXAMPLE 2. Let $k = \mathbb{Q}(\sqrt{4651})$ and $p = 3$. Then $\lambda_3^-(k^*) = 1$ and $n_0 = 1 \neq n_1 = n_2 = 2$ (see Table 1). Therefore it follows from Theorem 1 that $\lambda_3(k) = \mu_3(k) = 0$. Note that $|A_0| = 3 > 1 = |D_0|$. In order to conclude that $\lambda_3(k) = \mu_3(k) = 0$ for this k, we needed the information on the initial layer k_1 of k_∞/k before now (cf. [3], [7]). But we do not need such information now, therefore it seems that the invariant n_0 is more useful than n_1.

5. The proof of Theorem 2. First, we prove the following lemma.

LEMMA 7. *Let r and s be fixed non-negative integers. If $|D_{r+s}| = p^t|D_r|$, then*

$$n_0^{(r)} \geq \min\{n_0^{(r+s)} - t, n_2^{(r)} - t\}.$$

Proof. Note that $s \geq t$ and $d_{r+s} = p^t d_r$. Then we have

$$(\beta_{r+s}^{p^{s-t}}) = \mathfrak{p}_{r+s}'^{p^{s-t}d_{r+s}} = \mathfrak{p}_{r+s}'^{p^s d_r} = \mathfrak{p}_r'^{d_r} = (\beta_r) \quad \text{in } k_{r+s},$$

hence $(N_{r+s,r}(\beta_{r+s}))^{p^{s-t}} = (\beta_r)^{p^s}$. So $(N_{r+s,r}(\beta_{r+s})) = (\beta_r)^{p^t}$ in k_r. Therefore

$$\beta_r^{p^t} = N_{r+s,r}(\beta_{r+s})\varepsilon_r \quad \text{for some } \varepsilon_r \in E_r.$$

Taking the norm and expanding it in the p-adic form, we obtain

$$1 + p^{n_0^{(r)}+t}x_r' = 1 + p^{n_0^{(r+s)}}x_{r+s} + p^{n_2^{(r)}}y_r + \ldots, \quad x_r', x_{r+s} \in \mathbb{Z}_p^\times,\ y_r \in \mathbb{Z}_p.$$

This immediately implies Lemma 7. ∎

From now on, we consider the case where $A_0 = D_0$. Let $\overline{A}_n^{\Gamma}$ be the subgroup of A_n consisting of ideal classes which contain an ideal invariant under the action of $\mathrm{Gal}(k_n/k)$. Then the genus formula (cf. [16]) says that

$$|\overline{A}_n^{\Gamma}| = |A_0| \frac{p^n}{(E_0 : N_{n,0}(E_n))}.$$

If $A_0 = D_0$, then $\overline{A}_n^{\Gamma} = D_n$ for all $n \geq 0$ because $\overline{A}_n^{\Gamma} = i_{0,n}(A_0)D_n$, where $i_{0,n}$ denotes the natural map from the ideal group of k to the ideal group of k_n induced from the inclusion map. Hence we immediately obtain the following lemmas.

Lemma 8. *Let r be a fixed non-negative integer. Assume that $A_0 = D_0$. Then*

(i) $|D_r| = |D_0|p^{r-u_r} = |D_0|p^{n_2+r-n_2^{(r)}}$,

(ii) $n_2^{(r)} = n_2 + r - u$,

where u_r is the integer such that $p^{u_r} = (E_0 : N_{r,0}(E_r))$ and u is the integer such that $|D_r| = p^u|D_0|$.

Lemma 9. *Let r be a fixed non-negative integer. Assume that $A_0 = D_0$. Then $|D_{r+1}| = p|D_r|$ if and only if $n_2^{(r+1)} = n_2^{(r)}$.*

Proof. Let u_r be as in Lemma 8. Then $|D_{r+1}| = p|D_r|$ if and only if $u_{r+1} = u_r$. Hence the result follows from the definition of $n_2^{(r)}$. ■

Lemma 10. *Let r be a fixed non-negative integer. Assume that $A_0 = D_0$ and that $|D_{r+1}| = p|D_r|$. Then $n_0^{(r+1)} = n_0^{(r)}$ if and only if $n_0^{(r)} = n_2^{(r)}$. Namely, we have*

$$n_0^{(r+1)} = \begin{cases} n_0^{(r)} & \text{if } n_0^{(r)} = n_2^{(r)}, \\ n_0^{(r)} + 1 & \text{if } n_0^{(r)} \neq n_2^{(r)}. \end{cases}$$

Proof. This easily follows from Lemmas 4 and 9. ■

Proof of Theorem 2. (iv)⇒(iii)⇒(i) and (iv)⇒(ii)⇒(i) are trivial. Therefore it is sufficient to prove that (i)⇒(v)⇒(iv).

(i)⇒(v). Let r be a non-negative integer such that $n_0^{(r)} = r + 1$. Then $n_0^{(r+1)} = (r + 1) + 1$ because $(r + 1) + 1 \leq n_0^{(r+1)}$ and $n_0^{(r+1)} \leq n_0^{(r)} + 1$. Repeating this process, we conclude that $n_0^{(r+s)} = r + s + 1$ for all $s \geq 0$. We denote by u the integer such that $|D_r| = p^u|D_0|$. For $s \geq n_2 - 1 - u$, we put

$$|D_{r+s}| = p^t|D_r| = p^{t+u}|D_0|.$$

Now suppose that $t+u<n_2-1$. Then we have

$$n_0^{(r+s)}-t=r+s+1-t\geq r+n_2-u-t\geq r+2,$$
$$n_2^{(r)}-t=n_2+r-u-t\geq r+2$$

by Lemma 8(ii). It easily follows from Lemma 7 that

$$n_0^{(r)}\geq \min\{n_0^{(r+s)}-t, n_2^{(r)}-t\}\geq r+2,$$

which is a contradiction. Hence we must have $t+u=n_2-1$, so $|D_{r+s}|=|D_0|p^{n_2-1}$ for all $s\geq n_2-1-u$. Therefore Lemma 2 implies that $A_n^{\Gamma}=D_n$ for all $n\geq n_2^{(r)}-1$.

(v)⇒(iv). By Lemma 2, we have

$$|D_r|=|A_r^{\Gamma}|=|A_0|p^{n_2-1}=|D_0|p^{n_2-1}$$

for all sufficiently large r. Hence Lemma 8(i) shows that

$$|D_0|p^{n_2+r-n_2^{(r)}}=|D_0|p^{n_2-1},$$

which means that $n_2^{(r)}=r+1$ for all sufficiently large r.

The last assertion immediately follows from Lemma 1. This completes the proof of Theorem 2. ∎

6. Other useful results and some examples. In this section we shall give a few of easy results, which are useful when we cannot apply Theorems 1 and 2. First we prove the following.

Lemma 11. *If there exists an integer r_0 such that $|A_{r_0}^{\Gamma}|=|D_{r_0}|$ and $r_0\geq n_2-1$, then $A_n\simeq A_{r_0}$ for all $n\geq r_0$.*

Proof. Note that $N_{m,n}:A_m\to A_n$ and $N_{m,n}:D_m\to D_n$ are surjective for all $m\geq n\geq 0$ because k_∞/k is totally ramified at p. It follows from the assumption and Lemma 2 that $N_{m,n}:A_m^{\Gamma}\to A_n^{\Gamma}$ is isomorphic for all $m\geq n\geq r_0$. Hence, $N_{m,n}:A_m\to A_n$ is also isomorphic for all $m\geq n\geq r_0$. This completes the proof. ∎

Proposition 1. *Let k and p be as in Section 2. If $|D_r|=|A_0|p^{n_2-2}$ and $n_0^{(r)}\neq n_2^{(r)}$ for some $r\geq 0$, then $A_n\simeq A_{n_0^{(r)}}$ for all $n\geq n_0^{(r)}$, hence in particular $\lambda_p(k)=\mu_p(k)=0$.*

Proof. It follows from Lemma 6 that $|D_n|>|D_r|=|A_0|p^{n_2-2}$ for all $n\geq n_0^{(r)}$. Hence $|A_n^{\Gamma}|=|A_0|p^{n_2-1}=|D_n|$ for all $n\geq n_0^{(r)}$ by Lemma 2. Since $n_0^{(r)}\geq n_2-1$, the assertion immediately follows from Lemma 11. ∎

Example 3. Let $k=\mathbb{Q}(\sqrt{7753})$ and $p=3$. Then $n_0=1\neq n_1=n_2=2$, $\lambda_3^-(k^*)=2$ and $|A_0|=3>1=|D_0|$. Hence Theorems 1 and 2 cannot be applied to this k. However, $|D_1|=3=|A_0|$ and $n_0^{(1)}=2\neq n_2^{(1)}=3$ (see

Table 1). Therefore it follows from Proposition 1 that $A_n \simeq A_2$ for all $n \geq 2$, in particular $\lambda_3(k) = \mu_3(k) = 0$.

Lemma 6 asserts that $n_0^{(r)} \neq n_2^{(r)}$ implies $|D_r| < |D_{n_0^{(r)}}|$. However, the converse does not always hold (cf. Example 4). Thus the following proposition is sometimes useful. Here we note that, if A_n is cyclic for all $n \geq 0$ and if A_0 is trivial, then the converse is also true. In fact, for a fixed non-negative integer r, we see that $n_0^{(r)} = r + s$ if and only if $|D_r| = \ldots = |D_{r+s-1}| < |D_{r+s}|$ for $1 \leq s \leq n_2^{(r)} - r - 1$ in this situation (cf. Theorem 1 of [12]).

PROPOSITION 2. *Let k and p be as in Section 2. If $\lambda_p^-(k^*) = 1$, and $D_r \neq 1$ for some $r \geq 0$, then $\lambda_p(k) = \mu_p(k) = 0$.*

Proof. This immediately follows from the proof of Theorem 1 (or Lemma 3). ∎

EXAMPLE 4. Let $k = \mathbb{Q}(\sqrt{1129})$ and $p = 3$. Then $n_0 = n_1 = n_2 = 1$, $n_0^{(1)} = n_2^{(1)} = 2$ and $|A_0| = 9 > 3 = |D_0|$ (see Table 1). Hence Theorem 1 for $r = 0, 1$ and Theorem 2 cannot be applied to this k. But, since $\lambda_3^-(k^*) = 1$, it follows from Proposition 2 that $\lambda_3(k) = \mu_3(k) = 0$. Now, by Table 1 and Lemma 2, we see that $|A_1^{\Gamma}| = 9 = |D_1|$, so $|A_n^{\Gamma}| = |D_n| = |D_1|$ for all $n \geq 1$. Therefore Lemma 6 implies that $n_0^{(r)} = n_2^{(r)} = r + 1$ for all $r \geq 1$, so all $r \geq 0$. Hence we cannot apply Theorem 1 for all $r \geq 0$ to this k.

Finally we note that there exist some examples of k to which we cannot apply our theorems and propositions, but nevertheless we can verify Greenberg's conjecture for them by Lemma 11. Such examples are $k = \mathbb{Q}(\sqrt{6601})$, $k = \mathbb{Q}(\sqrt{6901})$ and so on.

7. Tables of basic numerical data of $k = \mathbb{Q}(\sqrt{m})$ for $p = 3$. We shall give a table of the fundamental data of $k = \mathbb{Q}(\sqrt{m})$ for $p = 3$ and positive square-free integers m's less than 10000 satisfying $m \equiv 1 \pmod 3$. The total number of such m's is exactly 2279. We find that there exist exactly 2042 m's which satisfy $A_0 = D_0$ and $n_0 = 1$. Greenberg's conjecture is valid for these k's by Corollary 2 to Theorem 2. Table 1 gives several useful data for 237 remaining m's. We can verify Greenberg's conjecture for 185 k's in Table 1 by applying our results. The asterisks in the column of $\lambda_3(k)$, the number of which is exactly 52, mean that Greenberg's conjecture cannot be verified by these data.

Concerning our method of computation, we refer to [11] and [13] for $n_0^{(1)}, n_2^{(1)}, |A_1|$ and $|D_1|$, to [14] for the 3-primary part A_0^{*-} of the ideal class group of $\mathbb{Q}(\sqrt{-3m})$, and to [4] for $\lambda_3^-(k^*)$. Note that $\lambda_3^-(k^*) = \lambda_3(\mathbb{Q}(\sqrt{-3m}))$. The rest is easily computed.

Addendum. Recently, after we have written this paper, we heard from H. Sumida that he verified Greenberg's conjecture for $p = 3$ and $m =$ 727, 2794, 4279, 4741, 5533, 7429, 7465, 7642, 9634 and 9691, which are marked with the asterisks in Table 1, by computing the Iwasawa polynomials associated with p-adic L-functions. He is now preparing the paper entitled "Greenberg's conjecture and the Iwasawa polynomial".

Table 1. All m's satisfying $A_0 \neq D_0$ or $n_0 > 1$: $1 \leq m \leq 10000$

m	n_0	n_1	n_2	$n_0^{(1)}$	$n_2^{(1)}$	$\lambda_3^-(k^*)$	A_0^{*-}	$\lvert D_0\rvert$	$\lvert A_0\rvert$	$\lvert D_1\rvert$	$\lvert A_1\rvert$	$\lambda_3(k)$
67	2	2	3	2	4	1	(3)	1	1	1	3	0
103	2	2	2	2	2	2	(3)	1	1	3	9	0
106	2	2	2	2	3	1	(3)	1	1	1	3	0
139	2	2	2	2	2	2	(3)	1	1	3	9	0
238	2	2	3	2	4	1	(3)	1	1	1	3	0
253	2	2	2	2	3	1	(3)	1	1	1	3	0
295	2	2	2	3	3	1	(3)	1	1	1	3	*
397	2	2	2	3	3	1	(3)	1	1	1	3	*
418	2	2	2	2	2	2	(3)	1	1	3	9	0
454	2	2	2	2	3	1	(3)	1	1	1	3	0
505	2	2	2	2	3	1	(3)	1	1	1	3	0
607	2	2	2	2	3	1	(9)	1	1	1	3	0
610	2	2	4	2	5	1	(3)	1	1	1	3	0
679	2	2	2	2	2	2	(3)	1	1	3	9	0
727	2	2	3	3	3	2	(9)	1	1	3	9	*
745	2	2	2	3	3	1	(3)	1	1	1	3	*
787	2	2	2	2	3	1	(9)	1	1	1	3	0
790	2	2	2	2	2	2	(3)	1	1	3	9	0
886	2	2	2	2	3	1	(3)	1	1	1	3	0
994	2	2	2	2	3	1	(3)	1	1	1	3	0
1102	2	2	2	2	3	1	(3)	1	1	1	3	0
1129	1	1	1	2	2	1	(3)	3	9	9	27	0
1153	2	2	2	2	2	2	(3)	1	1	3	9	0
1261	2	2	2	2	2	2	(3)	1	1	3	9	0
1294	2	2	2	2	3	1	(3)	1	1	1	3	0
1318	2	2	2	2	3	1	(3)	1	1	1	3	0
1333	2	2	2	2	3	1	(3)	1	1	1	3	0
1390	3	3	4	3	5	1	(3)	1	1	1	3	0
1462	2	2	2	2	3	1	(3)	1	1	1	3	0
1609	2	2	2	2	2	4	(3)	1	1	3	9	0
1642	2	2	2	2	2	2	(3)	1	1	3	27	0
1654	1	1	1	2	2	1	(3)	3	9	9	27	0
1669	2	2	2	2	3	1	(9)	1	1	1	3	0
1714	2	2	2	3	3	4	(3, 3)	3	3	3	9	*
1726	2	2	2	2	2	2	(3)	1	1	3	27	0
1738	2	2	2	3	3	1	(9)	1	1	1	3	*
1753	2	2	2	2	3	1	(3)	1	1	1	3	0
1810	2	2	2	2	3	1	(9)	1	1	1	3	0

Table 1 (cont.)

m	n_0	n_1	n_2	$n_0^{(1)}$	$n_2^{(1)}$	$\lambda_3^-(k^*)$	A_0^{*-}	$\|D_0\|$	$\|A_0\|$	$\|D_1\|$	$\|A_1\|$	$\lambda_3(k)$
1867	2	2	6	2	7	1	(3)	1	1	1	3	0
1894	2	2	3	2	4	1	(3)	1	1	1	3	0
1954	1	1	1	2	2	1	(3)	1	3	3	9	0
2029	2	2	2	3	3	1	(9)	1	1	1	3	$*$
2059	3	3	3	4	4	1	(3)	1	1	1	3	$*$
2122	2	2	2	2	3	2	(3)	1	1	1	9	0
2149	4	4	4	5	5	1	(3)	1	1	1	3	$*$
2158	2	2	2	2	3	1	(3)	1	1	1	3	0
2221	2	2	3	2	4	1	(3)	1	1	1	3	0
2230	2	2	2	2	3	2	(3,3)	3	3	3	9	0
2263	2	2	2	2	3	2	(3,3)	3	3	3	9	0
2371	2	2	2	2	3	1	(9)	1	1	1	3	0
2410	2	2	3	2	4	1	(3)	1	1	1	3	0
2419	1	1	1	2	2	1	(9)	1	3	3	9	0
2431	2	2	2	2	2	3	(3)	1	1	3	9	0
2515	2	2	2	2	3	1	(9)	1	1	1	3	0
2521	2	2	3	2	4	1	(3)	1	1	1	3	0
2593	2	2	3	2	4	1	(3)	1	1	1	3	0
2659	2	2	3	2	4	2	(3,3)	3	3	3	9	0
2701	3	3	5	3	6	1	(3)	1	1	1	3	0
2713	1	1	1	2	2	1	(9)	1	3	1	9	$*$
2737	2	2	2	2	3	1	(3)	1	1	1	3	0
2743	2	2	3	2	4	1	(3)	1	1	1	3	0
2794	2	2	3	3	3	2	(9)	1	1	3	9	$*$
2917	3	3	3	4	4	3	(3,3)	3	3	3	9	$*$
2971	1	1	1	2	2	1	(9)	1	3	3	9	0
3001	2	2	2	2	2	2	(3)	1	1	3	9	0
3094	2	2	2	2	2	2	(3)	1	1	3	9	0
3133	3	3	5	3	6	1	(3)	1	1	1	3	0
3190	2	2	2	2	3	1	(3)	1	1	1	3	0
3199	2	2	2	2	3	1	(3)	1	1	1	3	0
3226	2	2	2	2	3	1	(9)	1	1	1	3	0
3235	2	2	2	2	3	1	(9)	1	1	1	3	0
3277	2	2	2	2	3	1	(27)	1	1	1	3	0
3355	2	2	2	2	2	3	(3)	1	1	3	9	0
3391	2	2	4	2	5	2	(3,3)	3	3	3	9	0
3469	2	2	2	3	3	2	(3)	1	1	1	9	$*$
3490	2	2	2	3	3	1	(9)	1	1	1	3	$*$
3571	2	2	2	2	3	1	(3)	1	1	1	3	0
3667	2	2	2	2	3	2	(3,3)	3	3	3	9	0
3673	2	2	4	2	5	1	(3)	1	1	1	3	0
3739	2	2	2	3	3	1	(3)	1	3	1	9	$*$
3781	2	2	2	2	3	1	(9)	1	1	1	3	0
3787	2	2	2	2	2	2	(3)	1	1	3	9	0
3847	2	2	2	2	2	2	(3)	1	1	3	9	0
3895	2	2	3	2	4	1	(3)	1	1	1	3	0

Table 1 (cont.)

m	n_0	n_1	n_2	$n_0^{(1)}$	$n_2^{(1)}$	$\lambda_3^-(k^*)$	A_0^{*-}	$\|D_0\|$	$\|A_0\|$	$\|D_1\|$	$\|A_1\|$	$\lambda_3(k)$
3979	2	2	3	2	4	1	(3)	1	1	1	3	0
3997	2	2	2	2	3	1	(9)	1	1	1	3	0
4081	3	3	3	4	4	1	(3)	1	1	1	3	$*$
4099	2	2	2	2	3	2	(3)	1	1	1	27	0
4207	2	2	2	2	3	1	(9)	1	1	1	3	0
4210	2	2	2	2	3	1	(9)	1	1	1	3	0
4222	2	2	2	2	2	2	(3)	1	1	3	9	0
4237	2	2	2	2	3	1	(3)	1	1	1	3	0
4279	3	3	3	3	3	2	(3, 3)	3	3	9	27	$*$
4447	2	2	2	2	3	2	(3)	1	1	1	9	0
4471	1	1	1	2	2	1	(3)	1	3	3	9	0
4498	2	2	2	2	3	1	(3)	1	1	1	3	0
4519	2	2	2	2	3	1	(3)	1	1	1	3	0
4603	2	2	2	2	3	1	(27)	1	1	1	3	0
4615	2	2	3	2	4	1	(3)	1	1	1	3	0
4618	2	2	4	2	5	1	(3)	1	1	1	3	0
4651	1	2	2	2	3	1	(3)	1	3	3	9	0
4654	2	2	2	3	3	1	(3)	1	1	1	3	$*$
4681	2	2	2	2	3	1	(3)	1	1	1	3	0
4687	2	2	2	2	2	3	(3)	1	1	3	9	0
4711	2	2	2	2	3	1	(3)	1	1	1	3	0
4741	2	2	3	3	3	3	(9)	1	1	3	9	$*$
4789	2	2	2	3	3	1	(9)	1	1	1	3	$*$
4837	2	2	2	2	2	3	(3)	1	1	3	9	0
4867	2	2	2	2	3	1	(3)	1	1	1	3	0
4870	2	2	2	2	3	1	(9)	1	1	1	3	0
4954	1	1	1	2	2	1	(3)	3	9	9	27	0
4963	2	2	3	2	4	1	(3)	1	1	1	3	0
5005	2	2	2	2	2	2	(3)	1	1	3	9	0
5062	3	3	3	3	4	1	(3)	1	1	1	3	0
5083	2	2	2	2	3	1	(3)	1	1	1	3	0
5113	2	2	2	2	3	1	(3)	1	1	1	3	0
5149	2	2	2	2	3	1	(9)	1	1	1	3	0
5161	2	2	2	2	2	2	(3)	1	1	3	9	0
5182	2	2	2	2	3	1	(3)	1	1	1	3	0
5185	2	2	2	3	3	1	(3)	1	1	1	3	$*$
5365	2	2	2	2	2	2	(3)	1	1	3	9	0
5386	2	2	2	2	2	2	(3)	1	1	3	9	0
5407	2	2	2	2	2	2	(3)	1	1	3	27	0
5437	2	2	2	2	2	2	(3)	1	1	3	9	0
5458	2	2	2	2	2	2	(3)	1	1	3	9	0
5494	2	2	2	2	3	1	(3)	1	1	1	3	0
5530	2	2	2	3	3	2	(3)	1	1	1	9	$*$
5533	2	2	3	3	3	2	(9)	1	1	3	9	$*$
5611	3	3	3	3	3	3	(9)	1	1	3	9	$*$
5617	2	2	2	2	3	1	(9)	1	1	1	3	0

Table 1 (cont.)

m	n_0	n_1	n_2	$n_0^{(1)}$	$n_2^{(1)}$	$\lambda_3^-(k^*)$	A_0^{*-}	$\|D_0\|$	$\|A_0\|$	$\|D_1\|$	$\|A_1\|$	$\lambda_3(k)$
5647	2	2	3	2	4	1	(3)	1	1	1	3	0
5749	2	2	2	2	3	1	(27)	1	1	1	3	0
5902	2	2	2	2	3	1	(9)	1	1	1	3	0
5938	1	1	1	2	2	1	(27)	1	3	1	9	*
5971	2	2	3	3	3	3	(9)	1	1	3	27	*
6001	2	2	2	2	3	1	(9)	1	1	1	3	0
6169	2	2	2	3	3	1	(3)	1	1	1	3	*
6187	2	2	2	3	3	3	(3)	1	1	1	9	*
6202	2	2	2	3	3	1	(3)	1	1	1	3	*
6238	1	1	1	2	2	1	(3)	1	3	3	9	0
6271	2	2	2	3	3	1	(3)	1	1	1	3	*
6286	2	2	2	3	3	1	(9)	1	1	1	3	*
6295	2	2	2	2	2	2	(3)	1	1	3	9	0
6355	2	2	2	2	3	1	(3)	1	1	1	3	0
6403	2	2	2	2	3	1	(9)	1	1	1	3	0
6430	2	2	2	2	2	3	(3)	1	1	3	9	0
6451	2	2	2	2	2	2	(3)	1	1	3	27	0
6502	2	2	2	2	3	1	(9)	1	1	1	3	0
6559	2	2	4	3	4	2	(3, 3)	9	9	27	81	*
6601	1	1	1	2	2	2	(3)	1	3	3	9	0
6691	2	2	2	2	3	1	(3)	1	1	1	3	0
6730	2	2	2	2	3	1	(9)	1	1	1	3	0
6799	2	2	2	2	2	2	(3)	1	1	3	9	0
6871	2	2	2	3	3	1	(27)	1	1	1	3	*
6901	1	1	1	2	2	2	(3)	1	3	3	9	0
6907	2	2	2	2	3	1	(3)	1	1	1	3	0
6934	2	2	2	3	3	1	(9)	1	1	1	3	*
6949	2	2	2	2	2	2	(3)	1	1	3	9	0
6955	3	3	4	3	5	1	(3)	1	1	1	3	0
7006	3	3	3	3	4	3	(3, 3)	3	3	3	9	*
7051	2	2	2	2	3	1	(9)	1	1	1	3	0
7078	2	2	4	2	5	1	(3)	1	1	1	3	0
7234	1	1	1	2	2	2	(3)	1	3	3	9	0
7246	2	2	3	2	4	2	(9)	1	1	1	9	0
7294	2	2	2	2	3	1	(9)	1	1	1	3	0
7303	2	2	2	2	2	3	(3)	1	1	3	9	0
7309	2	2	2	3	3	1	(9)	1	1	1	3	*
7315	2	2	2	2	3	2	(3)	1	1	1	9	0
7321	2	2	2	3	3	1	(3)	1	1	1	3	*
7387	1	1	1	2	2	1	(9)	1	3	3	9	0
7429	2	2	3	3	3	2	(9)	1	1	3	9	*
7465	3	3	3	3	4	2	(3, 3)	9	9	9	27	*
7522	2	2	2	2	3	1	(3)	1	1	1	3	0
7582	2	2	2	3	3	1	(3)	1	1	1	3	*
7603	2	2	2	2	3	1	(27)	1	1	1	3	0
7621	2	2	2	2	3	1	(3)	1	1	1	3	0

Table 1 (cont.)

m	n_0	n_1	n_2	$n_0^{(1)}$	$n_2^{(1)}$	$\lambda_3^-(k^*)$	A_0^{*-}	$\lvert D_0\rvert$	$\lvert A_0\rvert$	$\lvert D_1\rvert$	$\lvert A_1\rvert$	$\lambda_3(k)$
7633	2	2	2	2	3	1	(9)	1	1	1	3	0
7639	1	1	1	2	2	1	(3)	1	3	3	9	0
7642	2	2	3	3	3	2	(27)	1	1	3	9	$*$
7705	2	2	2	2	2	2	(3)	1	1	3	9	0
7711	1	2	2	2	3	1	(3)	1	3	3	9	0
7726	2	2	2	2	3	3	(3, 3)	1	3	1	81	$*$
7753	1	2	2	2	3	2	(9)	1	3	3	27	0
7906	2	2	2	2	3	1	(9)	1	1	1	3	0
7951	2	2	3	2	4	1	(3)	1	1	1	3	0
7954	2	2	3	2	4	2	(3, 3)	3	3	3	9	0
7957	2	2	2	3	3	1	(3)	1	1	1	3	$*$
7969	3	3	3	3	4	1	(3)	1	1	1	3	0
7978	2	2	2	2	2	2	(3)	1	1	3	9	0
8011	2	2	2	2	3	1	(3)	1	1	1	3	0
8017	1	1	1	2	2	1	(3)	1	3	1	9	$*$
8095	2	2	2	2	3	1	(3)	1	1	1	3	0
8101	2	2	2	3	3	1	(3)	1	1	1	3	$*$
8137	2	2	2	2	3	2	(3)	1	1	1	27	0
8155	2	2	2	3	3	1	(3)	1	1	1	3	$*$
8194	2	2	2	2	2	4	(3)	1	1	3	9	0
8203	2	2	2	2	3	1	(3)	1	1	1	3	0
8209	2	2	2	2	2	2	(3)	1	1	3	9	0
8245	2	2	2	2	3	1	(3)	1	1	1	3	0
8365	2	2	2	2	3	1	(9)	1	1	1	3	0
8374	2	2	3	2	4	3	(3, 3)	3	3	3	27	0
8422	2	2	2	2	3	1	(3)	1	1	1	3	0
8545	1	1	1	2	2	1	(9)	1	3	3	9	0
8569	2	2	2	3	3	1	(3)	1	1	1	3	$*$
8599	2	2	2	2	3	1	(3)	1	1	1	3	0
8626	2	2	2	2	3	1	(3)	1	1	1	3	0
8713	2	2	3	2	4	2	(3, 3)	3	3	3	9	0
8755	2	2	2	2	3	1	(3)	1	1	1	3	0
8758	2	2	2	2	2	4	(3)	1	1	3	9	0
8782	1	1	1	2	2	1	(9)	1	3	1	9	$*$
8785	2	2	3	2	4	1	(3)	1	1	1	3	0
8809	2	2	4	2	5	1	(3)	1	1	1	3	0
8821	2	2	4	2	5	1	(3)	1	1	1	3	0
8854	1	1	1	2	2	2	(3)	1	3	3	9	0
8863	1	2	2	2	3	1	(3)	1	3	3	9	0
8893	2	2	2	2	2	2	(3)	1	1	3	9	0
8965	3	3	3	3	4	1	(3)	1	1	1	3	0
9019	2	2	2	2	3	1	(9)	1	1	1	3	0
9034	1	1	1	2	2	1	(27)	1	3	3	9	0
9058	2	2	2	3	3	1	(9)	1	1	1	3	$*$
9097	2	2	2	2	2	2	(3)	1	1	3	27	0
9103	2	2	2	2	3	1	(27)	1	1	1	3	0

Table 1 (cont.)

m	n_0	n_1	n_2	$n_0^{(1)}$	$n_2^{(1)}$	$\lambda_3^-(k^*)$	A_0^{*-}	$\lvert D_0\rvert$	$\lvert A_0\rvert$	$\lvert D_1\rvert$	$\lvert A_1\rvert$	$\lambda_3(k)$
9115	2	2	3	2	4	1	(3)	1	1	1	3	0
9145	2	2	2	2	3	1	(3)	1	1	1	3	0
9202	2	2	2	2	3	1	(3)	1	1	1	3	0
9274	4	4	5	4	6	1	(3)	1	1	1	3	0
9427	2	2	2	2	3	3	(3)	1	1	1	9	0
9463	2	2	3	2	4	1	(3)	1	1	1	3	0
9586	1	1	1	2	2	3	(3)	1	3	3	9	0
9634	3	3	4	3	5	2	(9, 3)	3	3	3	9	$*$
9679	4	4	6	4	7	1	(3)	1	1	1	3	0
9691	2	2	3	3	3	2	(9)	1	1	3	9	$*$
9754	2	2	4	2	5	1	(3)	1	1	1	3	0
9766	1	1	1	2	2	1	(3)	1	3	3	9	0
9790	2	2	2	2	3	4	(3, 3)	3	3	3	27	0
9814	4	4	4	5	5	1	(3)	1	1	1	3	$*$
9895	3	3	3	3	4	1	(3)	1	1	1	3	0

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