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Integers with no large prime factors

by

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1. Introduction. Let $P(n)$ denote the largest prime factor of an integer $n > 1$, and $P(1) = 1$. For real numbers $x, y \ge 2$, let $S(x,y) = \{n : 1 \le n \le x,\ P(n) \le y\}$ and $u = \log x/\log y$. Also, let

$$\Psi(x,y) = \sum_{n\in S(x,y)} 1 \quad \text{and} \quad \Psi_q(x,y) = \sum_{\substack{n\in S(x,y)\\(n,q)=1}} 1.$$

Estimates for the function $\Psi(x,y)$ are needed in various problems in number theory and the study of the function has been the object of numerous articles. Thus de Bruijn in [1] established the quantitative estimate

$$\Psi(x,y) = x\varrho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right), \tag{1.1}$$

for the range $x \ge 3$, $\exp\{(\log x)^{5/8+\varepsilon}\} \le y \le x$, where ε is any fixed positive number, and $\varrho(u)$, the *Dickman–de Bruijn function*, is defined as the continuous solution of the system

$$\varrho(u) = 1, \quad 0 \le u \le 1,$$
$$u\varrho'(u) = -\varrho(u-1), \quad u > 1.$$

Recently Hildebrand [7] showed that the asymptotic formula (1.1) remains valid in the range

$$x \ge 3, \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \le y \le x, \tag{1.2}$$

where $\log_2 x = \log\log x$. More recently Hildebrand and Tenenbaum [8] obtained an asymptotic formula for $\Psi(x,y)$ in the range $x \ge y \ge 2$.

The asymptotic behaviour for $\Psi_q(x,y)$ has been studied by several authors, including Norton [9], Hazlewood [6], Fouvry and Tenenbaum [4].

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Thus, it was shown in [4] that uniformly for

$$x \geq x_0(\varepsilon), \quad \exp\{(\log_2 x)^{5/3+\varepsilon}\} \leq y \leq x, \tag{1.3}$$

and

$$\log_2(q+2) \leq \left\{\frac{\log y}{\log(u+1)}\right\}^{1-\varepsilon},$$

we have the estimate

$$\Psi_q(x,y) = \frac{\varphi(q)}{q}\Psi(x,y)\left(1 + O\left(\frac{\log_2(qy)\log_2 x}{\log y}\right)\right), \tag{1.4}$$

where $\varphi(q)$ is Euler's function.

We improved the above result (unpublished) by showing that

$$\Psi_q(x,y) = \frac{\varphi(q)}{q}\Psi(x,y)\left\{1 + O\left(\frac{\log(\omega(q)+3)\log(u+1)}{\log y}\right)\right\}$$

holds uniformly in the range

$$x \geq x_0, \quad \exp\{c_1 \log x \log_3 x/\log_2 x\} \leq y \leq x$$

and

$$\omega(q) \leq \exp\{c_2 \log x/\log_2 x\},$$

where $\omega(n)$ denotes the number of distinct prime divisors of n.

Very recently Tenenbaum [12] improved the above result; he showed the following result:

Let c be an arbitrary positive constant. Under the conditions

$$P(q) \leq y \leq x, \quad \omega(q) \leq y^{c/\log(u+1)},$$

we have uniformly

$$\Psi_q(x,y) = \frac{\varphi(q)}{q}\Psi(x,y)\left(1 + O\left(\frac{\log(u+1)\log(\omega(q)+3)}{\log y}\right)\right). \tag{1.5}$$

The proof of the last assertion used a result in sieve theory. (For all relevant literature on the functions $\Psi(x,y)$ and $\Psi_q(x,y)$, see [9] and [4].)

The purpose of this paper is to estimate $\Psi_q(x,y)$ in a wider range in q.

Let q_y denote the product of the prime divisors of q that are $\leq y$. For $u > 1$, let $\xi = \xi(u)$ be the unique positive solution of $e^\xi = u\xi + 1$, and $\xi(1) = 0$, so that asymptotically

$$\xi(u) = \log u + \log_2 u + O(1).$$

Put $\beta = \beta(x,y) = 1 - \xi(u)/\log y$. Finally, let $c_0, c_1, c_2, \ldots$ denote positive absolute constants.

We now state our main result.

Theorem 1. *For*

$$x \geq x_0(\varepsilon), \quad (\log x)^{1+\varepsilon} \leq y \leq x, \tag{1.6}$$

$$\omega(q_y) \le y^{1/2}, \tag{1.7}$$

we have uniformly

$$\Psi_q(x,y) = \prod_{p|q,\, p\le y}(1-p^{-\beta})\Psi(x,y)\left\{1+O\left(\frac{\log(\omega(q_y)+3)}{\log(u+1)\log y}\right)\right. \tag{1.8}$$
$$\left. + O(\exp(-(\log y)^{3/5-\varepsilon}))\right\}.$$

Moreover, *if*

$$\omega(q_y) \le \exp\{c_3 \log y/\log(u+1)\}, \tag{1.9}$$

then the first error term in the right-hand side of (1.8) *may be replaced by* $O(\log(\omega(q_y)+3)/\log x)$.

From Theorem 1 we shall deduce the following corollary:

COROLLARY. *For* x, y *satisfying* (1.3) *and* $\omega(q_y) \le y^{1/2}$, *we have uniformly*

$$\Psi_q(x,y) = \prod_{p|q,\, p\le y}\left(1-\frac{1}{p^\beta}\right)x\varrho(u)\left(\frac{-\xi(u)\zeta(\beta)}{\beta\log y}\right)$$
$$\times\left\{1+O\left(\frac{\log(\omega(q_y)+3)}{\log y\log(u+1)}\right)\right\}.$$

Remark. From Theorem 1 we know that (1.5) in the ranges (1.6) and (1.9) is a consequence of (1.8) and Lemma 10 below.

2. Estimates for $\Pi(y,s)$. We write the complex variable s in the form $s=\sigma+it$ with real σ and t. Let

$$\Pi(y,s) = \prod_{p\le y}(1-p^{-s})^{-1}, \qquad y=[y]+1/2,$$
$$\sigma(t) = \log^{2/3}(|t|+2)\log_2^{1/3}(|t|+3),$$

and let $\zeta(s)$ be the Riemann zeta-function.

LEMMA 1. *There is an absolute constant* $c_4>0$ *such that*:

(i) *In the region* $\sigma \ge 1-c_4/\sigma(t)$, $\zeta(s)\ne 0$.

(ii) *In the region* $|t|\ge 1$, $\sigma\ge 1-c_4/\sigma(t)$,
$$\zeta(s) \ll \log^{2/3}(|t|+2)\log_2^{1/3}(|t|+3).$$

(iii) *In the region* $|t|\ge 1$, $\sigma \ge 1-c_4/2\sigma(t)$,
$$\log\zeta(s) \ll \log^{2/3}(|t|+2)\log_2^{1/3}(|t|+3).$$

Proof. By Richert [10], we have for $0\le\sigma\le 2$, $t\ge 2$,
$$\zeta(s) \ll (1+t^{100(1-\sigma)^{3/2}})(\log t)^{2/3}.$$

From this and applying Theorems 3.10 and 3.11 of Titchmarsh [13] with $\phi(t) = \frac{302}{3}\log_2 t$, $\theta(t) = (\log_2 t)^{2/3}/(\log t)^{2/3}$, the lemma follows.

To show Theorem 1 and Corollary, we shall need the estimate for the quantity $\Pi(y,s)$. Saias [11] proved that the estimate

$$\Pi(y,s) = \log y \exp\left\{\gamma + \int_0^{(1-s)\log y} \frac{e^v - 1}{v}\,dv\right\} \times (s-1)\zeta(s)\left\{1 + O_\varepsilon\left(\frac{1}{L(\varepsilon)}\right)\right\} \tag{2.1}$$

holds uniformly in the range

$$y \ge 2, \quad \max(1-(\log y)^{-2/5-\varepsilon}, 3/4) \le \sigma \le 2, \quad |t| \le L(\varepsilon),$$

where ε is any fixed positive number and

$$L(\varepsilon) = \exp\{(\log y)^{3/5-\varepsilon}\}. \tag{2.2}$$

From this we also have

$$\Pi(y,\beta+it) = \exp\{\gamma + I(\xi(u)) + w(u,-it\log y)\} \times (-\xi(u)\zeta(\beta+it))(1+O_\varepsilon(L(\varepsilon)^{-1})), \tag{2.3}$$

where

$$I(z) = \int_0^z \frac{e^v-1}{v}\,dv, \tag{2.4}$$

and

$$w(u,z) = \int_0^z \frac{e^{\xi(u)+v}}{\xi(u)+v}\,dv. \tag{2.5}$$

In [8], Hildebrand and Tenenbaum have given an upper estimate for $\Pi(y,s)$, but insufficient for our purposes. The following lemma gives a stronger upper bound for $\Pi(y,\beta+it)$. The method of proof is based on the method of Vinogradov [14].

Lemma 2. *For $2 \le u \le L(\varepsilon)$ and $t \ge 1/\log y$ we have uniformly*

$$|e^{w(u,-it\log y)}| \ll \exp\left\{-\frac{(1/10)ut^2}{(1-\beta)^2+t^2}\right\}. \tag{2.6}$$

Proof. Let us set $\eta = 1-\beta = \xi(u)/\log y$ and $a(t) = a(t,u,y) = \operatorname{Re} w(u,-it\log y)$. Then

$$a(t) = e^{\xi(u)} \int_0^{t\log y} \frac{x\cos x - \xi(u)\sin x}{\xi^2(u)+x^2}\,dx.$$

We first consider the case $u \geq u_0$ (u_0 sufficiently large). Using integration by parts we obtain

$$a(t) = e^{\xi(u)}\bigg\{\frac{t\log y\sin(t\log y) + \xi(u)\cos(t\log y)}{\xi^2(u) + (t\log y)^2} - \frac{1}{\xi(u)} + O\bigg(\frac{t^2}{(\eta^2+t^2)\xi^2(u)}\bigg) + 2\int_0^{t\log y}\frac{x^2\sin x + (1+\xi(u))x\cos x}{(\xi^2(u)+x^2)^2}\,dx\bigg\}. \tag{2.7}$$

Again using integration by parts we deduce that the last integral on the right-hand side of (2.7) is

$$\leq \frac{2\xi(u)(t\log y)\sin(t\log y)}{(\xi^2(u)+(t\log y)^2)^2} + O\bigg(\frac{t^2}{(t^2+\eta^2)\xi^2(u)}\bigg). \tag{2.8}$$

Put $\tan\theta = t/\eta$. Then from (2.7) and (2.8) we get

$$a(t) \leq e^{\xi(u)}\bigg\{\frac{\eta}{\sqrt{\eta^2+t^2}\xi(u)}\cos(t\log y - \theta) - \frac{1}{\xi(u)} + \frac{2\xi(u)(t\log y)\sin(t\log y)}{(\xi^2(u)+(t\log y)^2)^2} + O\bigg(\frac{t^2}{(\eta^2+t^2)\xi^2(u)}\bigg)\bigg\}. \tag{2.9}$$

If $t > \eta$, from (2.9) we obtain

$$a(t) \leq e^{\xi(u)}\bigg\{-\frac{1}{2}\cdot\frac{t^2}{(\eta^2+t^2)\xi(u)} + O\bigg(\frac{1}{\xi^2(u)}\bigg)\bigg\} \leq -\frac{(1/10)ut^2}{(\eta^2+t^2)}.$$

If $6/\log y < t \leq \eta$, we have $\sin(t\log y) \leq 1 \leq (t\log y)/6$. Hence, from (2.9) we have

$$a(t) \leq e^{\xi(u)}\bigg\{-\frac{1}{3}\cdot\frac{t^2}{(\eta^2+t^2)\xi(u)} + \frac{1}{\pi}\cdot\frac{t^2}{(\eta^2+t^2)\xi(u)} + O\bigg(\frac{t^2}{(\eta^2+t^2)\xi^2(u)}\bigg)\bigg\} \leq -\frac{(1/10)ut^2}{\eta^2+t^2}.$$

If $\pi/\log y < t \leq 6/\log y$, then $\sin(t\log y) \leq 0$. From this and (2.9), the desired estimate (2.7) is derived at once.

Finally, if $1/\log y \leq t \leq \pi/\log y$, then $\cos(t\log y - \theta) \leq \cos(\pi/4) = 1/\sqrt{2}$. From (2.9) we get

$$a(t) \leq e^{\xi(u)}\bigg\{-\frac{\eta^2}{3(\eta^2+t^2)} + O\bigg(\frac{t^2}{(\eta^2+t^2)\xi(u)}\bigg)\bigg\} \leq -\frac{(1/10)ut^2}{\eta^2+t^2}.$$

Thus (2.6) is proved in the case $u \geq u_0$.

In the case $2 \leq u \leq u_0$, we have to show that $a(t) \ll 1$, which follows easily from (2.9).

This completes the proof of Lemma 2.

LEMMA 3. *For $2 \le u \le L(\varepsilon)$ and $0 \le t \le 1/\log y$ we have uniformly*
$$a(t) \le -c_0 u (t \log y)^2,$$
where c_0 is a sufficiently small positive number.

Proof. It suffices to show
$$F(t) := a(t) + c_0 e^{\xi(u)}\left(\frac{\xi(u) t^2}{\eta^2}\right) \le 0.$$
By definition of $a(t)$ and the condition $0 \le t \log y \le 1$, we have
$$F'(t) \le e^{\xi(u)} \frac{t}{\eta^2}\left(\frac{1-(5/6)\xi(u)}{1+t^2/\eta^2} + 2c_0\xi(u)\right).$$
From this and noting that $1 + t^2/\eta^2 \le 1 + \xi^{-2}(u)$, $\xi(u) \ge \xi(2) > 1.25$ and c_0 has been chosen sufficiently small, we obtain $F'(t) < 0$ for $t > 0$. This provides the required inequality.

LEMMA 4. *For $2 \le u \le L(\varepsilon)$ and $|t| \le 1/\log y$ we have uniformly*
$$w(u, -it\log y) = -e^{\xi(u)}\left(\frac{it}{\eta}\right) - e^{\xi(u)}\left(\frac{\xi^2(u) - 2\xi(u) + 2}{\eta^2}\right)\frac{t^2}{2!} + O(u(t\log y)^3).$$

Proof. Write
$$\frac{\partial^n}{\partial z^n} w(u,z)\bigg|_{z=0} = w_n(u).$$
By the definition of $w(u,z)$, we have $w_0(u) = 0$, $w_1(u) = e^{\xi(u)}\xi^{-1}(u)$, $w_2(u) = -e^{\xi(u)}(\xi(u)-1)\xi^{-2}(u)$, and $(\partial^3/\partial z^3) w(u,z) \ll u$. From this and Taylor's theorem, the lemma is derived at once.

Remark. From Lemmas 1, 2 and formula (2.3) we have for $1/\log y \le |t| \le L(\varepsilon)$, and $2 \le u \le L(\varepsilon)$
$$|\Pi(y, \beta + it)| \ll \exp\{I(\xi(u)) - c_{10} u t^2/((1-\beta)^2 + t^2)\} \times \{(\log(|t|+2))^{2/3}(\log_2(|t|+3))^{1/3} + 1/t\}.$$
This improves on a result of [8].

3. Estimates for $\varphi(q_y, s)^{-1}$. Let
$$\varphi(q_y, s) = \prod_{p|q,\, p \le y} (1 - p^{-s})^{-1}.$$
If $\omega(q_y) \ge 2$, we choose K_q so that $\pi(K_q) = \omega(q_y)$, where $\pi(x)$ denotes the number of primes not exceeding x. If $\omega(q_y) \le 1$, we put $K_q = e$. Hence, we

have

$$\log K_q \asymp \log(\omega(q_y)+3).$$

We need some estimates for $\varphi(q_y,s)^{-1}$.

LEMMA 5. *For $u\ge 2$, $|t|\le (u^{1/3}\log y)^{-1}$, and $\omega(q_y)\le y^{1/2}$, we have uniformly*

$$\varphi(q_y,\beta+it)^{-1}=\varphi(q_y,\beta)^{-1}(1+itA+O(t^2A_0^2)), \tag{3.1}$$

where $A=A(q_y,\beta)$ is a real-valued function, and

$$A\ll \eta^{-1}(u\xi(u))^{1/2}(\log K_q/\log y)=:A_0.$$

Proof. We have

$$\frac{\varphi(q_y,\beta+it)^{-1}}{\varphi(q_y,\beta)^{-1}}=e^{itA+O(t^2B)},\quad\text{say,}$$

where

$$A:=\sum_{p\mid q,\,p\le y}\sum_{m=1}^{\infty}\frac{\log p}{p^{m\beta}},\qquad B:=\sum_{p\mid q,\,p\le y}\sum_{m=1}^{\infty}\frac{m\log^2 p}{p^{m\beta}}.$$

We first estimate the quantity A. If $\exp\{c_3\log y/\log(u+1)\}\le\omega(q_y)\le y^{1/2}$, by partial summation and the prime number theorem we obtain

$$\begin{aligned}A&\ll\sum_{p\le K_q}\frac{\log p}{p^{\beta}}=\int_2^{K_q}\frac{\log z}{z^{\beta}}\,d\pi(z)\ll e^{\eta\log K_q}+\int_2^{K_q}\frac{dz}{z^{\beta}}\\&\ll\eta^{-1}e^{\eta\log K_q}\ll\eta^{-1}(u\xi(u))^{1/2}(\log K_q/\log y).\end{aligned}\tag{3.2}$$

This provides the desired estimate.

If $\omega(q_y)\le\exp\{c_3\log y/\log(u+1)\}$, we have

$$A\ll\sum_{p\le K_q}\frac{\log p}{p^{\beta}}\ll\sum_{p\le K_q}\frac{\log p}{p}\ll\log K_q.\tag{3.3}$$

This provides a stronger estimate than the assertion of the lemma.

Similarly,

$$\begin{aligned}B&\ll\sum_{p\le K_q}\frac{\log^2 p}{p^{\beta}}\ll\int_2^{K_q}\frac{\log z}{z^{\beta}}\,dz+e^{\eta\log K_q}\log K_q\\&\ll\eta^{-1}\log K_q e^{\eta\log K_q}\ll\eta^{-1}\log y(u\xi(u))^{1/2},\end{aligned}\tag{3.4}$$

since $t^2B\ll 1$, for $|t|\le(u^{1/3}\log y)^{-1}$, so we have

$$e^{itA+O(t^2B)}=1+itA+O(t^2A_0^2).$$

This completes the proof of Lemma 5.

Lemma 6. *For* $u \ge 2$, $|t| \le 1/\log K_q$, *and* $\omega(q_y) \le \exp\{c_3 \log y/\log(u+1)\}$, *we have uniformly*

(i) $\varphi(q_y, \beta+it)^{-1} = \varphi(q_y,\beta)^{-1}(1+itA_1+O(t^2\log^2 K_q))$, *where* $A_1 = A_1(q_y,\beta)$ *is a real-valued function, and* $A_1 \ll \log K_q$.

(ii) $\frac{\partial}{\partial t}\varphi(q_y,\beta+it)^{-1} \ll \varphi(q_y,\beta)^{-1}\log K_q$.

(iii) $\frac{\partial^2}{\partial t^2}\varphi(q_y,\beta+it)^{-1} \ll \varphi(q_y,\beta)^{-1}\log^2 K_q$.

Proof. It is similar to the proof of Lemma 5.

Lemma 7. *For* $u \ge 2$, $|t| \le 1/\log y$, *and* $\omega(q_y) \le y^{1/2}$, *we have uniformly*

$$\varphi(q_y,\beta+it)^{-1} \ll \varphi(q_y,\beta)^{-1}\exp\{O(u^{1/2}(t\log y)^2)\}.$$

Proof. We have

$$\left|\frac{\varphi(q_y,\beta+it)^{-1}}{\varphi(q_y,\beta)^{-1}}\right| \ll \exp\left\{O\left(t^2\sum_{p\le K_q}\frac{\log^2 p}{p^\beta}\right)\right\}$$
$$\ll \exp\{O(u^{1/2}(t\log y)^2)\}$$

as wanted.

Lemma 8. (i) *If* $u \ge 2$ *and* $\omega(q_y) \le y^{1/2}$, *then we have uniformly*

$$\varphi(q_y,\beta+it)^{-1} \ll \varphi(q_y,\beta)^{-1}(\log^2 y)e^{O(\sqrt{u})}.$$

(ii) *If* $u \ge 2$ *and* $\omega(q_y) \le \exp\{c_3\log y/\log(u+1)\}$, *then we have uniformly*

$$\varphi(q_y,\beta+it)^{-1} \ll \varphi(q_y,\beta)^{-1}\log^2 K_q.$$

Proof. We have

$$\varphi(q_y,\beta+it)^{-1} \ll \exp\left\{\sum_{p\le K_q}\frac{1}{p^\beta}\right\}.$$

In the case (i), by partial summation and the prime number theorem we obtain

$$\sum_{p\le K_q}\frac{1}{p^\beta} \le \log\frac{1}{1-\beta} + O(u^{1/2}),$$

hence

$$\varphi(q_y,\beta+it)^{-1} \ll (\log y)e^{O(\sqrt{u})}.$$

Similarly

$$\varphi(q_y,\beta) \ll (\log y)e^{O(\sqrt{u})}.$$

These provide the desired estimate.

In the case (ii), we have

$$\sum_{p\le K_q}\frac{1}{p^\beta} = \sum_{p\le K_q}\frac{1}{p}(1+O((1-\beta)\log p)) = \log\log K_q + O(1).$$

Hence

$$\varphi(q_y, \beta + it)^{-1} \ll \log K_q.$$

Also

$$\varphi(q_y, \beta) \ll \log K_q.$$

These provide the assertion.

This completes the proof of Lemma 8.

LEMMA 9. *For $u \geq 2$ and $\omega(q_y) \leq y^{1/2}$, we have uniformly*

$$\varphi(q_y)/q_y \ll \varphi(q_y, \beta)^{-1} e^{O(\sqrt{u})}.$$

Proof. We have

$$\frac{\varphi(q_y)}{q_y} \cdot \frac{1}{\varphi(q_y, \beta)^{-1}} \ll e^{\Sigma},$$

where

$$\Sigma = \sum_{p|q,\, p \leq y} \left(\frac{1}{p^\beta} - \frac{1}{p}\right).$$

By partial summation and the prime number theorem we obtain

$$\Sigma \ll \int_2^{K_q} \frac{e^{\eta \log z} - 1}{z} \cdot \frac{dz}{\log z}$$
$$= \int_{\eta \log 2}^{1} \frac{e^w - 1}{w}\, dw + \int_1^{\eta \log K_q} \frac{e^w - 1}{w}\, dw \ll \sqrt{u}.$$

This provides the desired estimate.

LEMMA 10. *If $\omega(q_y) \leq \exp\{c_3 \log y / \log(u+1)\}$, then we have uniformly*

$$\varphi(q_y, \beta)^{-1} = \frac{\varphi(q_y)}{q_y}\left(1 + O\left(\frac{\log(u+1)\log K_q}{\log y}\right)\right).$$

Proof. By the same argument as in [8, p. 289], we obtain

$$0 < \prod_{p|q,\, p \leq y} \left(\log\left(1 - \frac{1}{p}\right) - \log\left(1 - \frac{1}{p^\beta}\right)\right)$$
$$\leq \int_\beta^1 \frac{d}{d\sigma}\left\{\sum_{p \leq K_q} \log\left(1 - \frac{1}{p^\sigma}\right)\right\} d\sigma$$
$$= \left(1 + O\left(\frac{1}{\log y}\right)\right) \int_0^{\eta \log K_q} \frac{e^s - 1}{s}\, ds + O(\eta) \ll \eta \log K_q.$$

From this, Lemma 10 follows.

4. Application of the sieve methods. Let

$$N_q(x) = \sum_{n\le x,\,(n,q)=1} 1 = x\left\{\frac{\varphi(q)}{q} + R_q(x)\right\}. \tag{4.1}$$

In this section we first give two lemmas on $N_q(x)$, which are obtained by the fundamental lemma of the sieve. Then we apply these results to estimate the integrals

$$I_A = \int_{-\infty}^{u-2} |R_q(y^v)|e^{v\xi(u)}\,dv, \qquad I_B = \int_{u-2}^{\infty} |R_q(y^v)|e^{v\xi(u)}\,dv.$$

LEMMA 11. *If $q\ge 1$, $P(q)\le X$ and $r=\log X/\log(\omega(q)+3)\ge 2$, then we have uniformly*

$$N_q(X) = X\frac{\varphi(q)}{q}\{1+O(e^{-(3/5)r\log r})+O(e^{-(1/2)\sqrt{\log X}})\}. \tag{4.2}$$

Proof. This is a simple modification of Tenenbaum's argument in [12]. We apply the fundamental lemma in the form given in [5, Ch. 4, Section 8]. For any $z\le X$ and $s=\log X/\log z$ we have

$$N_{q_z}(X) = X\frac{\varphi(q)}{q}\{1+O(e^{-s\log s+s\log_2 3s+2s})+O(e^{-\sqrt{\log X}})\}, \tag{4.3}$$

where $q_z=\prod_{p\mid q,p\le z} p$. We may assume that X is a sufficiently large positive number, the result being trivial otherwise. We select $z=(\omega(q)+\exp\sqrt{\log X})^{3/2}$. This implies $z\le X$, since $r\ge 2$. We also have

$$\begin{aligned}\frac{\varphi(q_y)}{q_y} &= \frac{\varphi(q)}{q}\left\{1+O\left(\left(1-\frac{1}{z}\right)^{-\omega(q)}-1\right)\right\} \\ &= \frac{\varphi(q)}{q}\{1+O(e^{-(1/2)\sqrt{\log X}})\}.\end{aligned} \tag{4.4}$$

By (4.3) and (4.4) we obtain

$$N_{q_z}(X) = X\frac{\varphi(q)}{q}\{1+O(e^{-(3/5)r\log r})+O(e^{-(1/2)\sqrt{\log X}})\}.$$

Thus, to finish the proof of the lemma, it suffices to show

$$N_{q_z}(X) - N_q(X) \ll Xe^{-(1/2)\sqrt{\log X}}.$$

The left-hand side equals

$$\sum_{d\mid q/q_z,\,d>1} \mu(d)N_{q_z}(X/d) \ll X\sum_{d\mid q/q_z,\,d>1} \mu^2(d)/d.$$

From this, the above estimate is derived at once.

This completes the proof of the lemma.

LEMMA 12. *For $P(q) \le y$ and $v \ge 2$, we have uniformly*

$$N_q(y^v) = y^v \frac{\varphi(q)}{q}\{1 + O(e^{-(1/5)v\log v}) + O(e^{-(1/2)v\log y})\}. \tag{4.5}$$

P r o o f. We apply the fundamental lemma in the form given in [3]. For any z and $s \ge 1$ we have

$$N_{q_z}(X) = X(\varphi(q_z)/q_z)\{1 + O(s^{-s/2})\} + O(z^s).$$

Now, upon selecting $X = y^v$, $z = y$, and $s = \log\sqrt{X}/\log z = v/2$, the result is derived at once.

Next apply Lemma 11 to estimate I_A.

LEMMA 13. *If $u_0 \le u \le (\log_2 y)^2$, then*

$$I_A \ll \frac{\varphi(q)}{q} \cdot \frac{\log(\omega(q)+3)}{\log y}\exp\{u^{20/21}\}. \tag{4.6}$$

P r o o f. Let $v_0 = 2\log(\omega(q)+3)/\log y$. We write $I_A = I_{A1} + I_{A2}$, where I_{A1} corresponds to the integration range $-\infty < v \le v_0$; we have

$$\begin{aligned} I_{A1} &\ll e^{v_0\xi(u)}\int_0^{v_0}\left(\frac{N_q(y^v)}{y^v} + \frac{\varphi(q)}{q}\right)dv \\ &\ll u\log(u+1)\left\{\frac{1}{\log y}\sum_{n\le y^{v_0},\,(n,q)=1}\frac{1}{n} + v_0\frac{\varphi(q)}{q}\right\}. \end{aligned}$$

The sum over n is

$$\begin{aligned} &\ll \prod_{p\le y^v,\,p\nmid q}\left(1+\frac{1}{p}\right) \ll v_0\log y\prod_{p\mid q,\,p\le y^v}\left(1-\frac{1}{p}\right) \\ &\ll v_0(\log y)(\varphi(q)/q). \end{aligned}$$

Hence

$$I_{A1} \ll (u^2)\frac{\varphi(q)}{q}\cdot\frac{\log(\omega(q)+3)}{\log y}.$$

This is acceptable.

For I_{A2}, applying Lemma 11 we have

$$\begin{aligned} R_q(y^v) \ll\ &\exp\left\{-\frac{11}{10}v\log v - \frac{1}{20}\cdot\frac{\log y}{\log(\omega(q)+3)}\right\}\frac{\varphi(q)}{q} \\ &+ \exp\left\{-\frac{1}{3}v\log y\right\}\frac{\varphi(q)}{q}. \end{aligned}$$

So for $u \le (\log_2 y)^2$ we obtain

$$I_{A2} \ll \frac{\varphi(q)}{q}\cdot\frac{\log(\omega(q)+3)}{\log y}\int_{v_0}^{u-2} e^{-(11/10)v\log v + v\xi(u)}\,dv.$$

If $v > u^{19/20}$, then $(11/10)\log v \ge (26/25)\xi(u)$. Hence the last integral is $\ll \exp\{u^{20/21}\}$. The desired result (4.6) now follows on collecting these estimates.

LEMMA 14. *If* $u_0 \le u \le (\log_2 y)^2$, *then*

$$I_B \ll \frac{\varphi(q)}{q} \cdot \frac{\log(\omega(q)+3)}{\log y} e^{-u/6}. \tag{4.7}$$

Proof. For $u-2 \le v \le \log y/(\log_3 y)^3$, it is easily seen that

$$e^{-(1/2)\sqrt{\log y^v}} \ll e^{-2v\xi(u)}(\log y)^{-2}.$$

Thus from Lemma 11 we deduce that

$$|R_q(y^v)| \ll \frac{\varphi(q)}{q} \cdot \frac{\log(\omega(q)+3)}{\log y} e^{-(1.05)v\xi(u)}. \tag{4.8}$$

If $v > \log y/(\log_3 y)^3$, applying Lemma 12 yields

$$\begin{aligned} |R_q(y^v)| &\ll \frac{\varphi(q)}{q}\{e^{-(1/5)v\log v} + e^{-(1/3)v\log y}\} \\ &\ll \frac{\varphi(q)}{q} \cdot \frac{\log(\omega(q)+3)}{\log y} e^{-2v\xi(u)}. \end{aligned} \tag{4.9}$$

Now (4.7) follows from the above two estimates.

5. Proof of Theorem 1: the case $u \le (\log_2 y)^2$. Let

$$\Lambda_q(x,y) = \begin{cases} x\int_{-\infty}^{\infty} \varrho(u-v)\, dR_q(y^v), & x \in \mathbb{R}\setminus\mathbb{Z}^+, \\ \Lambda_q(x+0,y), & x \in \mathbb{Z}^+. \end{cases}$$

Recall that $\pi(K_q) = \omega(q_y)$ for $\omega(q_y) \ge 2$ and $K_q = e$ for $\omega(q_y) \le 1$. By formulas (5.4), (5.5) and (5.8) of [4] we have

$$\Psi_q(x,y) = \Lambda_q(x,y) + O\Big(L(\varepsilon/2)^{-1}\Psi(x,y)\prod_{p\le K_q}(1+p^{-\beta+c/\log y})\Big),$$

where $L(\varepsilon)$ is defined by (2.2).

It is easily seen that the product over $p \le K_q$ is $\ll \exp\{(\log_2 y)^3\}$. So we obtain

$$\Psi_q(x,y) = \Lambda_q(x,y) + O(\Psi(x,y)L(\varepsilon)^{-1}). \tag{5.1}$$

We give the main steps of the proof of Theorem 1 in the form of four lemmas.

LEMMA 15. (i) *For $u \ge 2$ and $T \ge e^{\xi(u)}$ we have uniformly*

$$\varrho(u) = e^{\gamma - u\xi(u) + I(\xi(u))}J(u) + O(1/T), \tag{5.2}$$

where

$$J(u)=\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{itu+w(u,-it)}}{1-it/\xi(u)}\,dt, \tag{5.3}$$

and where $I(z)$, $w(u,z)$ *are defined by* (2.4), (2.5), *respectively.*

(ii) *For* $u\ge 2$, $0\le v\le u-2$ *and* $T\ge e^{\xi(u)}$ *we have uniformly*

$$\varrho(u-v)=e^{\gamma-u\xi(u)+I(\xi(u))}e^{v\xi(u)}K(u,v)+O(1/T), \tag{5.4}$$

where

$$K(u,v)=\frac{1}{2\pi}\int_{-T}^{T}\frac{e^{it(u-v)+w(u,-it)}}{1-it/\xi(u)}\,dt. \tag{5.5}$$

Write

$$I_q(x,y)=\frac{1}{2\pi}\int_{-T'}^{T'}\frac{e^{it\log x+w(u,-it\log y)}}{(\beta+it)\varphi(q_y,\beta+it)}(-\zeta(\beta+it))\,dt \tag{5.6}$$

and

$$Q(u)=e^{\gamma-u\xi(u)+I(\xi(u))}. \tag{5.7}$$

LEMMA 16. *For* $u_0\le u\le(\log_2 y)^2$, $\omega(q_y)\le y^{1/2}$ *and* $1/\log y\le T'\le 1$ *we have uniformly*

$$\begin{aligned}\Lambda_q(x,y)&=xQ(u)\xi(u)I_q(x,y)\\&\quad+O\bigg(\Psi(x,y)\varphi(q_y,\beta)^{-1}e^{-c_{11}u/\log^2(u+1)}\\&\qquad\times\bigg(\frac{\log(\omega(q_y)+3)}{\log y}+\frac{1}{T'\log y}\bigg)\bigg).\end{aligned} \tag{5.8}$$

Write

$$\begin{aligned}H_q^{(j)}(x,y)&=xQ(u)\xi(u)\frac{1}{2\pi}\int_{-T_j}^{T_j}\frac{e^{it\log x+w(u,-it\log y)}}{\beta+it}\\&\qquad\times\bigg(\frac{-\zeta(\beta+it)}{\varphi(q_y,\beta+it)}-\frac{-\zeta(\beta+it)}{\varphi(q_y,\beta)}\bigg)\,dt\qquad(j=1,2),\end{aligned} \tag{5.9}$$

where $T_1=1/\log y$, $T_2=1/\log K_q$.

LEMMA 17. *For* x, y *satisfying* (1.3), $u\ge u_0$ *and* $\exp\{c_3\log y/\log(u+1)\}\le\omega(q_y)\le y^{1/2}$ *we have uniformly*

$$H_q^{(1)}(x,y)\ll\Psi(x,y)\varphi(q_y,\beta)^{-1}(\log K_q/(\log y\log(u+1))). \tag{5.10}$$

LEMMA 18. *For* x, y *satisfying* (1.3), $\omega(q_y)\le\exp\{c_3\log y/\log(u+1)\}$ *and* $u\ge u_0$ *we have uniformly*

$$H_q^{(2)}(x,y)\ll\Psi(x,y)\varphi(q_y,\beta)^{-1}(\log K_q/\log x). \tag{5.11}$$

In the case $u_0 \le u \le (\log_2 y)^2$, where u_0 is a sufficiently large absolute constant, Theorem 1 follows easily from these lemmas and (5.1). In fact, by Lemma 16 with $T' = 1/\log y$ and (5.1) we have

$$\Psi_q(x,y) = xQ(u)\xi(u)\frac{1}{2\pi}\int\limits_{-1/\log y}^{1/\log y} \frac{e^{it\log x + w(u,-it\log y)}}{(\beta+it)\varphi(q_y,\beta+it)}(-\zeta(\beta+it))\,dt$$
$$+ O(\Psi(x,y)\varphi(q_y,\beta)^{-1}e^{-c_{11}u/\log^2(u+1)}).$$

When $q=1$, the last formula remains true. From this and Lemma 17, the desired estimate (1.7) is derived, when we assume $\exp\{c_3\log y/\log(u+1)\} \le \omega(q_y) \le y^{1/2}$.

If $\omega(q_y) \le \exp\{c_3\log y/\log(u+1)\}$, (1.7) is proved similarly.

If $1 \le u < u_0$, the assertion of Theorem 1 becomes, by Lemma 10,

$$\Psi_q(x,y) = \frac{\varphi(q_y)}{q_y}\Psi(x,y)\biggl(1 + O\biggl(\frac{\log(\omega(q_y)+3)}{\log y}\biggr)\biggr). \tag{5.12}$$

We first dispose of the case $y^{1/C} < \omega(q_y) \le y^{1/2}$, where C is sufficiently large absolute constant. The desired estimate (5.12) follows from

$$\Psi_q(x,x) \le 7(\varphi(q)/q)x$$

(see, for example, [5, p. 104]).

We may therefore suppose $\omega(q_y) \le y^{1/C}$. By the definition of $\Lambda(x,y)$ we have for $x \notin \mathbb{Z}^+$,

$$\Lambda_q(x,y) = x\int\limits_{-\infty}^{u} \varrho(u-v)\,dR_q(y^v)$$
$$= x\varrho(u-v)R_q(y^v)|_{-\infty}^{u} + x\int\limits_{-\infty}^{u-1} R_q(y^v)\varrho'(u-v)\,dv.$$

By Lemma 11 the first term of the right-hand side equals

$$x\varrho(u)(\varphi(q_y)/q_y)(1 + O(\log(\omega(q_y)+3)/\log y)).$$

By Lemma 11 we also deduce that, in the same way as in the proof of Lemma 13, the second term is

$$\ll x\varrho(u)(\varphi(q_y)/q_y)(\log(\omega(q_y)+3)/\log y).$$

By the above estimates and (5.1), (5.12) is proved for the case considered.

P r o o f o f L e m m a 15. By (1.9) of [2] we have

$$\varrho(u) = \frac{1}{2\pi i}\int\limits_{-i\infty}^{i\infty} e^{\gamma - uz + I(z)}\,dz \qquad (u \ge 1).$$

From this and (3.3) and (3.4) of [2] we obtain for $T \ge 1$, $u \ge 1$,

$$\varrho(u) = \frac{1}{2\pi i} \int_{-iT}^{iT} e^{\gamma - uz + I(z)}\, dz + O(1/T).$$

Also,

$$\operatorname{Re} I(iT) = \int_0^T \frac{\cos t - 1}{t}\, dt = -\log T + O(1)$$

and

$$I(\sigma + iT) - I(iT) \ll \frac{1}{T} \int_0^{\xi(u)} e^x\, dx \ll 1,$$

if $T \ge e^{\xi(u)}$ and $0 \le \sigma \le \xi(u)$. So we have

$$\varrho(u) = e^{\gamma - u\xi(u) + I(\xi(u))} \overline{J}(u) + O(1/T),$$

where

$$\overline{J}(u) = \frac{1}{2\pi} \int_{-T}^{T} e^{itu + I(\xi(u) - it) - I(\xi(u))}\, dt.$$

Obviously,

$$I(\xi(u) - it) - I(\xi(u)) = \int_0^{-it} \frac{e^{\xi(u)+w}}{\xi(u) + w}\, dw + \log\left(\frac{\xi(u)}{\xi(u) - it}\right).$$

From this and the definition of $w(u,z)$ we have $\overline{J}(u) = J(u)$, which proves (i).

The proof of (ii) is similar.

P r o o f o f L e m m a 16. Write $T^* = T' \log y$ and

$$K(u,v) = \int_{-T^*}^{T^*} + \int_{T^* \le |t| \le T} = K_1(u,v) + K_2(u,v), \quad \text{say.} \tag{5.13}$$

By the definition of $\Lambda_q(x,y)$ we have for $x \notin \mathbb{Z}^+$,

$$\begin{aligned} \Lambda_q(x,y) &= xQ(u) \int_{-\infty}^{u-2} e^{v\xi(u)} K_1(u,v)\, dR_q(y^v) \\ &\quad + xQ(u) \int_{-\infty}^{u-2} e^{v\xi(u)} K_2(u,v)\, dR_q(y^v) \\ &\quad + x \int_{u-2}^{u} \varrho(u-v)\, dR_q(y^v) + O\left(\frac{x}{T} \int_{-\infty}^{u-2} |dR_q(y^v)|\right) \\ &= G_1 + G_2 + G_3 + O(G_4), \quad \text{say.} \end{aligned} \tag{5.14}$$

We first estimate G_2. Changing the order of integration and using integration by parts we get

$$G_2 = xQ(u)\left\{\frac{\varphi(q)}{q}J_1(u) + e^{(u-2)\xi(u)}R_q(y^{u-2})J_1(2) - J_2\right\}, \tag{5.15}$$

where

$$J_1(b) = \frac{1}{2\pi}\int\limits_{T^*<|t|\le T} \frac{e^{itb+w(u,-it)}}{1-it/\xi(u)}\,dt \qquad (2 \le b \le u)$$

and

$$J_2 = \frac{1}{2\pi}\int\limits_{T^*<|t|\le T} e^{itu+w(u,-it)}\xi(u)\Big\{\int\limits_{-\infty}^{u-2} R_q(y^v)e^{v(\xi(u)-it)}\,dv\Big\}\,dt.$$

By using integration by parts again we further obtain

$$\begin{aligned} J_1(b) &\ll \left|\frac{e^{w(u,-it)}}{t}\right|_{t=T^*} + \int\limits_{T^*}^{T} \frac{|e^{w(u,-it)}|u^2}{t^2}\,dt \\ &\ll e^{-c_{11}u/\log^2(u+1)}(T^*)^{-1}. \end{aligned} \tag{5.16}$$

For J_2, changing the order of integration, then using integration by parts twice we see that the inner integral is

$$\begin{aligned} &\ll e^{-c_{11}u/\log^2(u+1)} + e^{\xi(u)}\xi(u)\left|\int\limits_{T^*}^{T} e^{w(u,-it)}e^{it(u-v-1)}\frac{dt}{\xi(u)-it}\right| \\ &\ll e^{-c_{11}u/\log^2(u+1)}. \end{aligned}$$

From this, and Lemmas 13 and 9, we get

$$\begin{aligned} J_2 &\ll e^{-c_{11}u/\log^2(u+2)}\int\limits_{0}^{u-2} |R_q(y^v)|e^{v\xi(u)}\,dv \\ &\ll \varphi(q_y,\beta)^{-1}(\log K_q/\log y)e^{-c_{12}u/\log^2(u+1)}. \end{aligned} \tag{5.17}$$

Combining (5.15)–(5.17) with (4.8) and using Lemma 9 we obtain

$$\begin{aligned} G_2 \ll{} & \Psi(x,y)\varphi(q_y,\beta)^{-1}e^{-c_{12}u/\log^2(u+1)} \\ & \times\left(\frac{1}{T^*} + \frac{\log K_q}{\log y}\right) =: E_1, \quad \text{say.} \end{aligned} \tag{5.18}$$

Also, by Lemmas 13 and 9 we easily get

$$G_3 \ll E_1. \tag{5.19}$$

Now we turn to estimating G_4 in (5.14). We have

$$G_4 \ll \frac{x}{T}\sum_{m\le y^u,\,(m,q)=1}\frac{1}{m} \tag{5.20}$$

$$\ll \frac{x}{T} \prod_{p \le y^u} \left(1 + \frac{1}{p}\right) \prod_{p \le y^u,\, p|q} \left(1 + \frac{1}{p}\right)^{-1}$$
$$\ll xT^{-1}(u \log y)(\varphi(q_y)/q_y) \ll E_1$$

if $T = e^{2u\xi(u)}(\log^2 y)$. Combining the above estimates yields

$$\Psi_q(x, y) = G_1 + O(E_1). \tag{5.21}$$

To finish the proof of the lemma, it remains to estimate G_1. Changing the order of integration (with t replaced by $t \log y$) we have

$$G_1 = xQ(u)\xi(u) \frac{1}{2\pi} \int_{-T'}^{T'} \frac{e^{it \log x + w(u, -it \log y)}}{\eta - it} \times \Big\{ \int_{-\infty}^{u-2} e^{v(\xi(u) - it \log y)}\, dR_q(y^v) \Big\}\, dt. \tag{5.22}$$

By Lemma 4.4 of [4] we have

$$\int_{-\infty}^{\infty} e^{v(\xi(u) - it \log y)} dR_q(y^v) = \sum_{m=0}^{\infty} \frac{(\xi(u) - it \log y)^m}{m!} \int_{-\infty}^{\infty} v^m\, dR_q(y^v)$$
$$= \prod_{p|q,\, p \le y} \left(1 - \frac{1}{p^{\beta - it}}\right) \frac{(-\eta + it)\zeta(\beta + it)}{\beta + it},$$

which implies that the main term of G_1 is $xQ(u)\xi(u)I_q(x, y)$.

We denote the error term of G_1 by G_1'. By using integration by parts and (4.9) we obtain

$$\begin{aligned} G_1' &= xQ(u)\xi(u) \frac{1}{2\pi} \int_{-T'}^{T'} \frac{e^{it \log x + w(u, -it \log y)}}{\eta - it} e^{-(u-2)\xi(u)} R_q(y^{u-2})\, dt \\ &\quad - xQ(u)\xi(u)(\log y) \frac{1}{2\pi} \int_{u-2}^{\infty} e^{v\xi(u)} R_q(y^v) \\ &\quad \times \left\{ \frac{1}{2\pi} \int_{-T'}^{T'} e^{it(u-v) \log y + w(u, -it \log y)}\, dt \right\} dv \\ &= G_{11}' + G_{12}', \quad \text{say.} \end{aligned} \tag{5.23}$$

By Lemma 3 we easily get

$$\frac{1}{2\pi} \int_{-1/\log y}^{1/\log y} e^{it(u-v) \log y + w(u, -it \log y)}\, dt \ll \frac{1}{\log y}.$$

Now suppose that $T' > 1/\log y$. By using integration by parts twice and using Lemma 2 we get

$$\frac{1}{2\pi} \int\limits_{1/\log y<|t|\le T'} e^{it(u-v)\log y+w(u,-it\log y)}\,dt \ll e^{-c_{11}u/\log^2(u+1)}\frac{1}{\log y}.$$

Thus, the above estimates and Lemmas 14 and 9 yield

$$\begin{aligned}(5.24)\qquad G'_{12} &\ll xQ(u)\xi(u)(I_B)\\ &\ll \Psi(x,y)\varphi(q_y,\beta)^{-1}e^{-c_{11}u/\log^2(u+1)}\frac{\log(\omega(q_y)+3)}{\log y}.\end{aligned}$$

Similarly, we also have

$$(5.25)\qquad G'_{11} \ll \Psi(x,y)\varphi(q_y,\beta)^{-1}e^{-c_{11}u/\log^2(u+1)}\frac{\log(\omega(q_y)+1)}{\log y}.$$

From (5.21)–(5.25) we obtain (5.8) and the proof of Lemma 16 is complete.

Proof of Lemma 17. To prove the lemma we need the following result (see, for example [13, p. 16]):

$$(5.26)\qquad \zeta(s) = \frac{1}{s-1} + \gamma + O(|s-1|), \qquad |t|\le 2,\ 0<\sigma\le 2,\ s\ne 1.$$

Moreover, it is easy to prove that

$$(5.27)\qquad \zeta'(s) = \frac{-1}{(s-1)^2} + O(1), \qquad |t|\le 2,\ 0<\sigma\le 2,\ s\ne 1.$$

We divide the range of integration into two parts: $|t|\le T_0$ and $T_0 < |t| \le 1/\log y$, where $T_0 = (u^{1/3}\log y)^{-1}$, the corresponding integrals being denoted by H_1 and H_2. By Lemmas 3 and 7 we have

$$\begin{aligned}(5.28)\qquad H_2 &\ll xQ(u)\xi(u)\varphi(q_y,\beta)^{-1}\eta^{-1}\\ &\quad\times \int\limits_{T_0}^{1/\log y} e^{-c_{11}u(t\log y)^2}e^{O(\sqrt{u}(t\log y)^2)}\,dt\\ &\ll \Psi(x,y)\varphi(q_y,\beta)^{-1}e^{-c_{13}u^{1/3}}.\end{aligned}$$

Now we estimate H_1. Lemma 4 yields

$$\begin{aligned}(5.29)\qquad & e^{it\log x+w(u,-it\log y)}\\ &= e^{-(1/2)w_2(u)(t\log y)^2}\left\{1-\frac{it}{\eta}+O\left(\frac{t^2}{\eta^2}\right)+O(u(t\log y)^3)\right\}.\end{aligned}$$

Expanding $\zeta(\beta+it)/\zeta(\beta)$ in the Taylor series, we get

$$(5.30)\qquad \zeta(\beta+it) = \zeta(\beta)\left\{1+\frac{\zeta'(\beta)}{\zeta(\beta)}(it)+O\left(\frac{t^2}{\eta^2}\right)\right\},$$

where

$$\frac{\zeta'(\beta)}{\zeta(\beta)} \asymp \frac{1}{\eta}.$$

By Lemma 5 we have

$$\varphi(q_y, \beta + it)^{-1} - \varphi(q_y, \beta)^{-1} = \varphi(q_y, \beta)^{-1}(itA + O(t^2 A_0^2)). \tag{5.31}$$

Also

$$\frac{1}{\beta + it} = \frac{1}{\beta}\left(1 - \frac{it}{\beta} + O(t^2)\right). \tag{5.32}$$

Collecting the above estimates we deduce that the integrand is

$$\begin{aligned}\varphi(q_y, \beta)^{-1} e^{-(1/2) w_2(u)(t\log y)^2} &\left(\frac{-\zeta(\beta)}{\beta}\right) \\ &\times\{itA + O(t^2 A_0 \eta^{-1}) + O(t^2 A_0^2) + O(t^3 A_0^2 \eta^{-1}) \\ &\qquad + O(u(t\log y)^3 (tA_0)) + O(u(t\log y)^3 (tA_0)^2)\}.\end{aligned} \tag{5.33}$$

We now integrate the last expression over the range $|t| \le T_0$ to get

$$\begin{aligned}H_q^{(1)}(x,y) \ll xQ(u)&\left(\frac{-\xi(u)\zeta(\beta)}{\beta\log y}\right) \\ &\times \frac{1}{\sqrt{u}}\left\{\frac{1}{\xi(u)}\left(\frac{\log K_q}{\log y}\right)^2 + \frac{1}{\sqrt{u\xi(u)}}\left(\frac{\log K_q}{\log y}\right)\right\}.\end{aligned}$$

It is well known that (for example, see (2.7) of [8])

$$\Psi(x,y) \sim x\varrho(u) \sim e^{-u\xi(u) + I(\xi(u))}\frac{1}{\sqrt{2\pi u}} \quad \text{as } u \to \infty.$$

Also, by (5.26),

$$\frac{-\xi(u)\zeta(\beta)}{\beta\log y} \asymp 1.$$

Thus, the desired estimate (5.10) is derived.

Proof of Lemma 18. We divide the range of integration into three parts: $|t| \le T_0$, $T_0 < |t| \le 1/\log y$, and $1/\log y < |t| \le 1/\log K_q$, the corresponding integrals being denoted by H_1', H_2' and H_3'. Write

$$Z(t) = \frac{e^{w(u, -it\log y)}(-\zeta(\beta + it))}{\beta + it}$$

and

$$\Phi(t) = \varphi(q_y, \beta + it)^{-1} - \varphi(q_y, \beta)^{-1}.$$

Thus, H_3' can be rewritten as

$$H_3' = xQ(u)\xi(u)\frac{1}{2\pi}\int\limits_{1/\log y < |t| \le 1/\log K_q} Z(t)\Phi(t)e^{it\log x}\,dt.$$

We have

$$\frac{d}{dt}Z(t) = e^{w(u,-it\log y)}\frac{e^{\xi(u)}(-i\log y)}{\xi(u)+it\log y}\cdot\frac{-\zeta(\beta+it)}{\beta+it}e^{-it\log y}$$
$$+ e^{w(u,-it\log y)}\left\{\frac{-\zeta'(\beta+it)i}{\beta+it}+\frac{-\zeta(\beta+it)(-i)}{(\beta+it)^2}\right\}$$
$$= Z_1(t)e^{-it\log y} + Z_2(t), \quad \text{say.}$$

By Lemma 2 and (5.26), (5.27) we have for $1/\log y \le |t| \le 1$,

$$Z(t) \ll t^{-1}e^{-c_{11}u/\log^2(u+1)},$$
$$Z_i(t) \ll t^{-2}e^{c_{11}u/\log^2(u+1)}, \quad i=1,2,$$

and

$$\frac{d}{dt}Z(t) \ll t^{-2}e^{-c_{11}u/\log^2(u+1)}.$$

Similarly

$$\frac{d}{dt}Z_i(t) \ll t^{-3}e^{-c_{11}u/\log^2(u+1)}, \quad i=1,2.$$

By using integration by parts twice and by Lemmas 2 and 8 we obtain

$$H_3' \ll \Psi(x,y)\varphi(q_y,\beta)^{-1}e^{-c_{11}u/\log^2(u+1)}(\log K_q/\log y). \tag{5.34}$$

Now we turn to H_2' and H_1'. We proceed as in the proof of Lemma 17 for H_2 and H_1 but using Lemma 6 instead of Lemmas 7 and 5. We obtain

$$H_2' \ll \Psi(x,y)\varphi(q_y,\beta)^{-1}(\log K_q/\log y)e^{-c_{13}u^{1/3}}$$

and

$$H_1' \ll \Psi(x,y)\varphi(q_y,\beta)^{-1}(\log K_q/\log x).$$

This provides the desired estimate.

6. Proof of Theorem 1: the case $u > (\log_2 y)^2$. We shall use the following notations:

$$\Phi(y,s) = \log \Pi(y,s),$$
$$\Phi_k(y,s) = \frac{\partial^k}{\partial s^k}\Phi(y,s), \quad k\ge 0,$$
$$\sigma_k = \Phi_k(y,\beta), \quad k \ge 0.$$

We notice that Lemmas 4, 8, 9, 10 and 13 of [8] remain true if α is replaced by β, where $\alpha=\alpha(x,y)$ is defined by (1.11).

Using a variant of Perron's formula and Lemma 9 of [8] and our Lemma 8 we get

$$(6.1)\quad \Psi_q(x,y)=\frac{1}{2\pi i}\int_{\beta-i\overline{T}}^{\beta+i\overline{T}}\frac{x^s\Pi(y,s)}{s\varphi(q_y,s)}\,ds$$
$$+O(xe^{-u\xi(u)+I(\xi(u))}(\log y)\varphi(q_y,\beta)^{-1}((\overline{T})^{-1/2}+e^{-c_{11}u/\log^2(u+1)})),$$

where

$$\overline{T}=(Y_\varepsilon^{-1}+e^{-c_{11}u/\log^2(u+1)})^{-2}\quad\text{and}\quad Y_\varepsilon=\exp\{(\log y)^{3/2-\varepsilon}\}.$$

Now we suppose that $\omega(q_y)\le y^{1/2}$ (the proof for the case $\omega(q_y)\le\exp\{c_3\log y/\log(u+1)\}$ is similar). By Lemma 8(ii) of [8], our Lemma 8 and the condition $u>(\log_2 y)^2$ we further have

$$(6.2)\quad \frac{1}{2\pi i}\Big\{\int_{\beta-i\overline{T}}^{\beta-i/\log y}+\int_{\beta+i/\log y}^{\beta+i\overline{T}}\Big\}\frac{x^s\Pi(y,s)}{s\varphi(q_y,s)}\,ds$$
$$\ll xe^{-u\xi(u)}\Pi(y,\beta)\varphi(q_y,\beta)^{-1}(\log^2 y)(\log\overline{T})e^{-c_{11}u/\log^2(u+1)}$$
$$\ll \Psi(x,y)\varphi(q_y,\beta)^{-1}e^{-c_{14}u/\log^2(u+1)}.$$

Thus we obtain

$$(6.3)\quad \Psi_q(x,y)=\frac{1}{2\pi i}\int_{\beta-i/\log y}^{\beta+i/\log y}\frac{x^s\Pi(y,s)}{s\varphi(q_y,s)}\,ds$$
$$+O(\Psi(x,y)\varphi(q_y,\beta)^{-1}(\log^{-N}x)).$$

When $q=1$, (6.3) gives

$$(6.4)\quad \Psi(x,y)=\frac{1}{2\pi i}\int_{\beta-i/\log y}^{\beta+i/\log y}\frac{x^s\Pi(y,s)}{s}\,ds+O(\Psi(x,y)/\log^N x).$$

Write

$$(6.5)\quad \overline{H}_q(x,y)=\frac{1}{2\pi i}\int_{\beta-i/\log y}^{\beta+i/\log y}\frac{x^s\Pi(y,s)}{s}(\varphi(q_y,s)^{-1}-\varphi(q_y,\beta)^{-1})\,ds.$$

We first estimate the contribution of the range $|t|\le T_0$ (recall that $T_0=(u^{1/3}\log y)^{-1}$). Expanding the function $\Phi(y,s)$ in a Taylor series around $t=0$, we get

$$\Phi(y,s)=\sigma_0+it\sigma_1-\frac{t^2}{2}\sigma_2+O(t^3\sigma_3).$$

We further get

$$x^s\Pi(y,s)=x\Pi(y,\beta)e^{-u\xi(u)-(1/2)t^2\sigma_2}\{1+O(t(\log x+\sigma_1))+O(t^3\sigma_3)\}.$$

By Lemma 13 of [8] we easily get

$$\log x + \sigma_1 = O(uL(\varepsilon)^{-1}) + O(1).$$

Thus, Lemma 5 shows that

$$\begin{aligned}\frac{x^s \Pi(y,s)}{s}&(\varphi(q_y,s)^{-1} - \varphi(q_y,\beta)^{-1})\\ &= xe^{-u\xi(u)}\Pi(y,\beta)\varphi(q_y,\beta)^{-1}e^{-(1/2)t^2\sigma_2}\\ &\quad\times \{itA + O(t^2A_0^2) + O(t(uL(\varepsilon)^{-1}+1)tA_0) + O(t^3\sigma_3 tA_0)\},\end{aligned}$$

where A is defined by Lemma 5 and $A \ll \eta^{-1}(u\xi(u))^{1/2}(\log K_q/\log y)$. From this and Lemma 4 of [8] we find that the contribution of the range $|t| \le T_0$ is

$$\ll xe^{-u\xi(u)}\Pi(y,\beta)\varphi(q_y,\beta)^{-1}\frac{1}{\sqrt{u}\log y}\left(\frac{\log K_q}{\log(u+1)\log y} + \frac{1}{L(\varepsilon)}\right). \tag{6.6}$$

It remains to estimate the contribution of the range $T_0 < |t| \le 1/\log y$. By Lemma 8(i) of [8] and Lemma 6, this contribution is

$$\begin{aligned}&\ll xe^{-u\xi(u)}\Pi(y,\beta)\varphi(q_y,\beta)^{-1}\int_{T_0}^{1/\log y} e^{-c_{14}u(t\log y)^2}(tA_0)\,dt \\ &\ll xe^{-u\xi(u)}\Pi(y,\beta)\varphi(q_y,\beta)^{-1}e^{-c_{15}u^{1/3}}\frac{\log K_q}{(\log y)^2}.\end{aligned} \tag{6.7}$$

By Theorem 1 of [8] we have

$$\Psi(x,y) \asymp xe^{-u\xi(u)}\Pi(y,\beta)\frac{1}{\sqrt{u}\log y}.$$

From this and (6.3)–(6.7), the desired estimate (1.8) is derived in the range considered.

7. Proof of Corollary. The Corollary is an immediate consequence of Theorem 1 and the following lemma.

LEMMA 19. *For x, y satisfying* (1.3) *we have uniformly*

$$\Psi(x,y) = x\varrho(u)\left(\frac{-\xi(u)\zeta(\beta)}{\beta\log y}\right)\left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{7.1}$$

Proof. First, consider the case $1 \le u < u_0$. We have

$$\frac{-\xi(u)\zeta(\beta)}{\beta\log y} = \left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right) = 1 + O\left(\frac{1}{\log x}\right).$$

The estimate (7.1) clearly follows from this and (1.1).

We may therefore suppose $u \ge u_0$. From (5.1) and Lemma 16 with $T' = 1$ and $q = 1$, we have

$$\Psi(x,y) = xQ(u)\xi(u)\frac{1}{2\pi}\int_{-1}^{1} \frac{e^{it\log x + w(u,-it\log y)}}{\beta+it}(-\zeta(\beta+it))\,dt + O\biggl(x\varrho(u)\biggl(e^{-c_{14}u/\log^2(u+1)}\frac{1}{\log x} + \frac{1}{L(\varepsilon)}\biggr)\biggr), \tag{7.2}$$

where $Q(u)$ is defined by (5.7).

Write

$$J(u,b) = \frac{1}{2\pi}\int_{-b}^{b} \frac{e^{it\log x + w(u,-it\log y)}}{\eta - it}\,dt.$$

By Lemma 15(i) with $T = e^{2u\xi(u)}(\log y)$, we have for $u \ge u_0$,

$$\varrho(u) = Q(u)\xi(u)J(u,T\log y) + O(Q(u)e^{-c_{14}u/\log^2(u+1)}(1/\log x)). \tag{7.3}$$

We divide the range of integration of $J(u,T\log y)$ in (7.3) into the parts: $|t| \le 1$ and $1 < |t| \le T\log y$. Using integration by parts we see that the contribution of the range $1 < |t| \le T\log y$ is

$$\ll e^{-c_{14}u/\log^2(u+1)}(1/\log x).$$

Thus, we further obtain

$$\varrho(u) = Q(u)\xi(u)J(u,1) + O\biggl(Q(u)e^{-c_{14}u/\log^2(u+1)}\frac{1}{\log x}\biggr). \tag{7.4}$$

To finish the proof of the lemma, it therefore suffices to show that

$$W := \int_{-1}^{1} e^{it\log x + w(u,-it\log y)}F(t)\,dt \ll \frac{1}{\sqrt{u}\log y}\cdot\frac{1}{\log x}, \tag{7.5}$$

where

$$F(t) = \frac{\zeta(\beta+it)}{\zeta(\beta)(1+it\beta^{-1})} - \frac{\eta}{\eta - it}.$$

We divide the range of integration in (7.5) into the three parts: $|t| \le T_0$, $T_0 < |t| \le 1/\log y$ and $1/\log y \le |t| \le 1$. The corresponding integrals are denoted by W_1, W_2 and W_3.

For $|t| \le 1$ we have

$$\begin{gathered} F(t) = \frac{\eta}{\eta - it}(1 + O(|\eta - it|) + O(\eta) + O(t)) - \frac{\eta}{\eta - it} = O(\eta), \\ F'(t) = O(1) \quad \text{and} \quad F''(t) = O(1/\eta). \end{gathered} \tag{7.6}$$

From this, Lemma 2 and using integration by parts twice we get

$$W_3 \ll e^{-c_{14}u/\log^2(u+1)}(1/\log^2 x). \tag{7.7}$$

By Lemma 3 and (7.6) we have

$$W_2 \ll e^{-c_{15}u^{1/3}}(1/\log^2 x). \tag{7.8}$$

To estimate W_1, we expand $F(t)$ in a Taylor series around $t = 0$, to get

$$F(t) = F'(0)(it) + O(t^2/\eta),$$

where

$$F'(0) = \frac{\zeta'(\beta)}{\zeta(\beta)} - \frac{1}{\beta} - \frac{1}{\eta} \ll 1.$$

From this and (5.29) we obtain

$$\begin{aligned} W_1 &= \frac{1}{2\pi} \int_{-T_0}^{T_0} e^{-(1/2)w_2(u)(t\log y)^2} \\ &\qquad \times \{F'(0)(it) + O(t^2/\eta) + O(ut^4 \log^3 y)\}\,dt \\ &\ll \frac{1}{\sqrt{u}\log y} \cdot \frac{1}{u\log y}. \end{aligned} \tag{7.9}$$

The desired estimate (7.5) now follows on collecting these estimates.

This completes the proof of Lemma 19.

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