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On the range of fractional parts $\{\xi(p/q)^n\}$

by

Leopold Flatto, Jeffrey C. Lagarias (Murray Hill, N.J.)
and Andrew D. Pollington (Provo, Ut.)

1. Introduction. A well-known unsolved problem in number theory concerns the distribution of $(3/2)^n \pmod 1$. This sequence is believed to be uniformly distributed, which is the case for almost all real numbers $\theta^n \pmod 1$, but it is not even known to be dense in $[0,1]$. One of the few positive results known for (non-integer) rational $\theta = p/q$ is that of Vijayaraghavan (1940), who showed that the set $(p/q)^n \pmod 1$ has infinitely many limit points. Vijayaraghavan later remarked that it was striking that one could not even decide whether or not $(3/2)^n \pmod 1$ has infinitely many limit points in $[0,1/2)$ or in $[1/2,1)$. Both these latter assertions would follow if one could show that

$$\limsup_{n\to\infty}\left\{\left(\frac{3}{2}\right)^n\right\} - \liminf_{n\to\infty}\left\{\left(\frac{3}{2}\right)^n\right\} > \frac{1}{2}.$$

This remains unsettled. However, in this paper we will show that

$$\limsup_{n\to\infty}\left\{\left(\frac{3}{2}\right)^n\right\} - \liminf_{n\to\infty}\left\{\left(\frac{3}{2}\right)^n\right\} \ge \frac{1}{3}, \tag{1.1}$$

as a special case of more general results.

One approach to such questions is to study the distribution of $\{\xi\theta^n\} := \xi\theta^n \pmod 1$, where $\xi > 0$ is an arbitrary real number. Already Pisot (1938), Chapitre IV, studied such quantities for certain algebraic θ, followed by Vijayaraghavan (1941)–(1948) and Pisot (1946). More recently Choquet (1981) and Pollington (1978), (1979), (1981) have separately studied the distribution of general sequences $\xi\theta^n \pmod 1$, and have shown that a wide variety of behaviors can occur. We shall obtain the result (1.1) by showing the stronger result that

$$\limsup_{n\to\infty}\left\{\xi\left(\frac{3}{2}\right)^n\right\} - \liminf_{n\to\infty}\left\{\xi\left(\frac{3}{2}\right)^n\right\} \ge \frac{1}{3} \tag{1.2}$$

holds for *all* real $\xi > 0$. This fact depends on the specific nature of $3/2$, because if $\theta = \lambda^2$ where λ is a totally real Pisot number, e.g. $\theta = (3+\sqrt{5})/2$, then for any $\varepsilon > 0$ one can find a positive real ξ with

$$\limsup_{n\to\infty}\{\xi\theta^n\} - \liminf_{n\to\infty}\{\xi\theta^n\} < \varepsilon. \tag{1.3}$$

More generally, for any Pisot or Salem number θ and any $\varepsilon > 0$ there exists ξ with $\|\xi\theta^n\| < \varepsilon$ for all $n = 0, 1, \ldots$, where $\|x\|$ is the distance of x to the nearest integer, cf. Bertin *et al.* (1992), Theorem 5.2.4. For $\theta = \lambda^2$ where λ is a totally real Pisot number one has $1 - \varepsilon < \{\xi\theta^n\} < 1$ for all $n = 0, 1, \ldots$ on taking $\xi = \theta^k$ for large enough k, which yields (1.3).

The main focus of the paper is the study of ξ, θ for which all $\xi\theta^n \pmod 1$ stay inside a subinterval $[s, s+t)$ which is strictly contained in $[0, 1)$. Let $Z_\theta(s, s+t)$ denote the set of all positive real ξ which have this property. We exclusively consider the case of (non-integer) rationals $\theta = p/q$, where $p > q \ge 2$ are relatively prime. Thus we study the set $Z_{p/q}(s, s+t)$ given by those positive real numbers ξ satisfying

$$s \le \left\{\xi\left(\frac{p}{q}\right)^n\right\} < s + t \quad \text{for all } n \ge 0. \tag{1.4}$$

We call members of $Z_{p/q}(s, s+t)$ *generalized Z-numbers*, extending the terminology of Mahler (1968), who used the term *Z-numbers* for the members of $Z_{3/2}(0, 1/2)$. It is known that $Z_{p/q}(s, s+t)$ is either countable (possibly empty) or else has cardinality the power of the continuum ([1]).

In this paper we study conditions guaranteeing that $Z_{p/q}(s, s+t)$ be countable. We consider the following properties:

(A) $Z_{p/q}(s, s+t)$ contains at most one number in each unit interval $[m, m+1)$ for all $m \in \mathbb{Z}^+$. (Here $\mathbb{Z}^+$ denotes the set of integers $m \ge 0$.)

(B) Property (A) holds and in addition there is some $\gamma < 1$ such that

$$\#\{\xi \le x : \xi \in Z_{p/q}(s, s+t)\} = O(x^\gamma) \quad \text{as } x \to \infty.$$

(C) $Z_{p/q}(s, s+t) = \emptyset$.

Properties (A) and (B) naturally arise out of the method of Mahler (1968), who originally considered $Z_{3/2}(0, 1/2)$. He conjectured that no Z-numbers exist, a problem which still remains unsolved. What Mahler proved was that properties (A) and (B) hold for Z-numbers, and he obtained the bound $O(x^\gamma)$ with $\gamma = \log_2 \frac{1+\sqrt{5}}{2} \cong .70$. Later Flatto (1992) improved the bound to $\gamma = \log_2(3/2) \cong .59$. In a complementary direction, Tijdeman (1972) found an elementary method which permits one to show for certain p/q and t that $Z_{p/q}(0, t)$ contains exactly one element in every unit interval $[n, n+1)$ for every integer $n \ge 1$.

([1]) This fact can be proved without recourse to the continuum hypothesis, using the rather simple topological structure of $Z_{p/q}(s, s+t)$, cf. Flatto (1992).

The essential idea of Mahler's method, as explained in Flatto (1992), is the observation that, for any Z-number ξ, the integer parts $g_n = [\xi(3/2)^n]$ and fractional parts $x_n = \{\xi(3/2)^n\}$ have the following three properties:

(i) The sequence $\{g_n\}$ is produced by iterating a fixed map $F_{3/2} : \mathbb{Z}^+ \to \mathbb{Z}^+$.

(ii) The sequence $\{2x_n\}$ is produced by iterating a fixed map $f_{3/2} : [0,1] \to [0,1]$.

(iii) Certain symbolic dynamics of the iterates $\{g_n\}$ for $F_{3/2}$ and of the iterates $\{2x_n\}$ for $f_{3/2}$ are identical.

The map $F_{3/2}$ is given by

$$F_{3/2}(m) = \begin{cases} 3m/2 & \text{if } m \equiv 0 \pmod 2, \\ (3m+1)/2 & \text{if } m \equiv 1 \pmod 2, \end{cases}$$

and the map $f_{3/2}$ is given by

$$f_{3/2}(x) = \tfrac{3}{2}x \pmod 1.$$

The latter is a special case of the well-studied *β-transformation*

$$f_\beta(x) = \beta x \pmod 1, \qquad 0 \le x < 1, \tag{1.5}$$

where $\beta > 1$. We call the properties (i)–(iii) above a "*decoupling condition*", because the integer parts and the (scaled) fractional parts of $\xi(3/2)^n$ are then separately describable by iterating independent maps $F_{3/2}$ and $f_{3/2}$. Property (iii) provides the connection between the integer and fractional parts.

More generally, we say that $Z_{p/q}(s, s+t)$ satisfies a "decoupling condition" if appropriate analogues of properties (i)–(iii) above hold, as given in Proposition 2.1 in Section 2. Flatto (1992) showed that a "decoupling condition" holds for $Z_{p/q}(0,t)$ whenever $0 \le t \le 1/q$.

This paper extends Mahler's method to more general sets $Z_{p/q}(s, s+t)$ where $s > 0$. Our first result gives conditions under which property (A) holds, and slightly stronger conditions under which property (B) holds. Let $\lceil x \rceil$ denote the smallest integer $n \ge x$.

Theorem 1.1. *Suppose that $p > q \ge 2$ are relatively prime integers, and that the interval $[s, s+t) \subset [0,1]$ is such that*

$$\text{(C1)} \quad 0 < t \le 1/q, \tag{1.6}$$

$$\text{(C2)} \quad \{(p-q)s\} \le q - pt. \tag{1.7}$$

Then the set $Z_{p/q}(s, s+t)$ contains at most one element in each unit interval $[m, m+1)$, for all nonnegative integers m. In addition

$$\#\{\xi : 0 \le \xi \le x \text{ and } \xi \in Z_{p/q}(s, s+t)\} = O(x^\gamma) \tag{1.8}$$

where $\gamma = \min(\log_q(p/q), \log_q(\lceil pt + \{(p-q)s\}\rceil))$ as $x \to \infty$.

This theorem is proved by establishing the "decoupling condition" for $Z_{p/q}(s,s+t)$ whenever conditions (C1) and (C2) hold. In particular, for any fixed rational $p/q>1$ the "decoupling condition" holds for *all* sufficiently short intervals, namely for all intervals $[s,s+t)$ with $0\le t\le 1/p$. The complicated conditions (C1) and (C2) are somewhat difficult to appreciate, but seem to be the natural limits of validity of the "decoupling condition". Of importance for our later results is that the general version of property (ii) of the "decoupling condition" involves a linear mod one transformation

$$f_{\beta,\alpha}(x)=\beta x+\alpha \pmod 1.$$

We note that the conditions (C1) and (C2) are interdependent as follows: For $p\le q^2-q$ condition (C1) implies (C2), while for $p\ge q^2$ condition (C2) implies (C1).

Theorem 1.1 usually verifies property (B), for it gives an exponent $\gamma<1$ unless both $p>q^2$ and $pt+\{(p-q)s\}>q-1$. However, Corollary 1.3a below shows that the bound $\gamma=1$ is actually sharp in some cases. At the other extreme, we show that property (C) holds in some circumstances (see Section 3 and Corollary 1.4a below).

Our second result shows that property (A) holds for some $Z_{p/q}(s,s+t)$ not covered by Theorem 1.1. It was originally developed in the thesis of Pollington (1978).

THEOREM 1.2. *Suppose that $p>q\ge 2$ are relatively prime integers, and that the interval $[s,s+t]\subset[0,1]$ is such that*

$$\text{(C1*)}\quad t\ge 1/q, \tag{1.9}$$

$$\text{(C2*)}\quad qt-1\le (p-q)s\le q-pt. \tag{1.10}$$

Then the set $Z_{p/q}(s,s+t)$ contains at most one element in each unit interval $[m,m+1)$, for all nonnegative integers m.

This result applies only when $q<p\le q^2$, because if $p>q^2$ then (C1*) gives $q-pt\le q-p/q<0$, which violates (C2*).

Theorem 1.2 is proved by establishing a "partial decoupling condition" for $Z_{p/q}(s,s+t)$, in which an analogue of property (i) holds, i.e. $g_n=[\xi(p/q)^n]$ is described by an iterated map $T_{p/q}:\mathbb{Z}^+\to\mathbb{Z}^+$, and furthermore there is a simple deterministic relation between g_n and $x_n=\{\xi(p/q)^n\}$. However, in these circumstances we do not know of any analogue of property (ii).

We illustrate Theorem 1.2 for $p/q=3/2$. The widest interval which Theorem 1.2 yields is $[1/5,4/5]$, obtained by taking $s=1/5$, $t=3/5$. This fact is complemented by a result of Pollington (1981) stating that, for $\gamma=4/65$, $Z_{3/2}(\gamma,1-\gamma)$ has positive Hausdorff dimension, hence is uncountable.

Our third result gives circumstances under which property (A) holds and property (B) does not, and is proved by the method of Tijdeman (1972).

THEOREM 1.3. *Let $p > q \geq 2$ be relatively prime integers with $p \geq 2q-1$ and suppose that a is an integer with $0 \leq a \leq p-2q+1$. Then for $\frac{q-1}{p-q} < t \leq \frac{p-q-a}{p-q}$ the set $Z_{p/q}\big(\frac{a}{p-q}, \frac{a}{p-q}+t\big)$ contains at least one element ξ in each interval $\big[m+\frac{a}{p-q}, m+\frac{a+q-1}{p-q}\big]$, for all nonnegative integers m.*

Combining this theorem with Theorem 1.1 immediately yields:

COROLLARY 1.3a. *Let $p > q \geq 2$ be relatively prime integers and suppose $p > q^2$, so that $\frac{q}{p} > \frac{q-1}{p-q}$. Then for*

$$\frac{q-1}{p-q} < t \leq \frac{q}{p}$$

and for any integer $0 \leq a \leq p-2q$, the set $Z_{p/q}\big(\frac{a}{p-q}, \frac{a}{p-q}+t\big)$ contains exactly one Z-number in each interval $[m, m+1)$, for all nonnegative integers m.

Finally we come to the most significant result of this paper.

THEOREM 1.4. *Let $p > q \geq 2$ be relatively prime integers. Then for all positive real ξ*

$$\limsup_{n\to\infty}\left\{\xi\left(\frac{p}{q}\right)^n\right\} - \liminf_{n\to\infty}\left\{\xi\left(\frac{p}{q}\right)^n\right\} \geq \frac{1}{p}. \tag{1.11}$$

This result implies that any interval containing all $\{\xi(p/q)^n\}$ for $n \geq 0$ is of length at least $1/p$, which yields (1.2).

The proof of Theorem 1.4 is based on showing that, for given p, q and $t = 1/p$, property (C) holds for a dense set of values of s in $[0, 1-1/p]$. The proof of this fact uses Theorem 1.1 and a detailed analysis of the iteration of linear mod one transformations. The methods in the proof can also be used to verify that property (C) holds for certain specific intervals $[s, s+1/p)$ by a finite computation. For example, we show:

COROLLARY 1.4a. *$Z_{3/2}(s, s+1/3) = \emptyset$ for $s = 0, 1/6, 1/3, 1/2,$ and $2/3$.*

The case $s = 0$ of this corollary was obtained in Pollington (1978).

To put Theorem 1.4 in more perspective, define for arbitrary real $\theta > 1$ the quantity

$$\Omega(\theta) := \inf_{\xi>0}(\limsup_{n\to\infty}\{\xi\theta^n\} - \liminf_{n\to\infty}\{\xi\theta^n\}).$$

Tijdeman (1972) shows by an intervals construction that $Z_\theta(0, 1/(\theta-1)) \neq \emptyset$, whence

$$\Omega(\theta) \leq \frac{1}{\theta-1}. \tag{1.12}$$

In particular, $\Omega(p/q) \le q/(p-q)$. In comparison Theorem 1.4 asserts that

$$\Omega(p/q) \ge 1/p.$$

For $q = 2$ the ratio of the upper to the lower bound is bounded, and approaches 2 as $p \to \infty$. This gives a sense in which Theorem 1.4 is close to best possible. As already mentioned, $\Omega(\theta) = 0$ when $\theta = \lambda^2$ for any totally real Pisot number λ. There are, however, only countably many θ such that

$$\Omega(\theta) \le \frac{1}{2(1+\theta)^2}, \tag{1.13}$$

see Bertin *et al.* (1992), Theorem 5.6.1. Boyd (1969) gives another related result ([2]).

In Section 2 we prove Theorems 1.1–1.3, and in Section 3 we analyze linear mod one transformations and prove Theorem 1.4 on the range of $\{\xi(p/q)^n\}$.

2. Generalized Z-numbers and linear mod one transformations. The idea underlying Mahler's method is, for certain special intervals $[s, s+t)$, to show that all $\xi \in \mathbb{R}^+$ for which

$$s \le \left\{\xi\left(\frac{p}{q}\right)^k\right\} < s+t \quad \text{for all } k \ge 0,$$

have the property that the integer and fractional parts of $\xi(p/q)^k$ *separately* obey iterations of simple maps on $\mathbb{Z}^+$ and $[0,1)$, respectively. This "decoupling" of integer and fractional parts permits their separate study. For Theorem 1.1 the relevant maps are special cases of $T_{\beta,\alpha} : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $T_{\beta,\alpha}(n) = \lceil \beta n + \alpha \rceil$, and the *linear mod one transformation*

$$f_{\beta,\alpha}(x) = \beta x + \alpha \pmod 1, \quad 0 \le x < 1.$$

If $\beta = p/q$ and $\alpha = a/q$ are both rational, with $a \in \mathbb{Z}^+$, then the function $T_{\beta,\alpha}$ has the alternative expression

$$T_{p/q,a/q}(g) = \frac{pg + a + i_j}{q} \quad \text{if } g \equiv j \pmod q, \tag{2.1}$$

where $0 \le i_j \le q-1$ is determined by

$$i_j \equiv -pj - a \pmod q. \tag{2.2}$$

For a general map $G : X \to X$ one prescribes a symbolic dynamics by an auxiliary map $S : X \to \Sigma$ onto a *symbol space* Σ. Letting $G^{(k)}(x)$ denote

([2]) Boyd shows that, for all integers $m \ge 3$, each interval $[m, m+1)$ contains uncountably many θ for which there exists a ξ such that $\|\xi\theta^n\| \le (2e\theta(\theta+1)(1+\log\xi))^{-1}$ for $n = 0, 1, 2, \ldots$

the kth iterate of $G(x)$, one associates to x the *itinerary*

$$I(x) := (x, G(x), G^{(2)}(x), \ldots)$$

and the *symbolic itinerary*

$$I_S^*(x) := (S(x), S(G(x)), S(G^{(2)}(x)), \ldots).$$

In interesting cases the symbolic itinerary $I_S^*(x)$ determines x. There is a natural symbolic dynamics associated with $T = T_{p/q,a/q}$, which assigns to each $n \in \mathbb{Z}^+$ the integer $S_T(n) = j$ where $0 \le j \le q-1$ and $n \equiv j \pmod q$; thus $\Sigma = \{0, 1, \ldots, q-1\}$. Linear mod one transformations $f = f_{\beta,\alpha}$ also have a natural symbolic dynamics, which assigns to each $x \in [0,1)$ the integer

$$S_f(x) = [\beta x + \alpha]. \tag{2.3}$$

In this case the symbol space $\Sigma = \{0, 1, \ldots, [\beta + \alpha]\}$. For the two maps $T = T_{p/q,a/q}$ and $f = f_{\beta,\alpha}$, the symbolic itineraries of $m \in \mathbb{Z}^+$ and of $x \in [0,1)$ will be referred to respectively as the *T-expansion of m* and *f-expansion of x*. For both these maps the symbolic itineraries of an input uniquely determines that input.

The proof of Theorem 1.1 is based on the following Proposition 2.1, which is a precise version of the "decoupling condition". To simplify notation, assume p, q and s are fixed, and let T and f stand for $T_{p/q,[(p-q)s]/q}$ and $f_{p/q,\{(p-q)s\}}$, respectively. Also let $\{a_n\}$ be the T-expansion of $g \in \mathbb{Z}^+$, i.e. $a_n = S_T(T^n(g))$, and let $\{b_n\}$ denote the f-expansion of $\theta \in [0,1)$, i.e. $b_n = S_f(f^n(\theta))$.

Proposition 2.1. *Let $p > q \ge 2$ be relatively prime integers and let $[s, s+t) \subset [0,1)$ be an interval satisfying*:

(C1) $0 < qt \le 1$,

(C2) $\{(p-q)s\} \le q - pt$.

Then $\xi \in Z_{p/q}(s, s+t)$ if and only if

(S1) *The quantity $\theta := q(\{\xi\} - s)$ has*

$$0 \le f^{(n)}(\theta) < qt \quad \textit{for all } n \ge 0. \tag{2.4}$$

(S2) *The T-expansion $\{a_n\}$ of $g = [\xi]$ and the f-expansion $\{b_n\}$ of $\theta = q(\{\xi\} - s)$ are related by*

$$\sigma(a_n) = b_n \quad \textit{for all } n \ge 0,$$

where σ is the permutation of $\{0, 1, \ldots, q-1\}$ given by

$$\sigma(i) \equiv -pi - [(p-q)s] \pmod q. \tag{2.5}$$

Furthermore, *if* (S1), (S2) *are satisfied then*

$$T^{(n)}(g) = \left[\xi\left(\frac{p}{q}\right)^n\right], \qquad f^{(n)}(\theta) = q\left(\left\{\xi\left(\frac{p}{q}\right)^n\right\} - s\right) \quad \textit{for all } n \ge 0. \tag{2.6}$$

Note that if $qt = 1$, then the condition (S1) automatically holds, hence is superfluous.

Proposition 2.1 is an immediate consequence of repeated application of the following Lemma 2.1. We set

$$g := [\xi], \qquad g' := \left[\frac{p}{q}\xi\right], \tag{2.7a}$$

and

$$\theta := q(\{\xi\} - s), \qquad \theta' := q\left(\left\{\frac{p}{q}\xi\right\} - s\right), \tag{2.7b}$$

to simplify notation.

Lemma 2.1. *Let $p > q \ge 2$ be relatively prime integers, and let $[s, s+t)$ be an interval satisfying conditions* (C1), (C2). *Suppose $\xi > 0$ satisfies $s \le \{\xi\} < s+t$, i.e. $0 \le \theta < qt$. Then*

$$s \le \left\{\frac{p}{q}\xi\right\} < s+t$$

holds if and only if

(i) $0 \le f(\theta) < qt$,
(ii) $\sigma(S_T(g)) = S_f(\theta)$.

Furthermore, (i) *and* (ii) *together imply that*

$$g' = T(g), \qquad \theta' = f(\theta). \tag{2.8}$$

Proof. From (2.7) we have

$$\xi = [\xi] + \{\xi\} = g + s + \theta/q, \tag{2.9}$$

where $g \in \mathbb{Z}$ and $0 \le \theta < qt \le 1$, the last inequality guaranteed by condition (C1). Also (2.7) gives

$$\frac{p}{q}\xi = \left[\frac{p}{q}\xi\right] + \left\{\frac{p}{q}\xi\right\} = g' + s + \frac{\theta'}{q} \tag{2.10}$$

where $g' \in \mathbb{Z}$ and $0 \le s + \theta'/q < 1$, whence

$$g' \in \mathbb{Z}^+ \quad \text{and} \quad -qs \le \theta' < -qs + q. \tag{2.11}$$

Multiplying (2.9) by p/q and subtracting from (2.10) yields

$$\left(g' - \frac{p}{q}g\right) + \left(1 - \frac{p}{q}\right)s + \frac{1}{q}\left(\theta' - \frac{p}{q}\theta\right) = 0.$$

Multiplying by q and rearranging terms yields

$$q(g' - T(g)) + i_g + \theta' = \frac{p}{q}\theta + \{(p-q)s\} = \left[\frac{p}{q}\theta + \{(p-q)s\}\right] + f(\theta), \tag{2.12a}$$

where by definition of T, $0 \le i_g < q$ satisfies

$$i_g \equiv -pg - [(p-q)s] \pmod q. \tag{2.12b}$$

Now g' and θ' are the solutions to (2.12) uniquely determined by the requirements (2.11). Equating the integer and fractional parts of both sides of (2.12a), we find that

$$0 \le \theta' < qt$$

holds if and only if θ and g satisfy the two conditions

$$0 \le f(\theta) < qt, \tag{2.13}$$

and

$$\left[\frac{p}{q}\theta + \{(p-q)s\}\right] - i_g \equiv 0 \pmod q, \tag{2.14}$$

the last condition being equivalent to $g' \in \mathbb{Z}$.

Now, by condition (C2),

$$0 \le \left[\frac{p}{q}\theta + \{(p-q)s\}\right] \le \frac{p}{q}\theta + \{(p-q)s\} < pt + \{(p-q)s\} \le q. \tag{2.15}$$

Because $0 \le i_g < q$, (2.14) is equivalent to

$$\left[\frac{p}{q}\theta + \{(p-q)s\}\right] = i_g. \tag{2.16}$$

Equations (2.13) and (2.16) are conditions (i), (ii) of Lemma 2.1.

Finally, if (2.13) and (2.16) hold then equating the integer and fractional parts of (2.12a) yields $g' = T(g)$, $\theta' = f(\theta)$, respectively. ∎

We remark that condition (C1) is crucial for Lemma 2.1, in that simple arguments show that (S1) fails when (C1) fails.

Lemma 2.1 will be shown below to imply property (A); the following lemma is needed to establish the density result (1.6) of Theorem 1.1, which yields property (B).

LEMMA 2.2. *Let $p > q \ge 2$ be relatively prime integers. Then for any integer $a \ge 0$, the map $T = T_{p/q,a/q} : \mathbb{Z}^+ \to \mathbb{Z}^+$ has the property that for each $k \ge 1$, the sequence of k symbols*

$$(S_T(g), S_T(T(g)), \ldots, S_T(T^{(k-1)}(g)))$$

is determined by $g \pmod{q^k}$ and each of the q^k possible symbol strings in $\{0, 1, \ldots, q-1\}^k$ occurs exactly once for $0 \le g \le q^k - 1$.

Proof. For any $g \in \mathbb{Z}^+$ and $i \in \mathbb{Z}^+$

$$T(g + bi) = T(g) + pi. \tag{2.17}$$

The integers $i+qj$ with $0 \le i < q-1$, $0 \le j \le q^{k-1}-1$ form a complete residue system $\pmod{q^k}$. Let $g \equiv i+qj \pmod{q^k}$, i.e. $g = i+qj+q^k m$, m an integer. By (2.17), $T(g) = T(i)+pj+pq^{k-1}m$, so

$$T(g) \equiv T(i)+pj \pmod{q^{k-1}}.$$

Since $(p,q)=1$, we see that, for any given i, $\{T(i)+pj : 0 \le j < q^{k-1}\}$ runs through a complete residue system $(\bmod\, q^{k-1})$. Both statements of the lemma then follow by induction on k, assuming the induction hypothesis for

$$(S_T(g'), S_T(T(g')), \ldots, S_T(T^{(k-2)}(g'))).$$

The base case $k=1$ is obvious. ∎

Lemma 2.2 asserts that all possible finite symbol sequences occur when iterating $T_{p/q,a/q}$ on the domain $\mathbb{Z}^+$.

Proof of Theorem 1.1. To prove the first assertion, if there is some $\xi \in Z_{p/q}(s,s+t)$ with $\xi \in [g,g+1)$ for $g \in \mathbb{Z}^+$, then by Proposition 2.1 the symbolic dynamics of f applied to θ is completely determined by the symbolic dynamics of T applied to $g=[\xi]$. It is well known that linear mod one transformations $f_{\beta,\alpha}$ have at most one input $0 \le \theta < 1$ with any given symbol sequence $(S_0,S_1,S_2,\ldots)$ (see Parry (1960), Rényi (1957)). As the map from ξ to $\theta = q(\{\xi\}-s)$ is a bijection from $[s,s+t]$ to $[0,qt) \subset [0,1)$, it follows that g uniquely specifies $\{\xi\}$, whence $\xi = g+\{\xi\}$ is uniquely determined.

To get bounds for the cardinality of $\{\xi \le x : \xi \in Z_{p/q}(s,s+t)\}$ we observe first that Lemma 2.2 shows that the all integers $0 \le g < q^k$ have distinct symbol sequences for $T(g)$ for their first k symbols. We proceed to bound the number of possible symbol sequences of length k that $f(x_0)$ could possibly have. For any $0<\gamma<1$, let $L^k_{\beta,\alpha}(0,\gamma)$ count the number of different symbol sequences of length k for $f_{\beta,\alpha}$, such that there is some θ with $0 \le f^{(i)}(\theta) < \gamma$, $0 \le i \le k-1$, having such a symbol sequence. Now (2.7) asserts that

$$0 \le f^{(i)}(\theta) < qt \quad \text{for all } i \ge 0, \tag{2.18}$$

hence

$$\#\{\alpha \le q^k : \alpha \in Z_{p/q}(s,s+t)\} \le L^k_{p/q,\{(p-q)s\}}(0,qt). \tag{2.19}$$

We bound this quantity in two different ways. First, the quantity $L^k_{\beta,\alpha}(0,qt)$ is bounded above by $L^k_{\beta,\alpha}(0,1)$, which is just the *lap number* $L_{k,\beta,\alpha}$ which counts all possible allowed symbol sequences of length k for $f_{\beta,\alpha}$. It is studied [3] in Flatto and Lagarias (1994), who show that for

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[3] See Flatto, Lagarias and Poonen (1994) for detailed information on the asymptotics of $L^k_{\beta,\alpha}$ as $k \to \infty$, in the special case that $\alpha = 0$.

$\beta > 1$ there is a constant $c_{\beta,\alpha}$ with

$$L_{k,\beta,\alpha} \le c_{\beta,\alpha}\beta^k. \tag{2.20}$$

Substituting this bound in (2.19) gives

$$\#\{\alpha \le q^k : \alpha \in Z_{p/q}(s, s+1)\} \le c\left(\frac{p}{q}\right)^k = cq^{k\log_q(p/q)}.$$

On taking $q^{k-1} \le x < q^k$ we obtain (2.6), with exponent $\gamma = \log_q(p/q)$.

An alternative bound is obtained by observing that the allowed symbol sequences satisfying (2.18) can only use symbols that appear in $[0, qt)$, and these are exactly $\{0, 1, \ldots, \ell - 1\}$ where $\ell = \lceil \frac{p}{q}(qt) + \{(p-q)s\}\rceil$. Thus

$$L^k_{p/q,\{(p-q)s\}}(0, qt) \le (\lceil pt + \{p-q\}s\rceil)^k. \tag{2.21}$$

Substituting this bound in (2.19) yields (2.6) with exponent $\gamma = \log_q(\lceil pt + \{(p-q)s\}\rceil)$. ■

It is clear that the bound (2.21) should be further improvable in special cases.

Proof of Theorem 1.2. Write

$$\xi\left(\frac{p}{q}\right)^n = g_n + x_n, \qquad n \ge 0,$$

where $g_n = [\xi(p/q)^n]$, $x_n = \{\xi(p/q)^n\}$. We claim that if $\xi \in Z_{p/q}(s, s+t)$, then

$$g_{n+1} = T_{p/q,0}(g_n) = \left\lceil \frac{p}{q} g_n \right\rceil, \tag{2.22}$$

and

$$x_{n+1} = \frac{p}{q}x_n - \frac{1}{q}a^*_n, \tag{2.23}$$

where

$$a^*_n = \begin{cases} 0 & \text{if } g_n \equiv 0 \pmod{q}, \\ q-k & \text{if } g_n \equiv k \pmod{q},\ 1 \le k \le q-1. \end{cases} \tag{2.24}$$

If this is shown, then (2.24) gives

$$x_0 = \frac{1}{p}\left[a^*_0 + \frac{q}{p}a^*_1 + \ldots + \left(\frac{q}{p}\right)^n a^*_n\right] + \left(\frac{q}{p}\right)^{n+1} x_{n+1}.$$

Since $(q/p)^{n+1}x_{n+1} \to 0$, we get

$$x_0 = \frac{1}{p}\left(\sum_{j=0}^{\infty}\left(\frac{q}{p}\right)^j a^*_n\right),$$

so x_0 is determined by the sequence $\{a_n^* : n = 0, 1, \ldots\}$, which is just the symbolic dynamics of the map $T_{p/q,0}$ applied to g_0. Thus there is at most one such ξ in each interval $[g, g+1)$ for $g \in \mathbb{Z}^+$, as required.

To establish the claims it proves convenient to rewrite the left and right inequalities of (C2*) as

$$\frac{p}{q}s + \frac{1}{q} \geq s + t, \tag{2.25}$$

and

$$\frac{p}{q}(s+t) \leq 1 + s, \tag{2.26}$$

respectively. We also use

$$g_{n+1} + x_{n+1} = \frac{p}{q}g_n + \frac{p}{q}x_n. \tag{2.27}$$

Assume now that $\xi \in Z_{p/q}(s, s+t)$, i.e. that

$$s \leq x_n < s + t \quad \text{for all } n \geq 0. \tag{2.28}$$

We prove the claim in two cases.

Case 1: $pg_n \equiv 0 \pmod q$. Here we must show that necessarily

$$g_{n+1} = \frac{p}{q}g_n, \quad x_{n+1} = \frac{p}{q}x_n. \tag{2.29}$$

By (2.27), $\frac{p}{q}g_n \leq g_{n+1} + x_{n+1}$. But $\frac{p}{q}g_n$ is an integer in this case, so

$$\frac{p}{q}g_n \leq [g_{n+1} + x_{n+1}] = g_{n+1}.$$

Hence $g_{n+1} = \frac{p}{q}g_n + j$, $j \geq 0$, for j some integer. This, together with (2.27), gives

$$x_{n+1} = \frac{p}{q}x_n - j. \tag{2.30}$$

We must show $j = 0$. If $j > 0$, then (2.26), (2.28) and (2.30) give

$$x_{n+1} = \frac{p}{q}x_n - j < \frac{p}{q}(s+t) - 1 \leq s,$$

contradicting $x_{n+1} \geq s$, thus proving (2.29).

Case 2: $pg_n \equiv k \pmod q$ for some $1 \leq k \leq q-1$. Here we must show that

$$g_{n+1} = \frac{p}{q}g_n + \frac{q-k}{q}, \quad x_{n+1} = \frac{p}{q}x_n - \frac{q-k}{q}. \tag{2.31}$$

By (2.27), $(pq_n - k)/q \leq g_{n+1} + x_{n+1}$. As $(pg_n - k)/q$ is an integer, we have $(pg_n - k)/q \leq g_{n+1}$. From this and (2.27),

$$g_{n+1} = \frac{p}{q}g_n - \frac{k}{q} + j, \quad x_{n+1} = \frac{p}{q}x_n + \frac{k}{q} - j,$$

for some integer $j \geq 0$. To obtain (2.31) we must show that $j = 1$.

Suppose first that $j = 0$. Then by the above and using (2.27), then (2.25),

$$x_{n+1} \geq \frac{p}{q} x_n + \frac{1}{q} \geq \frac{p}{q} s + \frac{1}{q} \geq s + t,$$

contradicting (2.28). Next suppose that $j \geq 2$. Then by the above and using (2.28) then (2.26),

$$x_{n+1} \leq \frac{p}{q} x_n + \frac{k}{q} - 2 < \frac{p}{q}(s+t) - 1 \leq s,$$

contradicting (2.28). Thus $j = 1$ is established. ∎

Remarks. 1. The proof of Theorem 1.2 is still valid with (C1*) replaced by (C1). However, when (C1) holds the condition (C2*) implies (C2). Thus when (C1), (C2*) hold, the proofs of Theorems 1.1 and 1.2 are both valid. Theorem 1.2 uses $T_{p/q,0}$ while Theorem 1.1 uses $T_{p/q,a/q}$ with $a = [(p-q)s]$, and (C1), (C2*) permit $a \neq 0$ to occur. Hence the two proofs combined establish the extra fact that if (C1), (C2*) hold, then $T_{p/q,a/q}$ and $T_{p/q,0}$ have identical itineraries when started at any integer g such that $[g, g+1)$ contains an element of $Z_{p/q}(s, s+t)$.

2. In comparing the proofs of Theorems 1.2 and 1.1, we see that in Theorem 1.2, $\{g_n\}$ is still prescribed by iterating a fixed map T on $\mathbb{Z}^+$, however $\{x_n\}$ is apparently no longer describable in terms of iterating a fixed map f on $[0,1]$, but remains deterministically coupled to g_0 via the symbolic itinerary of g_0 under T. We do not know whether the property that x_0, g_0 satisfy (2.22) and (2.23) for all $n \geq 0$ guarantees that $g_0 + x_0 \in Z_{p/q}(s, s+t)$.

Proof of Theorem 1.3. Let $n_0 \geq 0$ be an integer, and set

$$n_k = T^{(k)}_{p/q,a/q}(n_0) \quad \text{for all } k \geq 0.$$

Set $\beta = p/q$, and we have

$$\beta n_k + \frac{a}{q} \leq n_{k+1} \leq \beta n_k + \frac{a+q-1}{q}.$$

Dividing by β^{k+1}, we obtain

$$\frac{a}{q\beta^{k+1}} \leq \frac{n_{k+1}}{\beta^{k+1}} - \frac{n_k}{\beta^k} \leq \frac{a+q-1}{q\beta^{k+1}}. \tag{2.32}$$

Summation of the right inequality in (2.32) over k yields

$$\frac{n_k}{\beta^k} \leq n_0 + \left(\frac{a+q-1}{q}\right)\left(\frac{1}{\beta} + \frac{1}{\beta^2} + \ldots\right) = n_0 + \frac{a+q-1}{q(\beta-1)}. \tag{2.33}$$

Now (2.32) shows that the sequence $\{n_k/\beta^k : k = 0, 1, \ldots\}$ is monotonically increasing, and (2.33) shows that it is contained in the interval $\big[n_0, n_0 +$

$\frac{a+q-1}{q(\beta-1)}]$, so has a limit

$$\xi \in \left[n_0 + \frac{a}{q(\beta-1)}, n_0 + \frac{a+q-1}{q(\beta-1)}\right]. \tag{2.34}$$

As

$$\xi - \frac{n_k}{\beta^k} = \sum_{j=k}^{\infty} \left(\frac{n_{j+1}}{\beta^{j+1}} - \frac{n_j}{\beta^j}\right),$$

we conclude from (2.32) that

$$\frac{a}{q\beta^k(\beta-1)} = \frac{a}{q}\left(\frac{1}{\beta^{k+1}} + \frac{1}{\beta^{k+2}} + \ldots\right) \le \xi - \frac{n_k}{\beta^k} \le \frac{a+q-1}{q\beta^k(\beta-1)}.$$

Multiplying this by β^k, we obtain

$$\frac{a}{p-q} \le \left(\frac{p}{q}\right)^k \xi - n_k \le \frac{a+q-1}{p-q},$$

for all $k \ge 0$. ∎

Proof of Corollary 1.3a. As $0 \le a \le p - 2q$, we have

$$s + t \le \frac{p-2q}{p-q} + \frac{q}{p} < 1.$$

Also

$$tq \le q^2/p < 1, \qquad \{(p-q)s\} + pt = pt \le q,$$

so that conditions (C1), (C2) of Theorem 1.1 are satisfied. Corollary 1.3a then follows from Theorems 1.1 and 1.3, observing that (2.34) implies $\xi \in [n_0, n_0 + 1)$. ∎

3. The range of $\{\xi(p/q)^n\}$. We shall derive Theorem 1.4 as a direct consequence of the following result.

THEOREM 3.1. *Let $p > q \ge 2$ be relatively prime integers. Then the set of s such that $Z_{p/q}(s, s + 1/p) = \emptyset$ is dense in $[0, 1 - 1/p]$, and always includes $s = 0$ and $s = 1 - 1/p$.*

We remark that the conclusion of Theorem 3.1 conceivably holds for all $s \in [0, 1 - 1/p]$. However, we cannot prove this. The weaker statement above suffices to establish Theorem 1.4, as we now show.

Proof of Theorem 1.4. We argue by contradiction. Suppose the assertion is false, so that there exists some $\xi > 0$ with

$$\limsup_{n\to\infty} \left\{\xi\left(\frac{p}{q}\right)^n\right\} - \liminf_{n\to\infty} \left\{\xi\left(\frac{p}{q}\right)^n\right\} < \frac{1}{p}.$$

Then we may find $\xi' > 0$ and an interval $[s', s'+t')$ with $t' < 1/p$ such that

$$s' \le \left\{ x' \left(\frac{p}{q}\right)^n \right\} < s' + t' \quad \text{for all } n = 0, 1, 2, \ldots,$$

by choosing $\xi' = \xi(p/q)^k$ for a sufficiently large integer k. Thus $Z_{p/q}(s', s'+t') \ne \emptyset$.

However, using Theorem 3.1 we can always find an interval $[s, s+1/p)$ with $[s', s'+t') \subseteq [s, s+1/p)$ such that $Z_{p/q}(s, s+1/p) = \emptyset$. Namely, if $s' = 0$ take $s = 0$, and if $s' \ge 1 - 1/p$ take $s = 1 - 1/p$, while if $0 < s' < 1 - 1/p$, then choose some s slightly smaller than s' with $Z_{p/q}(s, s+1/p) = \emptyset$. Thus $Z_{p/q}(s', s'+t') \subseteq Z_{p/q}(s, s+1/p) = \emptyset$, a contradiction. ∎

Theorem 3.1 is an immediate consequence of the following two theorems.

THEOREM 3.2. *Let $p > q \ge 2$ be relatively prime integers and $f = f_{p/q,\{(p-q)s\}}$. Suppose there exist only a finite number of $x \in [0,1)$ such that*

$$0 \le f^n(x) < q/p \quad \textit{for all } n = 0, 1, 2, \ldots \tag{3.1}$$

Then $Z_{p/q}(s, s+1/p) = \emptyset$.

Let $R_{p/q}(s)$ denote the set of $x \in [0,1)$ such that (3.1) holds.

THEOREM 3.3. *Given $p > q \ge 2$ relatively prime integers, the set of s for which $R_{p/q}(s)$ is finite is dense in $[0, 1 - 1/p]$ and contains the two points $s = 0$ and $s = 1 - 1/p$.*

Proof of Theorem 3.2. We argue by contradiction. The interval $[s, s+1/p)$ satisfies conditions (C1) and (C2) of Proposition 2.1. Thus any Z-number $\xi \in Z_{p/q}(s, s+1/p)$ has associated with it a quantity $\theta := q(\{\xi\} - s)$ such that $0 \le f^n(\theta) < q/p$ for $n = 0, 1, 2, \ldots$, whence θ falls in the finite set given by the hypothesis above. Each $\theta_k := f^{(k)}(\theta)$ also lies in this finite set, thus repetition must occur, so some θ_k is a purely periodic point of f, and θ_k has a purely periodic f-expansion $\{b_k\}$. Now $\xi' := \xi(p/q)^k$ is automatically a Z-number since ξ is, and Proposition 2.1 states that $\theta_k = g(\{\xi(p/q)^k\} - s)$. Suppose l is the period of the f-expansion of θ_k, and consider $\xi'' := \xi(p/q)^{k+l}$. Now ξ'' is also a Z-number, and since $\theta_k = \theta_{k+l}$, property (S2) of Proposition 2.1 for ξ' and for ξ'' shows that $g' = [\xi']$ and $g'' = [\xi'']$ have *identical T-expansions*. But $\xi' < \xi''$ and Theorem 2.1 states there is at most one Z-number in each interval $[m, m+1)$, hence we conclude $g' \ne g''$. This contradicts the uniqueness of T-expansions of different integers, hence we conclude $Z_{p/q}(s, s+1/p) = \emptyset$ after all. (The uniqueness of T-expansions follows from Lemma 2.2, for any $g \ne g'$ there exists some power q^j such that $g \not\equiv g' \pmod{q^j}$, whence their T-expansions differ somewhere in the first j symbols.) ∎

Next we prove Theorem 3.3. The proof is lengthy and is based on several preliminary results concerning linear mod one transformations (Theorems 3.4 and 3.5).

THEOREM 3.4. *Let $f(x) = f_{\beta,\alpha}(x) = \{\beta x+\alpha\}$, with $\beta > 1$ and $0 \le \alpha < 1$, and let $S_{\beta,\alpha}$ denote the set of points $x \in [0, 1)$ for which $0 \le f^{(n)}(x) < 1/\beta$ for all $n \ge 0$. Suppose that $0 \notin S_{\beta,\alpha}$, thus there is a smallest positive integer N with $f^{(N)}(0) \ge 1/\beta$. Then $S_{\beta,\alpha}$ is a finite set. If $f^{(N)}(0) > 1/\beta$ then $S_{\beta,\alpha}$ contains exactly N elements, and they are cyclically permuted under the action of f. If $f^{(N)}(0) = 1/\beta$ then $S_{\beta,\alpha}$ is empty.*

For fixed β, we call the values α where $S_{\beta,\alpha}$ is not a finite set *exceptional values*. Some conjectures concerning exceptional values and the structure of the corresponding sets $S_{\beta,\alpha}$ are stated at the end of this section.

Theorem 3.4 is proved by a detailed analysis of the inverse images of the interval $I_0 = [1/\beta, 1)$ under f. Let S^{c} denote the complement of $S = S_{\beta,\alpha}$ in $[0, 1)$, i.e.,

$$S^{\mathrm{c}} := \{0 \le x < 1 : \{f^{(n)}(x) : n \ge 0\} \text{ intersects } I_0\}.$$

In the rest of this section the term *interval* always means one which is closed on the left and open on the right. We prove three lemmas which gradually build up a precise description of S^{c}, from which Theorem 3.4 will follow. Lemma 3.1 expresses S^{c} as an infinite union of disjoint sets $I_n := f^{(-n)}(I_0)$ and shows that S^{c} contains N intervals $I_0, I_1, \ldots, I_{N-1}$. To obtain S^{c} we "glue" on to the I_j's an infinite sequence of intervals L_i and R_i contiguous to these on their left and right sides, respectively, which are described in Lemmas 3.2 and 3.3. These fill in the gaps between the intervals $I_0, \ldots, I_{n-1}$ except for the finite set S, and show that none of the I_i are contiguous intervals. The intervals I_i, L_i and R_i are depicted in Figure 3.2 below.

We begin by collecting obvious properties of the function f restricted to the interval $[0, 1/\beta)$. The graph of this restriction of f is sketched in Figure 3.1.

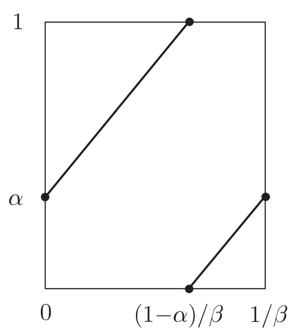


Fig. 3.1. Restriction of $f = f_{\beta,\alpha}$ to $[0, 1/\beta)$

The restricted f is linear of slope β mapping $[0,(1-\alpha)/\beta)$ onto $[\alpha,1)$ and $[(1-\alpha)/\beta,1/\beta)$ onto $[0,\alpha)$, which exhibits the key fact that f is a bijection from $[0,1/\beta)$ onto $[0,1)$. In what follows we let f^{-1} denote the inverse of this restriction, i.e. for $x\in[0,1)$,

$$f^{-1}(x)=\begin{cases}\dfrac{1}{\beta}x+\dfrac{1-\alpha}{\beta}, & 0\le x<\alpha,\\ \dfrac{1}{\beta}x-\dfrac{\alpha}{\beta}, & \alpha\le x<1.\end{cases}$$

Thus f^{-1} is piecewise linear and continuous except on the left at α. We let $f^{-n}:=(f^{-1})^{(n)}$.

LEMMA 3.1. (1) *If $I_0=[1/\beta,1)$ then the sets $I_n=f^{-n}(I_0)$ are nonempty, mutually disjoint sets and $S^{\mathrm{c}}:=[0,1)-S=\bigcup_{i=0}^{\infty}I_i$.*

(2) *If N is minimal with $f^{(N)}(0)\in I_0$, then the sets I_k for $0\le k\le N-1$ are each nonempty intervals contained in the open interval $(0,1)$, and $\alpha\in I_{N-1}$.*

Proof. (1) The sets I_k are nonempty by virtue of f^{-1} having domain $[0,1)$, and by definition $S^{\mathrm{c}}=\bigcup_{i=0}^{\infty}I_i$. We establish mutual disjointness by induction on n, it being trivially true for $n=0$. Suppose now that $I_0,f^{-1}(I_0),\ldots,f^{-n}(I_0)$ are mutually disjoint. Apply f^{-1} to conclude that $f^{-1}(I_0),\ldots,f^{-(n+1)}(I_0)$ are mutually disjoint, since f^{-1} is one-to-one. However all these sets lie in $[0,1/\beta)$, hence are disjoint from $I_0=[1/\beta,1)$, completing the induction step.

(2) Since $f^{(N)}(0)=f^{(N-1)}(\alpha)\in I_0$ where N is minimal with this property, we have $\alpha\in I_{N-1}$ and $0\in I_N$. Now we prove by induction on $k\ge 0$ that each I_k is an interval contained in $(0,1)$. It is certainly true for $k=0$. Suppose I_k is an interval in $(0,1)$ and $0\le k\le N-2$. Then $\alpha\notin I_k$ by the mutual disjointness property of the I_k, so I_k does not overlap the discontinuity point of f^{-1}, whence I_{k+1} is an interval. ∎

The intervals $\{I_i : 0\le i\le N-1\}$ are not contiguous, as we will show later. The interval I_{N-1} contains the point α, and we set $L_{-1}=I_{N-1}\cap[0,\alpha)$ and $R_{-1}=I_{N-1}\cap[\alpha,1)$, which we call the left and right halves of I_{N-1}. Certainly $R_{-1}\ne\emptyset$, since $\alpha\in R_{-1}$, and $L_{-1}\ne\emptyset$ except when α is the left endpoint of I_{N-1}, which happens exactly when $f^{(N)}(0)=1/\beta$. Now set

$$L_n:=f^{-(n+1)}(L_{-1}),\qquad R_n:=f^{-(n+1)}(R_{-1})\qquad\text{for } n\ge 0.$$

LEMMA 3.2. *The sets $\{I_k : 0\le k\le N-1\}$, $\{L_k : k\ge 0\}$ and $\{R_k : k\ge 0\}$ are all mutually disjoint intervals. The R_k's are nonempty. The L_k's are all nonempty unless $L_{-1}=\emptyset$, in which case all L_k's are empty.*

Proof. Since $I_{N-1}=L_{-1}\cup R_{-1}$, a disjoint union, $I_{N+n}=L_n\cup R_n$ for all $n\ge 0$ is also a disjoint union. The property that each L_n and R_n

is an interval is proved by induction on n. It is true for $n = 0$ because the intervals L_{-1} and R_{-1} do not cross over the discontinuity point of f^{-1}. Since $\alpha \in I_{N-1}$, Lemma 3.1 gives $\alpha \notin I_{N+n} = L_n \cup R_n$. Thus the induction step is completed by observing that no L_n or R_n contains the discontinuity point α. The assertions that all R_k are nonempty and that all L_k are nonempty if and only if L_{-1} is nonempty, follow from f^{-1} having domain $[0, 1)$. ∎

We now come to the main Lemma 3.3, which describes the precise way the intervals I_k, L_k, and R_k's interlace. Roughly speaking $L_k, L_{k+N}, L_{k+2N}, \ldots$ line up contiguously to the left of I_k while $R_k, R_{k+N}, R_{k+2N}, \ldots$ line up contiguously to its right. Here $R_0, R_N, \ldots$ are contiguous to 0 and are viewed as sitting to the right of I_0 under the $\pmod 1$ interpretation. The content of Lemma 3.3 is pictured in Figure 3.2.

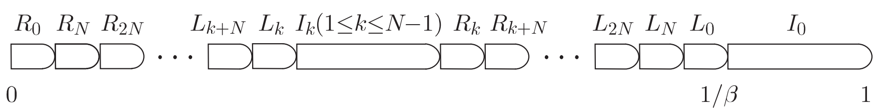


Fig. 3.2. How intervals in S^{c} interlace

LEMMA 3.3. (1) *Each interval L_k is contiguous with and to the left of I_k for $0 \le k \le N-1$. For all $k \ge 0$, L_{k+N} is contiguous with and to the left of L_k.*

(2) *The interval R_0 has 0 for left endpoint, and each other R_k is contiguous with and to the right of I_k for $1 \le k \le N-1$. For all $k \ge 0$, R_{k+N} is contiguous with and to the right of R_k.*

Proof. (1) We may assume $L_{-1} \ne \emptyset$, otherwise there is nothing to prove. Since f^{-1} has positive slope and maps $[0, \alpha)$ onto $[(1-\alpha)/\beta, 1/\beta)$, and L_{-1} is a subinterval of $[0, \alpha)$ having the right end point α, we conclude that $L_0 = f^{-1}(L_{-1})$ is a subinterval of $[(1-\alpha)/\beta, 1/\beta)$ having the right end point $1/\beta$. Thus L_0 is contiguous with and to the left of I_0. The intervals L_k, I_k, $0 \le k \le N-2$ are disjoint from R_{-1} and thus do not contain α. It follows from $L_{k+1} = f^{-1}(L_k)$, $I_{k+1} = f^{-1}(I_k)$ and an induction argument that L_k is contiguous with and to the left of I_k for $0 \le k \le N-1$. Since L_{-1} is the left half of I_{N-1}, we conclude that L_{N-1} is contiguous and to the left of L_{-1}. As $\alpha \notin L_{N-1}, L_{-1}$, we obtain $L_N = f^{-1}(L_{N-1})$ is contiguous and to the left of $L_0 = f^{-1}(L_1)$. Finally, $L_{k+1} = f^{-1}(L_k)$ and none of L_k's contain α, so another induction argument on k shows that L_{k+N} is contiguous with and to the left of L_k, for $k \ge 0$.

(2) The proof of (2) is similar to that of (1), except for minor details. ∎

Observe that, since the R_k's are nonempty, we conclude from Lemma 3.3(2) that no two intervals $\{I_k : 0 \le k \le N-1\}$ are contiguous.

Proof of Theorem 3.4. Lemma 3.3 implies that the set S^c "glues" together into $N+1$ connected components, the first being the union of $R_0, R_N, R_{2N}, \ldots$, the second being I_0 and all its neighboring $L_0, L_N, L_{2N}, \ldots$, and the other $N-1$ components consisting of each I_k for $1 \le k \le N-1$, together with all of its neighboring R_{k+jN} and L_{k+jN}. The complement S thus consists of at most N components.

We show that each of the components of S is degenerate, i.e., is either a point or the empty set, by proving S has Lebesgue measure $\lambda(S) = 0$. This amounts to showing $\lambda(S^c) = 1$. Now $\lambda(I_0) = 1 - 1/\beta$, and since f^{-1} is piecewise linear with slope $1/\beta$, we have $\lambda(I_k) = \beta^{-k}(1 - 1/\beta)$. By Lemma 3.1 all the I_k are disjoint, whence

$$\lambda(S^c) = \sum_{k=0}^{\infty} \lambda(I_k) = (1 - 1/\beta) \sum_{k=0}^{\infty} \beta^{-k} = 1. \tag{3.2}$$

It remains to decide under what circumstances the components of S are points or the empty set. Let p_n denote the left endpoint of R_n for $n \ge 0$, and set

$$q_k = \lim_{j\to\infty} p_{k+jN}, \qquad 0 \le k \le N-1.$$

If $f^{(N)}(0) = 1/\beta$ then by Lemma 3.2 all the L_k's are empty, in which case each q_k coincides with a left endpoint of some I_j, $0 \le j < N-1$, hence $S = \emptyset$ in this case. If $f^{(N)}(0) \ne 1/\beta$ then by Lemma 3.2 all the L_k's are nonempty, and Figure 3.2 puts into evidence the fact that each q_k cannot be in S, so $S = \{q_0, q_1, \ldots, q_{N-1}\}$. A continuity argument shows that $f^{-1}(q_k) = q_{k+1}$, except $f^{-1}(q_{N-1}) = q_0$, so $f(q_{k+1}) = q_k$ except $f(q_0) = q_{N-1}$, whence the q_i are cyclically permuted by f. ■

The next theorem shows that the condition $0 \notin S_{\beta,\alpha}$ of Theorem 3.4 is often true, i.e. for any given $\beta > 1$, this condition holds for a dense set of α. In fact we have the somewhat stronger result:

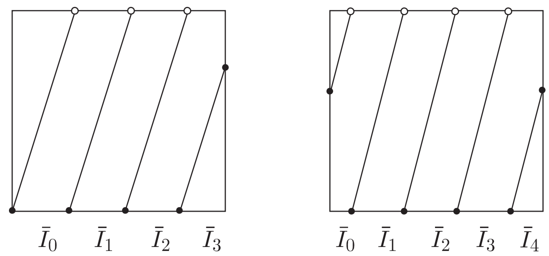
THEOREM 3.5. *For any $\beta > 1$ and any δ with $0 < \delta < 1$, let $K_\delta := K_\delta(\beta)$ denote the set of $\alpha \in [0,1)$ such that there is some finite N with $f_{\beta,\alpha}^{(N)}(0) \in [1-\delta, 1)$. Then K_δ is dense in $[0,1)$.*

In what follows we treat β as fixed and abbreviate $f_{\beta,\alpha}$ to f_α. The proof depends on a study of the f_α-expansion of 0 as α varies, which we give in Lemmas 3.4 and 3.5 below.

We use the symbolic dynamics of $f_{\beta,\alpha}$. For $0 \le \alpha < 1 - \{\beta\}$ the graph of $f_{\beta,\alpha}$ consists of $\lceil \beta \rceil$ straight line segments, while for $1 - \{\beta\} \le \alpha < 1$ it consists of $\lceil \beta \rceil + 1$ straight line segments. Let the jth piece be

$$\bar{I}_j := \{0 \le x < 1 : [\beta x + \alpha] = j\}$$

and represent it with the symbol j, see Figure 3.3.



(a) $0<\alpha<1-\{\beta\}$ (b) $1-\{\beta\}\le\alpha<1$

Fig. 3.3. Graphs of $f_{\beta,\alpha}$ for $\beta = 3\frac{1}{3}$

In all cases the symbol set is contained in $\{0,1,\ldots,\lceil\beta\rceil\}$, although $\lceil\beta\rceil$ never occurs if $0\le\alpha<1-\{\beta\}$.

LEMMA 3.4. *Let $\beta>1$ be fixed. For any integer $n\ge 0$, let $C_0,C_1,\ldots,C_n$ be an arbitrary sequence from the alphabet $\{0,1,\ldots,\lceil\beta\rceil\}$. Let $J[C_0C_1\ldots C_n] := \{\alpha\in[0,1) : f_\alpha^{(k)}(0)\in \bar I_{C_k}, 0\le k\le n\}$. Then*

(i) *$J[C_0C_1\ldots C_n]$ is an interval closed on the left and open on the right. (Possibly $J[C_0\ldots C_n]=\emptyset$.)*

(ii) *There is a constant $A[C_0C_1\ldots C_n]$ depending on $C_0,C_1,\ldots,C_n$ and β, such that if $\alpha\in J[C_0C_1\ldots C_n]$ then*

$$f_\alpha^{(n+1)}(0) = A[C_0C_1\ldots C_n] + (1+\beta+\ldots+\beta^n)\alpha. \tag{3.3}$$

Proof. We prove (i) and (ii) simultaneously by induction on n. For $n=0$, $f_\alpha^{(0)}(0)=0$ for all α, hence $J[0]=[0,1)$, while $J[i]=\emptyset$ for all $i\ge1$. Also $f_\alpha^{(n)}(0)=\alpha$ so we take $A[0]=0$ and (ii) holds. For the induction step, suppose (i) and (ii) are true for n. By (3.3) for $0\le j\le\lceil\beta\rceil$,

$$\begin{aligned} J[C_0\ldots C_nj] &= J[C_0\ldots C_n]\cap\{\alpha\in J[C_0\ldots C_n] : f_\alpha^{(n+1)}(0)\in I_j\}\\ &= J[C_0\ldots C_n]\cap\{\alpha : (j-\alpha)/\beta \le A[C_0\ldots C_n]\\ &\quad + (1+\beta+\ldots+\beta^n)\alpha < (j+1-\alpha)/\beta\}\\ &= J[C_0\ldots C_n]\cap\left[\frac{j-\beta A[C_0\ldots C_n]}{1+\beta+\ldots+\beta^{n-1}}, \frac{j+1-\beta A[C_0\ldots C_n]}{1+\beta+\ldots+\beta^{n-1}}\right), \end{aligned}$$

hence (i) holds for $J[C_0C_1\ldots C_nj]$.

We also note for later use that the set of values of j for which

$$\frac{j-\beta A[C_0C_1\ldots C_n]}{1+\beta+\ldots+\beta^{n-1}} \in \operatorname{Int}(J[C_0C_1\ldots C_n]) \tag{3.4}$$

form a consecutive string of integers (possibly empty). If l is the length of this string, then $j[C_0\ldots C_n]$ is partitioned into $l+1$ distinct nonempty intervals $J[C_0C_1\ldots C_nj]$.

To verify (ii) suppose $\alpha \in J[C_0C_1 \ldots C_n j]$. Then $\alpha \in J[C_0 \ldots C_n]$ so by induction hypothesis (ii)

$$f^{(n+1)}(0) = A[C_0 \ldots C_n] + (1 + \beta + \ldots + \beta^n)\alpha.$$

Now $f^{(n+1)}(0) \in \bar{I}_j$ so

$$\begin{aligned} f^{(n+2)}(0) &= f(f^{(n+1)}(0)) = \alpha + \beta f^{(n+1)}(0) - j \\ &= (\beta A[C_0 \ldots C_n] - j) + (1 + \beta + \ldots + \beta^{n+1})j \end{aligned}$$

and (ii) holds with $A[C_0 \ldots C_n j] = \beta A[C_0 \ldots C_n] - j$. This completes the induction step. ∎

LEMMA 3.5. *For all choices of $n+1$ symbols $C_0C_1 \ldots C_n$ from $\{0, 1, \ldots \ldots, \lceil\beta\rceil\}$ the Lebesgue measure*

$$\lambda(J[C_0C_1 \ldots C_n]) \le \frac{1}{1 + \beta + \ldots + \beta^n}.$$

Proof. Let $V[C_0C_1 \ldots C_n]$ denote the total variation of $f_\alpha^{(n+1)}(0)$ as α ranges over $J[C_0C_1 \ldots C_n]$. By Lemma 3.4(ii),

$$V(C_0C_1 \ldots C_n) = (1 + \beta + \ldots + \beta^n)\lambda(J[C_0C_1 \ldots C_n]). \tag{3.5}$$

However, $0 \le f_\alpha^{(n+1)}(0) < 1$ for all α and all n, hence

$$V(C_0C_1 \ldots C_n) \le 1. \tag{3.6}$$

Comparing (3.5) and (3.6) proves the lemma. ∎

Proof of Theorem 3.4. Let Δ_n be the partition of $[0, 1)$ consisting of the nonempty intervals $J[C_0C_1 \ldots C_n]$ and set

$$\|\Delta_n\| := \max\{\lambda(J[C_0C_1 \ldots C_n]) : J(C_0 \ldots C_n) \in \Delta_n\}.$$

Then $\lim_{n\to\infty} \|\Delta_n\| = 0$ by Lemma 3.5. It therefore suffices to show for any $J[C_0 \ldots C_n] \ne \emptyset$ that $J[C_0 \ldots C_n] \cap K_\delta \ne \emptyset$ for sufficiently small $\delta > 0$, since $K_\delta \subseteq K_{\delta'}$ whenever $\delta \le \delta'$.

We refer to any $J[C_0 \ldots C_n] \ne \emptyset$ as a J^{n+1}-*interval*. Lemma 3.5 implies that there must be a smallest integer $m \ge n$ such that $J[C_0 \ldots C_n]$ is not identical to a J^{m+2}-interval, i.e. $J[C_0 \ldots C_n] \equiv J[C_0 \ldots C_m]$ but $J[C_0 \ldots C_m]$ splits into at least two different J^{m+2}-intervals. As shown in the proof of Lemma 3.4 there must then be some j with $1 \le j \le [\beta] + 1$ with

$$\frac{j - \beta A[C_0 \ldots C_m]}{1 + \beta + \ldots + \beta^{m+1}} \in \mathrm{Int}(J[C_0C_1 \ldots C_m]).$$

Hence for any sufficiently small $\delta > 0$,

$$\alpha_\delta := \frac{j - \beta A[C_0 \ldots C_m] - \delta/2}{1 + \beta + \ldots + \beta^{m+1}} \in \mathrm{Int}(J[C_0C_1 \ldots C_m]).$$

Now Lemma 3.4 yields

$$f_{\alpha_\delta}^{(m+1)}(0) = A[C_0C_1\ldots C_m] + (1+\beta+\ldots+\beta^m)\alpha_\delta = \frac{j-\alpha_\delta-\delta/2}{\beta}.$$

Thus for all sufficiently small δ,

$$f_{\alpha_\delta}^{(m+2)}(0) = f_{\alpha_\delta}\left(\frac{j-\alpha_\delta-\delta/2}{\beta}\right) = 1-\frac{\delta}{2} > 1-\delta,$$

hence $\alpha_\delta \in K_\delta$ and $\alpha_\delta \in J[C_0C_1\ldots C_m]$. ∎

We now complete the:

Proof of Theorem 3.3. By their definitions, $R_{p/q}(s) = S_{\beta,\alpha}$ with $\beta = p/q$ and $\alpha = \{(p-q)s\}$. Combining Theorems 3.4 and 3.5 we see that $S_{\beta,\alpha}$ is finite for a dense set of α in $[0,1]$. Under the map $\alpha = \{(p-q)s\}$, this dense set of α pulls back to a set of s dense on $[0,1-1/\beta]$ for which $R_{p/q}(0)$ is finite.

It remains to check that $R_{p/q}(s)$ is finite for $s = 0$ and $s = 1-1/p$. For $s = 1-1/p$, $\alpha = q/p$. Thus $f(0) = f_{\beta,1/\beta}(0) = 1/\beta$, and we conclude from Theorem 3.4 that $R_{p/q}(1-1/p) = S_{\beta,1/\beta}$. For $s=0$, $\alpha=0$. Thus $f(x) = \{\beta x\} = \beta x$ for $0 \le x < 1/\beta$. For $0 < x < 1/\beta$, $f^{(k)}(x) = \beta^k x$ until some iterate occurs with $f^{(k)}(x) > 1/\beta$, which must happen since $\beta > 1$. Hence $R_{p/q}(0) = S_{\beta,0}$ consists of 0 alone and so is again finite. ∎

Proof of Corollary 1.4a. The case $p/q = 3/2$ and $t = 1/3$, and the case $s = 0$, follow by the proof above. For $s = 1/6, 1/3, 1/2, 2/3$, the result follows using the criterion of Theorem 3.2, since $f^{(N)}(0) \ge 2/3$ for $N = 3,2,3,1$ respectively. ∎

Finally, we state some open questions. Let

$$V_{p/q} := \{s : Z_{p/q}(s, s+1/p) \ne \emptyset\},$$

and let E_β denote the set of exceptional values for β, i.e.

$$E_\beta := \{\alpha : S_{\beta,\alpha} \text{ is an infinite set}\}.$$

Theorem 3.2 says that if $s \in V_{p/q}$ then $\alpha = \{(p-q)s\} \in E_{p/q}$. As far as we know the set $V_{p/q}$ is empty for all p/q. However we doubt that $E_{p/q}$ is empty. Indeed, we make the following two conjectures.

(I) For all β, E_β is a nonempty perfect set.

(II) For all $\alpha \in E_\beta$, $S_{\beta,\alpha}$ is a nonempty perfect set.

Theorem 3.4 shows that all $S_{\beta,\alpha}$ have Lebesgue measure zero. Possibly the same is true for all E_β.

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AT&T BELL LABORATORIES
MURRAY HILL, NEW JERSEY 07974, U.S.A.
E-mail: LEOPOLD@RESEARCH.ATT.COM
JLC@RESEARCH.ATT.COM

BRIGHAM YOUNG UNIVERSITY
PROVO, UTAH 84602, U.S.A.
E-mail: ANDY@HAMBLIN.MATH.BYU.EDU

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