ACTA ARITHMETICA
LXXIII.3 (1995)

Solutions of $x^3 + y^3 + z^3 = nxyz$

by

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Introduction. The diophantine equation

$$x^3 + y^3 + z^3 = nxyz \tag{1}$$

has only trivial solutions for three (probably) infinite sets of n-values and some other n-values ([7], Chs. 10, 15, [3], [2]). The main set is characterized by: $n^2 + 3n + 9$ is a prime number, $n - 3$ contains no prime factor $\equiv 1 \pmod 3$ and $n \neq -1, 5$. Conversely, equation (1) is known to have non-trivial solutions for infinitely many n-values. These solutions were given either as "1 chains" ([7], Ch. 30, [4], [6]), as recursive "strings" ([9]) or as (a few) parametric solutions ([3], [9]).

For a fixed n-value, (1) can be transformed into an elliptic curve with a recursive solution structure derived by the "chord and tangent process". Here we treat (1) as a quaternary equation and give new methods to generate infinite chains of solutions from a given solution $\{x, y, z, n\}$ by recursion. The result of a systematic search for parametric solutions suggests a recursive structure in the general case.

If x, y, z satisfy various divisibility conditions that arise naturally, the equation is completely solved in several cases.

Solutions based on the relation to other diophantine equations. We assume that $xyz \neq 0$ unless explicitly stated otherwise and $(x, y, z) = 1$. From (1) we immediately have

$$xyz \mid x^3+y^3+z^3-3xyz, \quad \text{i.e.} \quad xyz \mid (x+y+z)(x^2+y^2+z^2-xy-yz-zx).$$

If we restrict x, y or z to be a factor in one of these algebraic factors (or a combination of such cases), the connection between (1) and other well-known diophantine equations becomes evident and progress can be made.

Case (a). $x \mid x + y + z$, i.e. $x \mid y + z$.

Theorem A. *Assume that* $\{x, y, z, n\}$ *is a solution of* (1) *satisfying* $x \mid y + z$ *and* $\{x, y, z\} \neq \{-1, 1, M\}$, $\{M, 1, -M^3 - 1\}$ *or* $\{M, 1, -M^2 +$

$M-1\}$ *for any* M. *A new non-trivial solution* $\{x', y', z', n'\}$ *of* (1) *is then given by* $x' = (y^2 + (x^3 + y^3)/z)/x$, $y' = y$, $z' = z(x'^3 + y'^3)/(x^3 + y^3)$, $n' = (z'^2 + (x^3 + y^3)/z)/x'y'$.

Proof. x' is integral as $(y^2 + (x^3 + y^3)/z) \equiv (y^2 + nxy - z^2) \equiv (y^2 - z^2) \equiv 0 \pmod{x}$, due to the assumption $x \mid y + z$. To show that z' is an integer and that $x'y'z' \mid x'^3 + y'^3 + z'^3$ is straightforward but the calculations are lengthy. ■

Remark 1. The condition $\{x, y, z\} \neq \{-1, 1, M\}$ makes z' well-defined, while the other conditions are necessary and sufficient to make $x'y'z' \neq 0$.

Remark 2. It is easy to show that $x' \mid y' + z'$. Iterated use of the algorithm (while interchanging y and z) then in general gives an infinite sequence of solutions.

Case (b). $xy \mid x+y+z$. By repeating the algorithm in Theorem A (and alternating between x and y) it is possible in general to generate infinitely many solutions of (1) as x, y in this case is a solution of an equation

$$x^2 + y^2 + c = Nxy \tag{2}$$

in which the constants c and N are uniquely determined by one solution $\{x, y, z, n\}$ of equation (1). From any solution of equation (2) it is generally possible to generate an infinite sequence of solutions ([1], [5]) and to each of these solutions corresponds a solution of (1). In order to classify the solutions of (1) (that satisfy (b)) in a suitable way, we define a solution u, v of (2) in such a sequence as *minimal* if $|uv| \le |xy|$ for any solution x, y in the sequence.

THEOREM B. *All valid combinations of* c, N *in equation* (2) *and their associated minimal solution* $\{x, y\}$ *are given in Table* 1 (where the corresponding z and the first n-values associated with each c, N-combination are also listed). *Any solution of equation* (1) *satisfying* $xy \mid x + y + z$ *can be derived from exactly one of these solutions.*

Proof. Let $\{x, y, z\}$ be any solution of (1) satisfying $xy \mid x + y + z$ and the conditions in Theorem A. Define $c = (x^3 + y^3)/z$; clearly c is an integer $(\neq 0)$ and $(x, c) = (y, c) = 1$. Now $xy \mid x+y+z$ implies $x \mid y^2+c$ and $y \mid x^2+c$. Thus

$$x^2 + y^2 + c = Nxy$$

for some (integer) N. Now fix c, N and use the algorithm $x' = (y^2 + c)/x$, $y' = y$ to generate a new solution x', y'. As in Theorem A define $z' = (x'^3 + y'^3)/c$ and $n' = (z'^2 + c)/x'y'$. Then z' $(\neq 0)$ and n' are integers and $x'^3 + y'^3 + z'^3 = n'x'y'z'$. It is easy to show that $y' \mid x' + z'$ and from Theorem A we have $x' \mid y' + z'$ so $x'y' \mid x' + y' + z'$ and we can repeat the

Table 1

c	N	$\{x,y\}$	z	n
2	4	$\{1,1\}$	1	$3, 66, 13971, \ldots$
1	3	$\{1,1\}$	2	$5, 41, 1769, \ldots$
3	4	$\{1,2\}$	3	$6, 978, 196026, \ldots$
9	7	$\{1,2\}$	1	$\ldots, 112901, 41, 5, 2309,$ $5366105, \ldots$
14	8	$\{1,3\}$	2	$\ldots, 92454, 19, 6, 10995,$ $43270246, \ldots$
5	3	$\{2,3\}$	7	$9, 261, 12105, \ldots$
35	8	$\{2,3\}$	1	$\ldots, 696174, 154, 6, 1410,$ $5773726, \ldots$
$-M^3-1$	$M-M^2$	$\{1,M\}$ $(M\neq 0,\pm 1)$	-1	$\ldots, -M^4+3M, -M^2,$ $-M^8+5M^7-10M^6$ $+7M^5+5M^4-9M^3$ $-M^2+6M, \ldots$
-1	M	$\{0,1\}$ $(M\neq 0)$	-1	$*, M^5+2M^2, \ldots$
$-M-1$	$M-1$	$\{-1,1\}$ $(M\neq 0,-1)$	0	$*, M^3-2M^2+3M-3, \ldots$
0	-2	$\{-1,1\}$ $(M\neq 0)$	M	$-M^2$
$-(u^2-uv+v^2)$	1	$\{u,v\}, \{v,v-u\}$ or $\{v-u,-u\}$, where u,v arbitrary	$-(x+y)$	3

* : arbitrary n with trivial solution.

algorithm if $\{x',y',z'\}$ satisfies the conditions in Theorem A. From $(x,y)=1$ it is easy to deduce that $(x',y')=1$.

We now turn to the problem of determining all values of c, N and the associated minimal solutions and realize first that any minimal solution must be found among the solutions that satisfy

$$|Nv-u|\geq |u| \quad \text{and} \quad |Nu-v|\geq |v|$$

or the equivalent conditions if $uv\neq 0$:

$$|(v^2+c)/u|\geq |u| \quad \text{and} \quad |(u^2+c)/v|\geq |v|,$$

as these inequalities must be satisfied by a minimum of $|xy|$ in a solution sequence:

$$\ldots, (x^2+c)/y,\ x,\ y,\ (y^2+c)/x,\ \ldots$$

where two consecutive terms constitute a solution of equation (2).

Assume that u,v is a minimal solution of $x^2+y^2+c=Nxy$ and $0<u\leq v$. The corresponding z equals w, assuming $w>0$. From above we have $uv \mid u+v+w$.

First we consider the simple case $u=v$; then $u=v=1$ as $(u,v)=1$ and the only valid combinations of c,N are $1,3$ and $2,4$, corresponding to $w=2,1$ and $n=5,3$.

Next we assume $u < v$ (which satisfies the first inequality) and get from the second inequality, $(u^2 + c)/v \geq v$, i.e. $u^2 + (u^3 + v^3)/w \geq v^2$ or $w \leq v + u^2/(v - u)$. However, $w = Auv - u - v$ for some (integer) $A > 0$ and we have

$$Auv \leq 2v + u + u^2/(v - u) = (2v^2 - uv)/(v - u),$$

i.e.

$$1 \leq A \leq (2v - u)/(u(v - u)) = 2/u + 1/(v - u).$$

The only combinations of A, u and $v - u$ that also make $\{u, v, w\}$ a solution of equation (1) are

A	u	$v-u$
$1,2$	2	1
2	1	2
$2,3$	1	1

which gives

c	N
$35,5$	$8,3$
14	8
$9,3$	$7,4$

It is easy to verify that these solutions are minimal and every c, N-combination corresponds to a "type III sequence", defined in [5].

The other cases with $uv \neq 0$: $u > 0$, $v < 0$, $w > 0$ or $u < 0$, $v < 0$, $w > 0$ can be treated in the same way but require more details.

The case $uv = 0$ also occurs and the minimal solution must be $\{u, v\} = \{0, 1\}$ as $(v, w) = 1$. Thus $w = -1$, $c = -1$ and N arbitrary. For every N we have a "type I sequence" [5].

The last four c, N-combinations in Table 1 cover the three exceptions in Theorem A and a degenerate sequence into a 6-cycle when $n = 3$.

From [1] and [5] we know that all solutions of (2) are given by the solution sequences above. As every sequence must contain a minimal solution and a sequence is uniquely defined by c, N and one solution, it follows that all solutions of equation (1) satisfying $xy \mid x + y + z$ can be found this way and the solution $\{x, y, z, n\}$ of (1) can be deduced from the particular equation (2) with $c = (x^3 + y^3)/z$ and $N = (x^2 + y^2 + c)/xy$. ∎

We now consider two subcases of (b) where all solutions of equation (1) are given by a finite number of specific and parametric solutions:

Theorem B1. *If $xy \mid x + y + z$ and $z \mid x^2 - xy + y^2$ in equation* (1), *then the only solutions $\{x, y, z, n\}$ are*: $\{1, 1, 1, 3\}$, $\{M, 3 - M, -3, 3\}$, $\{1, 2, 1, 5\}$, $\{1, 2, 3, 6\}$, $\{2, 3, 1, 6\}$, $\{2, 3, 7, 9\}$, $\{M, 1, -1, -M^2\}$, $\{-M+1, M+2, -M^2 - M - 1, -M^2 - M - 4\}$ *and* $\{M, 1, -M^2 + M - 1, M^3 - 2M^2 + 3M - 3\}$.

Proof. $xy \mid x + y + z$ implies that $z = Axy - x - y$ for some A. Now if $z \mid x^2 - xy + y^2$ then $x^3 + y^3 + z'^3 = n'xyz'$, where $z' = (x^2 - xy + y^2)/z$ from Theorem C below. Then $x \mid y + z'$ and $y \mid x + z'$, i.e. $xy \mid x + y + z'$ and $z' = Bxy - x - y$ for some B. Finally, $x^2 - xy + y^2 = zz' = (Axy - x$

$-\,y)(Bxy - x - y)$, which gives $ABxy - (A+B)(x+y) + 3 = 0$. Writing this as: $(Ax-2)(By-2) + (Ay-2)(Bx-2) = 2$ and considering all sign combinations of $(Ax-2), (By-2), \ldots$ and A, B, x, y gives the statement. ■

THEOREM B2. *Assume that* $xyz \mid x + y + z$ *in equation* (1). *Then it has only the solutions* $\{x, y, z, n\}$: $\{1,1,1,3\}$, $\{x, y, -x-y, 3\}$, $\{1,1,2,5\}$, $\{1,2,3,6\}$ *and* $\{-1, 1, M, -M^2\}$.

Proof. $x + y + z = Axyz$ for some (integer) A. Thus $A = 1/(xy) + 1/(yz) + 1/(zx)$. If $|x|$, $|y|$ and $|z| \ge 2$ then $|A| \le 3/4$, which implies $A = 0$ and $n = 3$. Now assume $A \ne 0$. Then at least one of $|x|$, $|y|$ or $|z|$ is 1 and we choose $y = 1$. If $Ax = 1$ or $Az = 1$, we get the solution $\{-1, 1, M, -M^2\}$. On the other hand, if $Ax \ne 1$ and $Az \ne 1$, $|z| = |(x+1)/(Ax-1)| = |-1/A + (1-1/A)/(Ax-1)| \le 3$ and analogously $|x| \le 3$, which gives the remaining solutions. ■

Case (c). $x \mid x^2 + y^2 + z^2 - xy - yz - zx$, i.e. $x \mid y^2 - yz + z^2$.

THEOREM C. *If* $x \mid y^2 - yz + z^2$ *in* (1), *then a new solution is given by*: $x' = (y^2 - yz + z^2)/x$, $y' = y$, $z' = z$, $n' = (n(y+z) - 3x')/x$.

Proof. It is easy to see that $x' \mid y^3 + z^3$, $y \mid x'^3 + z^3$ and $z \mid x'^3 + y^3$. As $(x', y, z) = 1$, it follows that $x'yz \mid x'^3 + y^3 + z^3$. ■

Remark. Making a minor extension of the results in [9] it is possible, from a solution of equation (1) that satisfies (c), to generate new such solutions with associated n-values n_k and n'_k given by:

$$(3) \qquad n_{k+1} = n_k n - n_{k-1} n' + n_{k-2} \quad \text{and} \quad n'_k = n_{-k}, \quad \text{where } n_1 = n,\ n_0 = 3,\ n_{-1} = n'.$$

The associated x, x', y, z can be calculated from the relation $x\alpha + x'\beta = y + z\alpha\beta$ where α, β is a complete pair of units (see [9]) in the Galois cubic field determined by $u^3 - n_k u^2 + n'_k u - 1 = 0$.

EXAMPLE. In [9] it was shown that the solution $\{M^2 \pm M + 1, 2, M^2 + 5\}$ is associated with the unit α^2 in a Galois cubic field if α is a unit satisfying the minimal equation $u^3 + (M+1)u^2 + (M-2)u - 1 = 0$. The solutions associated with the units α^3 and α^4 respectively are $\{-M^4 - M^3 - 6M^2 - 4M - 7, M^4 - 2M^3 + 6M^2 - 5M + 4, -9, -M^3 - 6M - 4\}$ and $\{M^6 + M^5 + 10M^4 + 11M^3 + 28M^2 + 27M + 13, M^6 - 3M^5 + 12M^4 - 19M^3 + 30M^2 - 21M + 9, 2M^2 - 2M + 38, M^4 + 8M^2 + 4M + 13\}$.

Case (d). $xy \mid x^2 + y^2 + z^2 - xy - yz - zx$.

Thomas and Vasquez [9] (Th. 2) proved by induction:

THEOREM D. *If* $xy \mid x^2 + y^2 + z^2 - xy - yz - zx$, *then in general it is possible to repeat the algorithm in Theorem* C *and generate an infinite sequence of solutions.*

R e m a r k. The condition equivalent to $x \mid y^2 - yz + z^2$ takes a simple form: $|P| = 1$, if equation (1) is transformed into (cf. [3], p. 55):

$$aP^3 + bQ^3 + cR^3 = (n+6)PQR, \qquad \text{where } abc = n^2 + 3n + 9 \text{ and } (P, Q, R) = 1.$$

If $x \mid y^2 - yz + z^2$ and $y \mid z^2 - zx + x^2$, we can assume $P = Q = 1$ without loss of generality and solve the equation system:

$$a + b = (n+6)R - cR^3, \qquad ab = (n^2 + 3n + 9)/c.$$

This leads (if $cR^2 \neq 4$) to the equation:

$$u^2 - c(cR^2 - 4)v^2 = (cR^2 - 3)^3 \tag{4}$$

whose associated Pellian $U^2 - c(cR^2 - 4)V^2 = 4$ has the solution $\{U, V\} = \{cR^2 - 2, R\}$. The sign of u should be chosen thus: $u \equiv 1 \pmod{cR^2 - 4}$ to give a solution of (1) with $n = cR^2 - 2 + 2(u-1)/(cR^2 - 4)$. There are parametric solutions of (4), e.g. if $c = -1$, $R = M$: $\{u, v\} = \{3M^2 + 13, M^2 + 7\}$, corresponding to the solution $\{\pm M^2 + 4M \pm 7, M^3 + 3M, -M^2 - 8\}$ of (1), and if $c = -144M^4 - 132M^2 - 37$, $R = M$ we have $\{u, v\} = \{-864M^8 - 1224M^6 - 618M^4 - 135M^2 - 11, 12M^4 + 7M^2 + 1\}$.

We now solve equation (1) completely in two subcases of (d) by also imposing divisibility conditions on z:

THEOREM D1. *If $xy \mid x^2 + y^2 + z^2 - xy - yz - zx$ and $z \mid x + y$ in equation* (1), *then all solutions $\{x, y, z, n\}$ are given by*: $\{1, -1, M, -M^2\}$, $\{1, 1, -1, -1\}$, $\{1, 1, -2, 3\}$, $\{1, 3, -4, 3\}$, $\{1, 1, 1, 3\}$, $\{1, 1, 2, 5\}$ *and solutions generated from these by means of Theorem* D.

P r o o f. We study the equation

$$x^2 + y^2 + z^2 - xy - yz - zx = Nxy \tag{5}$$

with fixed z and $z \mid x+y$. It is easy to show that any solution of (1) satisfying the divisibility conditions in Theorem D1 must also satisfy equation (5) for some integer $N \equiv -3 \pmod{z^2}$ and vice versa, as the conditions $z \mid x + y$ and $N \equiv -3 \pmod{z^2}$ are equivalent.

We define a solution x, y of (5) to be minimal if $|xy|$ is a minimum when repeating the operations $x' = (y^2 - yz + z^2)/x$ and $y' = (z^2 - zx + x^2)/y$ and we note that z (fixed) satisfies $z \mid x' + y$ and $z \mid x + y'$ when making either operation, so z also divides $u + v$ if u, v is a minimal solution of (5). If we know z it is clear from the discussion in [5] that there are only a finite number of possible N-values and that we can find all solutions of (5) from the minimal solutions by the operations above.

Put $\{x, y, z\} = \{u, v, w\}$ and assume u, v minimal. Then we have the necessary conditions:

$$|u'| = (v^2 - vw + w^2)/|u| \geq |u| \quad \text{and} \quad |v'| = (w^2 - wu + u^2)/|v| \geq |v|.$$

If $u, v, w > 0$ we have immediately: $u^2 + v^2 \leq u^2 + v^2 + 2w^2 - uw - vw$ or $u + v \leq 2w$. The assumption $w \mid u + v$ implies $w \leq u + v$ and we thus have $1 \leq (u + v)/w \leq 2$. We define (the integer) $k = (u + v)/w$ and deduce from (5) that $uv \mid k^2 - k + 1$. Here $k = 1, 2$ and we find the minimal solutions:

k	u	v	w	N
1	1	1	2	1
2	1	1	1	0

We now consider the most complicated case: $u, v < 0$ and $w > 0$. Putting $u = kw - v$ into the first of the necessary conditions gives: $(k^2 - 1)w \leq u(2k - 1)$. Analogously we have $(k^2 - 1)w \leq v(2k - 1)$. In this case $k \leq -1$ and $(k^2 - 1)w \geq 0$, $u(2k - 1) \geq 0$ and $v(2k - 1) \geq 0$. Consequently, $((k^2 - 1)w)^2 \leq uv(2k - 1)^2$ but $uv \mid k^2 - k + 1$ implies $uv \leq k^2 - k + 1$ and we get:

$$((k^2 - 1)w)^2 \leq (2k - 1)^2(k^2 - k + 1). \tag{6}$$

If $k = -1, -2$ or -3 we have the minimal solutions:

k	u	v	w	N
-1	-1	-1	2	9
-1	-1	-3	4	13
-2	-1	-1	1	4

If $k < -3$, from (6) we get $w \leq |2k - 1|(\sqrt{k^2 - k + 1})/(k^2 - 1) < 3$ for all $k \leq -4$. Thus $w \leq 2$ and if we put $w = 1, 2$ into equation (5) and solve it by using Mills' [5] method, no additional minimal solutions occur. The remaining case: $u > 0$, $v < 0$ and $w > 0$, is easily resolved. The only possible k-value is $k = 0$ and we find a class of minimal solutions:

k	u	v	w	N
0	1	-1	M	$-(M^2 + 3)$

(any $M \neq 0$). ■

R e m a r k 1. Using the technique in Theorem D1 it is possible to find all minimal solutions of (5) in the more general case with $z = z_1 z_2$, where $z_1 \mid x + y$ and $z_2 \mid x^2 - xy + y^2$ and z_2 given.

R e m a r k 2. Using the notation in [9], Theorem D1 gives all strings satisfying $a + c \equiv 0 \pmod{d}$.

THEOREM D2. *If* $xyz \mid x^2+y^2+z^2-xy-yz-zx$ *in equation* (1), *then the only solutions* $\{x,y,z,n\}$ *are*: $\{1,1,1,3\}$, $\{-1,-1,1,-1\}$ *and* $\{x,y,z,4(x+y+z)+3\}$, *which represents any solution generated from* $\{-1,-1,1,-1\}$ *by repeated application of Theorem* C.

Proof. By studying the equation

$$x^2+y^2+z^2-xy-yz-zx = Nxyz \tag{7}$$

and using virtually the same arguments as Mordell [7] (pp. 106–110) did when discussing $x^2+y^2+z^2-axyz=b$, the statement can be proved; however a lot of detail is necessary. ∎

Remark 1. Straus' and Swift's [8] "finiteness result" implies that only a finite number of N-values are possible in equation (7) and the minimal solutions can also be derived from the equation(s) (5) in [8].

Remark 2. Using the notation in [9], Theorem D2 gives all strings with $a+b\equiv c \pmod{d}$.

Case (e). $x|y+z$ and $y|z^2-zx+x^2$.

THEOREM E. *The combination of Theorems* A *and* C *in general gives an infinite sequence of solutions*; *thus if* $x \mid y+z$ *and* $y \mid z^2-zx+x^2$ *in* (1) *then* $\{x,y,z\}$, $\{x',y',z'\} = \{(y^2+(x^3+y^3)/z)/x, y, z(x'^3+y^3)/(x^3+y^3)\}$, $\{x'',y'',z''\} = \{x', (z'^2-z'x'+x'^2)/y', z'\}, \ldots$ *are new solutions.*

Necessary and sufficient conditions for this are: $x=(A+B-1)/f$, $y=(A^2-A+1)/f$, $z=(AB-1)/f$ *and* $AB-1 \mid (A+1)f^2$, *where* $f=(A+B-1, AB-1, A^2-A+1)$.

Proof. It is easy to show that $x' \mid y'+z'$ and $y' \mid z'^2-z'x'+x'^2$ in Theorems A and C if $x \mid y+z$ and $y \mid z^2-zx+x^2$, which proves the first part of the statement. From Theorem A we have $y+z=Ax$ or $y=Ax-z$. Theorem C gives $y'=(z^2-zx+x^2)/y$ and we find $x \mid y'+z$, i.e. $y'=Bx-z$. Combining these expressions we get $yy'=z^2-zx+x^2=(Ax-z)(Bx-z)$, i.e. $x(AB-1)=z(A+B-1)$ implying $x=(A+B-1)/f$ and $z=(AB-1)/f$, where $f=(A+B-1, AB-1)$. Also $y=Ax-z=(A^2-A+1)/f$. The necessity of the divisibility condition is easily derived from $z \mid x^3+y^3$ and it is straightforward to prove the sufficiency of the conditions. ∎

Remark. From Theorem E we have $f \mid A^2-A+1$, which implies $AB-1 \mid (A+1)(A^2-A+1)^2$ and $(A+1)(A^2-A+1)^2+C=ACB$ for some integer C. Equations of this type have been discussed by Mohanty [6] (Theorem 8) and in general infinitely many solutions can be generated.

EXAMPLES. The most easily found starting-points are given by Theorem B1. A less obvious example is $\{M^2-M+3, M^2-3M+3, M^4-3M^3+6M^2-5M+3, M^4-2M^3+4M^2+3\}$, which can be derived from the solution

$\{2, M^2 \pm M + 1, M^2 + 5\}$, using the technique in the remark above. Two interesting solutions are $\{-M^2, M^4 - 3M^3 + 6M^2 - 6M + 3, M^5 - 3M^4 + 6M^3 - 8M^2 + 6M - 3, -M^4 + 3M^3 - 6M^2 + 9M - 6\}$ and $\{M, 1, -1, -M^2\}$ as they generate sequences that do not contain auto-generating solutions, i.e. solutions that only change the value of the parameter when Theorem A is applied.

Parametric solutions. Now we give additional parametric solutions of (1) by using a method that is independent of divisibility conditions. We do this by first transforming equation (1) into

$$aP^3 + bQ^3 + cR^3 = (n+6)PQR \tag{8}$$

([3], p. 55), then putting $a = n^2+3n+9$, $b = c = 1$ and assuming n, P, Q, R to be polynomials of appropriate type with unknown coefficients, and finally solving the resulting equation system which is straightforward in principle but computationally cumbersome. The method can be used for equation (1) directly, but the transformation decreases the magnitude of the solutions considerably as $x = (aP^3 - 3PQR)/f$ where $f \mid n - 3$. Using this method, a search has been made for parametric solutions where the polynomial $n(M)$ only contains even powers, $P(M)$ only even or only odd powers and $Q(M)$, $R(M)$ satisfy $Q(M) = R(-M)$ or $Q(M) = -R(-M)$. Defining g_P as the degree of the polynomial P, it is fairly easy to see that the possible values of $g_Q - g_P$ are: $g_n/2$, g_n, $g_n + 1 + 2k$ ($k \geq 0$). The solutions obtained this way are unique if we disregard trivial transformations of the parameter M and in several instances it has been possible to determine all solutions of this kind. In Table 2 we give the associated n-values (# = M an odd integer; * = no solution exists; + = other solutions may exist).

The n-values of the solutions with $g_P = 0$ are identical with the first consecutive values generated by the algorithm (3) in the remark to Theorem C, with start values $n_1 = 3$, $n_0 = 3$ and $n_{-1} = -(M^2 + 15)/4$, and $g_Q = |k|$ (of Theorem C) in these cases. Furthermore, $\{P, Q, R\} = \{1, Q_k, Q_{-k} = Q_k(-M)\}$ where Q_k, can be found from:

$$Q_{k+1} = Q_k(-(M+3)/2) - Q_{k-1}((M-3)/2) + Q_{k-2} \tag{9}$$

with $Q_1 = -(M+3)/2$, $Q_0 = 3$ and $Q_{-1} = (M-3)/2$.

When $g_P = 1$, the n-values (except if $n = -M^2$ or $(M^4 + 15M^2 + 48)/4$) can be generated by a similar (reverse and shifted) algorithm with $n_{k+1} = (-(M^2 + 15)/4)n_k - 3n_{k-1} + n_{k-2}$ and $n_0 = 3$, $n_{-1} = 3$ and $n_{-2} = -(M^2 + 15)/4$. Here $\{P, Q, R\} = \{M, Q_k, -Q_k(-M)\}$ where Q_k is generated by (9) with $Q_1 = M^2 + 3M + 27$, $Q_0 = (-3M + 27)/2$ and $Q_{-1} = (-M^2 - 6M + 27)/4$.

Table 2

	$g_P = 0$	$g_P = 1$
$g_n = 2$		
$g_Q - g_P =$		
$g_n/2$	$-(M^2+15)/4(\#)$	$-(M^2+24)$ or $-M^2$
g_n	$(M^2+33)/2(\#)$	$-(M^2+15)/4(\#)$
g_n+1	$(M^2+255)/4(\#)$	3
$\geq g_n+3$	$*$	$*$
$g_n = 4$		
$g_Q - g_P =$		
$g_n/2$	$(M^4+30M^2+129)/16(\#)$	$(M^4+15M^2+48)/4$ or $(M^4+39M^2+336)/4$
g_n	$(M^4+102M^2+2049)/8(\#+)$	$-(M^4+30M^2+33)/16(\#+)$
g_n+1	$(15M^4+1010M^2+16383)/16(\#+)$	$(-3M^4-82M^2+21)/16(\#+)$
$g_n = 6$		
$g_Q - g_P =$		
$g_n/2$	$-(M^6+45M^4+531M^2+1023)/64(\#)$	$-(M^6+54M^4+873M^2+3840)/16(\#+)$
g_n	$(M^6+207M^4+9711M^2+131073)/32(\#+)$	$-(M^6+81M^4+1467M^2+51)/64(\#+)$
g_n+1	$(21M^6+2485M^4+90615M^2+1048575)/64(\#+)$	$(-3M^6-187M^4-2865M^2+15)/32(\#+)$
$g_n = 8$		
$g_Q - g_P =$		
$g_n/2$	$(M^8+60M^6+1158M^4+7484M^2+8193)/256(\#+)$	$(M^8+69M^6+1635M^4+14999M^2+39936)/64(\#+)$

PROBLEM 1. The algorithm (9) can be derived from the results in [9] if $g_P = 0$, but what group structure makes it valid also for $g_P = 1$?

PROBLEM 2. Do similar algorithms exist for $g_P = 2, 3, \ldots$?

EXAMPLES. $\{x, y, z, n\} =$

- $\{(M^2+147)/4, (M^4 \pm 6M^3 + 36M^2 \pm 98M + 147)/16, (M^2+33)/2\}$,
- $\{M^5+21M^3+81M, \pm M^4 + 12M^3 \pm 81M^2 + 324M \pm 729, -M^2-24\}$,
- $\{81(M^4+1134M^2+263169)/16, (\pm M^7 - 27M^6 \pm 729M^5 - 11259M^4 \pm 142155M^3 - 1213785M^2 \pm 7105563M - 21316689)/128, (M^2+255)/4\}$,
- $\{(M^7+15M^5+60M^3+48M)/2, -M^5 \pm 3M^4 - 12M^3 \pm 28M^2 - 48M \pm 48, (M^4+15M^2+48)/4\}$ and
- $\{(M^{11}+78M^9+2193M^7+26256M^5+119232M^3+139968M)/2, -M^9 \pm 15M^8 - 171M^7 \pm 1349M^6 - 8400M^5 \pm 41040M^4 - 158112M^3 \pm 466560M^2 - 979776M \pm 1259712, (M^4+39M^2+336)/4\}$

(where upper/lower signs belong to y, z resp.).

Solutions with $n = M^2 + 5$ and $n = -M^2 - 8$ appear in the cases with $g_P = 4, 7$ and $g_Q - g_P = 2, 1$ respectively but they are not necessarily unique solutions. If we again study equation (8) (where P, Q, R are of the same type as above) and now suppose $a = 1$, $b = kM^2 + lM + m$, $c = kM^2 - lM + m$ ($abc = n^2 + 3n + 9$), they are the only solutions when $g_n = 2$, $g_Q = g_R = 0$ and $g_P = 0, 1$, in addition to a solution with $n = -M^2$:

n	P	Q	R	k	l	m	x	y, z
$M^2 + 5$	2	1	-1	1	1	7	2	$M^2 \pm M + 1$
$-M^2 - 8$	M	1	-1	-1	1	-7	$M^3 + 3M$	$\pm(M^2 + 7) + 4M$
$-M^2$	M	1	-1	-1	3	-3	M	± 1

Sylvester's "Theorem of Derivation" implies that also the equation $aP^3 + bQ^3 + cR^3 = (n+6)PQR$ with $a = (kM^2+lM+m)(kM^2-lM+m)$, $b = c = 1$ is solvable.

Table 3. Solutions of $x^3 + y^3 + z^3 = nxyz$, $0 < n < 81$

x	y	z	n	x	y	z	n
1	1	1	3	70	151	629	38
1	1	2	5	−9	1	2	40
1	2	3	6	1	2	9	41
2	3	7	9	−819	19	554	44
5	7	18	10	−845	38	367	47
9	13	38	13	9	13	77	51
2	7	13	14	2	7	27	53
−7	1	3	15	2	43	57	54
−70	9	31	16	19	91	310	57
5	18	37	17	−13559153	1513300	1950953	62
13	42	95	18	−3775	247	903	63
1	5	9	19	−1338039	119479	232736	64
−61	13	14	20	1	3	14	66
2	13	21	21	1133	7525	23517	67
9	38	91	26	2	57	73	69
27	43	182	29	−1478979	27083	896668	70
2	21	31	30	−67	7	9	71
−37	1	27	31	89200900157319	2848691279889518	1391526622949983	73
−97	14	19	35	133	2502	4607	74
−151	7	78	36	−45	2	13	76
				−52	5	7	77

Numerical solutions. In [3] (p. 62) a table of solutions of (1) for n-values between -81 and $+80$ was given. By searching all combinations of (8) with $\min(|aP^3|, |bQ^3|, |cR^3|) \le 3^{21}$ if at least two of a, b, c are greater than 1 and $P \le 3^{10}$ if $b = c = 1$, solutions have been found for additional n-values in that range: $n = 64, 70, 73, -32, -48, -50, -56, -65, -67$ (see Tables 3 and 4). In [3] (p. 58) the case $n = 64$ was incorrectly marked as unsolvable.

Acknowledgements. The author is very thankful to Docent Hans Riesel for his valuable suggestions and advice.

Table 4. Solutions of $x^3 + y^3 + z^3 = nxyz$, $-81 \le n < 0$

x	y	z	n	x	y	z	n
−1	1	1	−1	−19	67	234	−44
−1	1	2	−4	−52	21	223	−45
−1	1	3	−9	−8	5	43	−46
−4	1	7	−10	−9	196	221	−47
−4	9	19	−11	−1580745	28843468	39825737	−48
−3	14	19	−12	−1	1	7	−49
−1	1	4	−16	−5920704	1980727	24383561	−50
−1	7	9	−17	−7	76	163	−55
−37	7	78	−21	−63278951	329267696	1064663271	−56
−1	4	9	−22	−1	3	13	−57
−2	1	7	−24	−6244	817	17739	−59
−1	1	5	−25	−3	2	19	−60
−28	109	279	−27	−1	1	8	−64
−325	362	1813	−28	−354485	4094597	9326853	−65
−9	74	127	−29	−127	3423	4432	−66
−72252	401791	927041	−32	−308584	93733	1399411	−67
−3	13	35	−33	−35	914	1251	−68
−7	4	31	−34	−9	1	26	−72
−1333	14220	23233	−35	−715	13483	24577	−73
−1	1	6	−36	−10	7	73	−76
−52	19	193	−37	−2394	853	12581	−77
−1581475	28251	1934524	−38	−823	43	1764	−79
−217	2692	4345	−40	−1	1	9	−81

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Received on 24.2.1994
and in revised form on 5.9.1994 (2569)